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SENSITIVITY OF NORMAL THEORY TESTS FOR EQUALITY OF VARIANCES AND CO-VARIANCE MATRICS AGAINST KURTOSIS CO-EFFICIENT

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ABSTRACT: *The normal theory tests for the equality of k variances and co-variance matrices are described. The size of the likelihood ratio test both in univariate and multi-variate case is obtained. It is shown that the size of the test is influenced by kurtosis when the parent distribution is non-normal; and it is also influenced by the increase in groups (k) and dimensions (p).*

1. INTRODUCTION

The first approach to the problem of testing the equality of k variances under normality was made by Neyman & Pearson (1931), using the likelihood ratio statistic, which is approximately null distributed as chi-square with (k-1) degrees of freedom. For small samples the test has considerably greater sizes (observed significance levels) than the desired nominal levels.

Bartlett (1937) then suggested modifications to the likelihood ratio test which improve the approximation to chi-square. But the investigations carried out by Nair (1938) and Bishop and Nair (1939) showed that the criterion is still not always adequate if some of the degrees of freedom are 1,2 or 3. The 5% and 1% points of Bartlett's criterion, are given in Table-32 (Pearson & Hartley, 1970) permitting degrees of freedom as low as 2.

The above mentioned methods were discussed by Bishop and Nair (1939), and further refinements were made both by Hartley (1940) and Box (1949). The F-test is also available when only the equality of two population variances is to be tested. Plackett (1946) is a good review paper. Kendall & Stuart (1967, p465-69); 1968 p97-105) and Plackett (1960, Chapter 5) are also worth seeing.

Cochran (1941) introduced a statistic for quick assessment, which is best in the situation when just one of several populations is suspected to have a larger variance.

Hartley (1950a) derived a statistic which compares the largest and the smallest of the sample variances under consideration. David (1952) has given more accurate percentage points for this statistic.

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Cadwell (1953) introduced a statistic by comparing the largest and the smallest sample ranges. The percentage points were given by Leslie & Brown (1966). Bartlett & Kendall (1946) compared the homogeneity of variance of k populations by using analysis of variance. Hartley (1950b) compared the power of some of these tests empirically. Gartside (1972) compared the size and power of these methods.

These testing criteria discussed above, for the equality of k variances are derived under the assumption of normality for the random variables. A desirable characteristic of a test is that the significance level and the power of the test should be insensitive or "robust" to departures from this assumption since many random variables are not normally distributed. This was realized by Pearson and Adyanthaya (1929) and Pearson (1932). A crucial question in statistical methodology is therefore: Are statistical procedures derived under normality, robust with respect to departures from normality? Here robustness refers to Type I error, i.e. a statistical procedure is robust if the Type I error is not affected seriously by the departures from the assumption of normality, (Tan, 1982).

There are various studies on robustness on the effects of non-normality upon univariate normal-theory procedures. Pearson (1931) pointed out the sensitivity to non-normality of the tests for comparing two variances. Geary (1947), Finch (1950) and Gayen (1950a) confirmed the findings of Pearson (1931). These authors agreed that this test is particularly sensitive to changes in kurtosis coefficient from the normal-theory value of zero Box (1953) showed that this sensitivity is even greater when the number of variances to be compared exceeds two. In general terms, the main findings are that the tests are non-robust.

Gayen (1950b) and Tiku (1964, 1975) empirically suggested that the F-test is affected seriously by the departures from normality. These results confirm the early findings by Box (1953), Geary (1947) and Pearson (1932) among others .A good discussion was provided by Kendall and Stuart (1979), II, Chapter 31.

The test for the equality of k covariance matrices under normality was derived by Wilks (1932), using the likelihood ratio statistic, approximately null distributed as chi-square with (k-1)p(p+1)/2 degrees of freedom. A modification is given by Box (1949), which is a generalization of the Bartlett test for homogeneity of variances. Korin (1969) has prepared Tables of the upper 5% critical values of the Box (1949) criterion for the case of equal sample sizes. These have been reproduced by Pearson (1969). Hopkins & Clay (1963), Ito (1969), Mardia (1974) and Layard (1972, 1974) studied the effect of non-normality on the test and found it to be sensitive to non-normality. Ito (1969) and Mardia (1974) proved that these normal-theory tests are affected by the kurtosis coefficient of the parent distribution. There are a few other tests (Pillai, 1955; Bagai, 1962) based on the covariance matrices, but those are also non-robust.

The object of this paper is to look at the sensitivity of the normal theory likelihood ratio test of equality of variances and covariance matrics against departures of Kurtosis co-efficient from the normal theory value of zero. The effect on the size of the test for the increasing groups (k) in the univariate case and increasing dimensions (p) as well in the multivariate case is also to be looked at. In section 2 the univariate case and in section 3 the multivariate case is considered. In both of the situations after defining the null hypothesis the size of the likelihood ratio test against non-zero kurtosis co-efficient is obtained. The numerical examples are given to establish the theoretical facts. The summary and conclusions are given in section 4.

2. TESTS FOR EQUALITY OF VARIANCES

In this section hypothesis under study in described. Some tests for equality of variances are mentioned. The relationship between Likelihood-Ratio and F-Tests is proved. Size of the Likelihood Ratio Test under departures from normality is determined. Examples are given in support of theory.

2.1 Hypothesis of Interest

Consider k normal populations with $X_i \rightarrow N(\mu_i, \sigma_i^2)$, i=1,...,k. Suppose X_{ie}; e=1,...,n_i is a random sample of size n, from X_i i=1,...,k. The hypotheses of interest are:

$$H_{01} : \sigma_1^2 = \ldots = \sigma_k^2 \text{ vs } H_{A1} : \sigma_1^2 \neq \ldots \neq \sigma_k^2$$

($X_i \rightarrow N(\mu_i, \sigma_i^2)$, means that X_i has a normal distribution with mean μ_i and variance σ_i^2).

2.2 Description of Tests

a. Likelihood Ratio Test

In 1931, Neyman and Pearson obtained the likelihood ratio test statistic for testing H_{01} vs H_{A1}. Using the notation above the likelihood ratio test statistic is:

$$L = -2\ln\lambda = n \quad \ln s^2 - \sum_{i=1}^{k} n_i \ln s_i^2 . \qquad 2.1$$

where

$$n = \sum_{i=1}^{k} n_i$$

$$s_i^2 = \frac{1}{n_i}\sum_{e=1}^{n_i}(x_{ie} - \overline{x}_i)^2$$

$$s^2 = \frac{1}{n}\sum_{i=1}^{k}\sum_{e=1}^{n_i}(x_{ie} - \overline{x}_i)^2$$

$$\overline{x}_i = \frac{1}{n_i}\sum_{e=1}^{n_i} x_{ie} \qquad 2.2$$

Neyman and Pearson (1931) proved that in the limit when n_i are large $-21n\lambda$ given in 2.1 is approximately distributed as χ^2 with (k-1) degrees of freedom under H_{01}. In particular, when k=2

$$L = -2\ln\lambda = n_i \ln\left(\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}\right) - (n_1 \ln s_1^2 + n_2 \ln s_2^2)$$

is distributed as χ_1^2 (where χ_1^2 means χ^2 with 1 degree of freedom) under H_{01}, as above for $k \geq 2$.

b. Bartlett's Test

Bartlett (1937) proposed an alternative test statistic. It is actually a modified form of the Neyman and Pearson (1931) likelihood ratio test statistic described by 2.1. The modification is only that the n_i are replaced by m_i (degrees of freedom) whenever these appear in the method. We denote this modified method by $M = -21n\lambda^*$.

$$M = -2\ln\lambda^* = m.\ln s^{*2} - \sum_{i=1}^{k} m_i \ln s_i^{*2} \qquad 2.3$$

where

$$m_i = n_i^{-1}$$

$$m = \sum_{i=1}^{k} (n_i - 1) = n - k$$

$$s^{*2} = \frac{1}{m} \sum_{i=1}^{k} \sum_{e=1}^{n_i} (x_{ie} - \bar{x}_i)^2 \qquad 2.4$$

$$s_i^{*2} = \frac{1}{m_i} \sum_{e=1}^{n_i} (x_{ie} - \bar{x}_i)^2 \qquad 2.5$$

The test statistic described by 2.2 is approximately distributed as χ^2_{k-1} under H_{01} for large n_i. Bartlett (1937) showed that for small samples MC^{-1} is more closely approximated by χ^2_{k-1} under H_{01}, where C is:

$$C = 1 + \frac{1}{3(k-1)} \left(\sum_{i=1}^{k} \frac{1}{m_i} - \frac{1}{m} \right)$$

where k = 2

$$M = -2 \ln \lambda^* = m \ \ln \left(\frac{m_1 s_1^{*2} + m_2 s_2^{*2}}{m_1 + m_2} \right) - (m_1 \ln s_1^{*2} + m_2 \ln s_2^{*2})$$

is distributed approximately as χ^2_1 under H_{01} for large n_i

c. Cochran's Test

A simpler test than those that had been proposed earlier is proposed by Cochran in 1941, for quick assessment (before the use of computers). He suggested a ratio of the largest to the sum of the k variance estimates given in 2.5.

$$R = \frac{s^{*2} \max}{\sum_{i=1}^{k} s_i^{*2}}$$

The 5% and 1% significant points for R for the values of k and m are given in Table-31a by Pearson and Hartley (1970).

d. Hartley's Test

Hartley (1950a) proposed an even simpler test than that proposed by Cochran (1941). When all m_i are equal for quick assessment of heterogeneity among different population variances Hartley (1950a) suggested ratio of the largest to the smallest variance estimate as a statistic, i.e.

$$H = \frac{s^{*2}\max}{s^{*2}\min}$$

The 5% and 1% significant points of H are given in Table-31 by Pearson and Hartley (1970).

e. F-Test

Under normality, for k=2 when s_1^{*2} and s_2^{*2} alone are available, the heterogeneity of variances of two populations is usually tested by

$$F = \frac{s_1^{*2}}{s_2^{*2}} \qquad 2.6$$

The procedure is to reject H_{01} at significance level α if

$$F_1 \leq F \leq F_2$$

where

$$F_1 = F(\alpha_1 | m_1, m_2)$$

$$F_2 = F(\alpha_2 | m_1, m_2)$$

$$\alpha = \alpha_1 + \alpha_2$$

Theorem 1.1 gives the relationship between F 2.6 and M 2.2.

2.3 Relationship Between M and F – Tests

The M and F tests described by 2.2 and 2.6, respectively have the relation when k=2:

$$e^{-M} = \{\frac{m}{m_1 F + m_2}\}^m F^{m_1}$$

Proof;

$$e^{-M} = \frac{\prod_{i=1}^{2} (s_i^{*2})^{m_i}}{(s^{*2})^{m_o}}$$

$$e^{-M} = \left(\frac{s_1^{*2}}{s^{*2}}\right)^{m_1} \left(\frac{s_2^{*2}}{s^{*2}}\right)^{m_2}$$

Since from 2.4

$$s^{*2} = \frac{m_1 s_1^{*2} + m_2 s_2^{*2}}{m_1 + m_2}$$

Thus

$$e^{-M} = \left\{\frac{m s_1^{*2}}{m_1 s_1^{*2} + m_2 s_2^{*2}}\right\}^{m_1} \left\{\frac{m s_2^{*2}}{m_1 s_1^{*2} + m_2 s_2^{*2}}\right\}^{m_2}$$

$$e^{-M} = \left\{\frac{m}{m_1 + m_2 \frac{1}{F}}\right\}^{m_1} \left\{\frac{m}{m_1 F + m_2}\right\}^{m_2}$$

$$e^{-M} = \left\{\frac{m}{m_1 F + m_2}\right\}^{m} F^{m_1}$$

$M = 0$ ($s_1^{*2} = s_2^{*2}$) corresponds to the value $F = 1$. For $M > 0$, the upper tail of the M-distribution corresponds to two tails of the F-distribution, but not necessarily be divided equally between the tails. The case when $m_1 = m_2$, where $F_2 = 1/F_1$, then the M-test corresponds to the F-test with equal tails.

2.4 Size of the Likelihood Ratio Test Statistic under Departures from Normality

Let

$$\beta_{2.i} = \frac{\mu_{4.i}}{\mu^2_{2.i}}$$

where

$$\mu_{r.i} = E[X_i - \mu_i]^r$$

Following Box (1953) and Mardia (1974) define as a measure of kurtosis

$$\gamma_{2.i} = \beta_{2.i} - 3$$

Assume common kurtosis in all populations

$$\gamma_{2.i} = \gamma_2$$

Then from Box (1953) for large n_i under H_{01}

$$-2\ln\lambda \to (1+\frac{\gamma_2}{2})\ \chi^2_{k-1} \qquad 2.7$$

and thus

$$E(-2\ln\lambda) = (k-1)(1+\frac{\gamma_2}{2})$$

where -21nλ is given in 2.1

Therefore, the Neyman and Pearson (1931) likelihood ratio test and the Bartlett (1937) test described by 2.1 and 2.3 respectively are sensitive to changes in γ_2 from the normal-theory value of zero. For $\gamma_2 > 0$, H_{01} will be rejected too frequently under H_{01}. The point to note is that influence of γ_2 becomes greater as k increases. It is verified by the following examples. In Example-1: $\gamma_2 > 0$, and in Example-2: $\gamma_2 < 0$.

2.5 Numerical Examples

Example-1

Let us consider the exponential distribution.

$\gamma_2 = 6$

$(1+\frac{\gamma_2}{2}) = 4$

$\alpha = 0.05$

From 2.7 probability of rejecting H_{01} when H_{01} is true is:

$$p(reject\ H_{01} / H_{01}\ true) = p((1+\frac{\gamma_2}{2})\chi^2_{k-1} > \chi^2_{(k-1,.05)})$$

For

K=2 $p(reject\ H_{01} / H_{01}\ true) = p(4\chi^2_1 > 3.8) = p(\chi^2_1 > .95) \doteq .33$

K=4 $p(reject\ H_{01} / H_{01}\ true) = p(4\chi^2_3 > 7.8) = p(\chi^2_3 > 1.95) \doteq .58$

K=6 $p(reject\ H_{01} / H_{01}\ true) = p(4\chi^2_5 > 11.1) = p(\chi^2_5 > 2.78) \doteq .73$

Example-2

Now we consider uniform distribution.

$\gamma_2 = -1.2$

$(1+\frac{\gamma_2}{2}) = .4$

$\alpha = 0.05$

From 2.7, similar to Example 1 for

$k = 2$ $p(reject\ H_{01} / H_{01}\ true) = p(.4\chi^2_1 > 3.8) = p(\chi^2_1 > 9.5) \doteq .002$

$k = 4$ $p(reject\ H_{01} / H_{01}\ true) = p(.4\chi^2_3 > 7.8) = p(\chi^2_3 > 19.5) \doteq .002$

REMARKS

1. In Example-1, the case of the exponential distribution is discussed, where $\gamma_2 = 6$. The Likelihood ratio test rejects the null hypothesis Ho_1 too frequently under Ho_1. For k = 2 the size of the test is 0.33 and it increases rapidly with the increase in the number of groups (K). For six groups the size is 0.73 against $\alpha = 0.05$.
2. In Example-2, the case of the uniform distribution is considered. For this distribution $\gamma_2 = -1.2$, that is less than zero. Now the effect on the size of the likelihood ratio test is in reverse order. The size of the test is 0.002 for k = 2 and 4 against $\alpha = 0.05$. The test rejects the null hypothesis Ho_1 too infrequently under Ho_1.

3. TESTS FOR EQUALITY OF CO-VARIANCE MATRICES

In this section hypothesis under study is discussed. The most commonly used tests for the Equality of Co-variance Matrices under normality assumption are discussed. The size of the Likelihood Ratio Test under departure of kurtosis co-efficient from the normal theory value of zero is obtained. An important result stated by Muirhead (1982) without proof is proved. The ϵ-contaminated normal distribution is considered. By a numerical example it is shown that for kurtosis co-efficient of the sampled distribution greater than zero. Ho_2 is rejected too frequently under Ho_2.

3.1 Hypothesis of Interest

Consider k multivariate normal populations with X_i as $N_p(\underline{\mu}_i, \underline{\Sigma}_i)$. Suppose $X_{ie} = [X_{ie1} \cdots X_{iep}]^T$ e=1,...,n is a random sample of size n, from $\underline{X}_i$, i=1,...,k. We wish to test the hypotheses:

$$H_{02} : \underline{\Sigma}_1 = \ldots = \underline{\Sigma}_k \text{ vs } H_{A2} : \underline{\Sigma}_1 \neq \ldots \neq \underline{\Sigma}_k$$

3.2 Description of Tests

a. Likelihood Ratio Test

If $X_{ie} = (x_{ie1}, \ldots, x_{iep})^T$: i=1,...$n_i$ is a random vector sample of size n_i from a p-variate normal distribution with unknown mean vector $\mu_i = (\mu_{i1}, \ldots, \mu_{ip})^T$ and covariance matrices $\underline{\Sigma}_i$ The likelihood ratio test statistic (Wilks, 1932).

$$-2\ln\lambda = n\ln|\underline{S}| - \sum_{i=1}^{k} n_i \ln|\underline{S}_i| \qquad 3.1$$

is distributed approximately as χ^2 with (k-1) p(p+1)/2 degrees of freedom under H_{02} for large n_i, where $\underline{S}_i$ are the sample estimated covariance matrices based on n_i observations and $\underline{S}$ the corresponding pooled covariance matrix based on $n = \sum_{i=1}^{k} n_i$ observations.

$$\underline{S}_i = \frac{1}{n_i}\sum_{e=1}^{n_i}(\underline{x}_{ie} - \bar{\underline{x}}_i)\,(\underline{x}_{ie} - \bar{\underline{x}}_i)^T \qquad 3.2$$

$$\underline{S} = \frac{1}{n}\sum_{i=1}^{k_i} n_i \underline{S}_i$$

$$\underline{\overline{x}}_i = \frac{1}{n_i}\sum_{e=1}^{n_i} \underline{x}_{ie} \qquad 3.3$$

when k=2

$$-2\ln\lambda = n\ \ln\left|\frac{n_1\underline{S}_1 + n_2\underline{S}_2}{n_1 + n_2}\right| - (n_1\ \ln|\underline{S}_1| + n_2\ \ln|\underline{S}_2|) \qquad 3.4$$

is distributed approximately as χ^2 with ½p(p+1) degrees of freedom under H_{02} for large n_i.

b. Box Test

As mentioned earlier, Box (1949) modified the likelihood ratio test statistic of Wilks (1932). He proposed to replace n_i by $m_i = n_i^{-1}$, the degrees of freedom, and n by m where:

$$m = \sum_{i=1}^{k}(n_i - 1) = n - k$$

Box (1949) proved that:

$$-2\ln\lambda^* = m\ln\left|\underline{S}^*\right| - \sum_{i=1}^{k} m_i \ln\left|\underline{S}_i^*\right|$$

is also distributed approximately as χ^2 with ½p(p+1)(k-1) degrees of freedom under H_{02} for large n_i, where $\underline{S}_i^*$ are the unbiased sample estimated covariance matrices based on m_i degrees of freedom, and $\underline{S}^*$ the corresponding unbiased pooled covariance matrix based on m degrees of freedom.

$$\underline{S}_i^* = \frac{1}{m_{oi}}\sum_{e=1}^{n_i}(\underline{x}_{ie} - \underline{\overline{x}})^T\ \ (\underline{x}_{ie} - \underline{\overline{x}})^T.$$

$$\underline{S}^* = \frac{1}{m}\sum_{i=1}^{k} m_i \underline{S}_i^*$$

$\underline{\overline{x}}_i$ described by 3.3

The χ^2 approximation of $-2C\ln\lambda^*$ is not so good in particular when m_i are small. Box (1949) showed that if the scale factor C is introduced, where:

$$C = 1 - \frac{2p^2 + 3p - 1}{6(p+1)(k-1)}\left(\sum_{i=1}^{k}\frac{1}{m_i} - \frac{1}{m}\right)$$

Then $-2C\ell n\lambda^x$ gives a better approximation to χ^2 variate with ½p(p+1)(k-1) degrees of freedom under H_{02}. The approximation appears to be good if k and p do not exceed four or five and each m_i is not less than 20 (Morrison, 1976). When k=2;

$$-2\ln\lambda = m\ \ln\left|\frac{m_1\underline{S}^*_1 + m_2\underline{S}^*_2}{m_1 + m_2}\right| - (m_1\ \ln\left|\underline{S}^*_1\right| + m_2\ \ln\left|\underline{S}^*_2\right|)$$

is approximately distributed as χ^2 with ½p(p+1) degrees of freedom under H_{02} for large n_i.

3.3 Size of the Likelihood Ratio Test Statistic under Departures from Normality

As in the univariate case the size of the likelihood ratio test of equality of k covariance matrices is seriously influenced by kurtosis. Before showing this fact we prove a result stated by muirhead (1982) (without proof) in the following Theorem.

Theorem

For samples of size n_1 and n_2 from two p-variate normal distributions with covariance matrices $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$, write;

$n_i = K_i n$ i=1,2 with $(K_1+K_2) = 1$ and $\underline{S}_i = \underline{\Sigma} + (n K_i)^{-½}\underline{Z}_i$

where $\underline{\Sigma}$ is the common value of $\underline{\Sigma}_1$ and $\underline{\Sigma}_2$ under H_{02}. Then the likelihood ratio test statistic described by 3.4 has the following expansion when H_{02} is true for K_1, K_2 fixed.

$$-2\ln\lambda = ½K_2 tr(\underline{\Sigma}^{-1}\underline{Z}_1)^2 + ½K_1 tr(\underline{\Sigma}^{-1}\underline{Z}_2)^2 - (K_1K_2)^{½}$$

$$tr(\underline{\Sigma}^{-1}\underline{Z}_1\,\underline{\Sigma}^{-1}\underline{Z}_2) + O_p(n^{-½})$$

Proof

Let $g(\underline{S}) = \ln|\underline{S}|$ where $\underline{S}$ is sample covariance matrix based on n′ observations.

$$\frac{\partial g(\underline{S})}{\partial S_{rs}} = \frac{1}{|\underline{S}|}\frac{\partial|\underline{S}|}{\partial S_{rs}} = S^{-1}_{rs} \text{ if } r = s$$

$$= 2S^{-1}_{rs} \text{ if } r \neq s$$

$$\frac{\partial^2 g(\underline{S})}{\partial S_{rs}\partial S_{uv}} = -S^{-1}_{ru}\, S^{-1}_{vs} \text{ if } r = s,\ u = v$$

$$= -2S^{-1}_{ru}\, S^{-1}_{vs} \text{ if } r \neq s,\ u = v$$

$$= -[S^{-1}_{ru}\, S^{-1}_{vs} + S^{-1}_{rv}\, S^{-1}_{su}] \text{ if } r = s,\ u \neq v$$

$$= -2[S^{-1}_{ru}\, S^{-1}_{vs} + S^{-1}_{rv}\, S^{-1}_{su}] \text{ if } r \neq s,\ u \neq v$$

(Morrison, 1976,p.74)

Since

$$\underline{S} = \underline{\Sigma} + O_p(n'^{-1/2})$$

by Taylor series expansion

$$\ln|\underline{S}| = \ln|\underline{\Sigma}| + \sum_{rs} \Sigma^{-1}_{rs}(S_{rs} - \Sigma_{rs})$$

$$-\tfrac{1}{2} \sum_{rs,uv} \Sigma^{-1}_{ru}\Sigma^{-1}_{vs}(S_{rs} - \Sigma_{rs})(S_{uv} - \Sigma_{uv}) + O_p(n'^{-3/2}) \qquad 3.5$$

(e.g Fuller, 1976,p:192)

Now

$$\sum_{rs} \Sigma^{-1}_{rs}(S_{rs} - \Sigma_{rs}) = \mathrm{tr}\{\underline{\Sigma}^{-1}(\underline{S} - \underline{\Sigma})\} \qquad 3.6$$

and

$$\sum_{rs,uv} \Sigma^{-1}_{ru}\Sigma^{-1}_{vs}(S_{rs} - \Sigma_{rs})(S_{uv} - \Sigma_{uv}) = \mathrm{tr}[\underline{\Sigma}^{-1}(\underline{S} - \underline{\Sigma})\ \ \underline{\Sigma}^{-1}(\underline{S} - \underline{\Sigma})] \qquad 3.7$$

Thus from 3.5, 3.6 and 3.7 we can write

$$ln|\underline{S}| = ln|\underline{\Sigma}| + tr\{\underline{\Sigma}^{-1}(\underline{S} - \underline{\Sigma})\} - \tfrac{1}{2}\mathrm{tr}\{\underline{\Sigma}^{-1}(\underline{S} - \underline{\Sigma})\,\underline{\Sigma}^{-1}(\underline{S} - \underline{\Sigma})\} + O_p(n^{-3/2}) \qquad 3.8$$

$\underline{S}_1$ is a sample covariance matrix based on n_1 observations, so we can immediately write from 3.8

$$\ln\left|\underline{S}_1\right| = \ln\left|\underline{\Sigma}\right| + tr\{\underline{\Sigma}^{-1}(\underline{S}_1 - \underline{\Sigma})\} - ½tr\{\underline{\Sigma}^{-1}(\underline{S}_1 - \underline{\Sigma})\underline{\Sigma}^{-1}(\underline{S}_1 - \underline{\Sigma})\} + O_p(n_{01}^{-3/2}) \quad 3.9$$

Since

$$(\underline{S}_1 - \underline{\Sigma}) = (nK_1)^{½}\underline{Z}_1$$

Then 3.9 becomes

$$\ln\left|\underline{S}_1\right| = \ln\left|\underline{\Sigma}\right| + tr\{\underline{\Sigma}^{-1}(nK_1)^{-½}\underline{Z}_1\}$$
$$- ½tr\{\underline{\Sigma}^{-1}(nK_1)^{-½}\underline{Z}_1\underline{\Sigma}^{-1}(nK_1)^{-½}\underline{Z}_1\} + O_p(n_1^{-3/2}) \quad 3.10$$

$n_1 = nK_1$

To get $n_1 \ln\left|\underline{S}_1\right|$ we multiply both sides of equation 3.10 by n_1 and have

$$n_1 \ln\left|\underline{S}_1\right| = nK_1 \ln\left|\underline{\Sigma}\right| + (nK_1)^{½} tr(\underline{\Sigma}^{-1}\underline{Z}_1) - ½tr(\underline{\Sigma}^{-1}\underline{Z}_1)^2 + O_p(n^{-½}) \quad 3.11$$

Similarly, for the second sample covariance matrix we can write

$$n_2 \ln\left|\underline{S}_2\right| = nK_2 \ln\left|\underline{\Sigma}\right| + (nK_2)^{½} tr(\underline{\Sigma}^{-1}\underline{Z}_2) - ½tr(\underline{\Sigma}^{-1}\underline{Z}_2)^2 + O_p(n^{-½}) \quad 3.12$$

As we defined earlier $\underline{S}$ is a pooled covariance matrix based on n observations by 3.2 i.e.

$$\underline{S} = \frac{n_1\underline{S}_1 + n_2\underline{S}_2}{n_1 + n_2} = n^{-1}(n_1\underline{S}_1 + n_2\underline{S}_2) \quad 3.13$$

Since

$$\underline{S}_1 = \underline{\Sigma} + (n\,K_1)^{-½}\underline{Z}_1$$

$$\underline{S}_2 = \underline{\Sigma} + (n\,K_2)^{-½}\underline{Z}_2$$

Then 3.13 becomes

$$\underline{S} = n^{-1}[n K_1\{\underline{\Sigma} + (n K_1)^{-½}\underline{Z}_1\} + n K_2\{\underline{\Sigma} + (n K_2)^{-½}\underline{Z}_2\}]$$

$$= n^{-1}[n K_1\{\underline{\Sigma} + (n K_1)^{½}\underline{Z}_1\} + n K_2\{\underline{\Sigma} + (n K_2)^{-½}\underline{Z}_2\}]$$

$$= (K_1\underline{\Sigma} + K_2\underline{\Sigma}) + n^{-1}[(n K_1)^{½}\underline{Z}_1 + (n K_2)^{½}\underline{Z}_2]$$

$$= \underline{\Sigma}(K_1 + K_2) + n^{-1}[(n K_1)^{½}\underline{Z}_1 + (n K_2)^{½}\underline{Z}_2] \qquad 3.14$$

From 3.14

$$(\underline{S} - \underline{\Sigma}) = n^{-1}[n K_1)^{½}\underline{Z}_1 + (n K_2)^{½}\underline{Z}_2] \qquad 3.15$$

From 3.8 and 3.15 $\ln|\underline{S}|$ **follows as;**

$$\ln|\underline{S}| = \ln|\underline{\Sigma}| + \mathrm{tr}\{\underline{\Sigma}^{-1} n^{-1}[\{(n K_1)^{½}\underline{Z}_1 + (n K_2)^{½}\underline{Z}_2\}]$$

$$-½\mathrm{tr}[\underline{\Sigma}^{-1} n^{-1}\{(n K_1)^{½}\underline{Z}_1 + (n K_2)^{½}\underline{Z}_2\}]^2 + O_p(n^{-3/2})$$

$$n \ln|\underline{S}| = n \ln|\underline{\Sigma}| + n\ \mathrm{tr}\{\underline{\Sigma}^{-1} n^{-1}(n K_1)^{½}\underline{Z}_1 + \underline{\Sigma}^{-1} n^{-1}(n K_2)^{½}\underline{Z}_2)\}$$

$$-\frac{n}{2} tr\{\underline{\Sigma}^{-1} n^{-1}(n K_1)^{½}\underline{Z}_1 + \underline{\Sigma}^{-1} n^{-1}(n K_2)^{½}\underline{Z}_2\}^2 + O_p(n^{-½})$$

$$= n \ln|\underline{\Sigma}| + \mathrm{tr}\{\underline{\Sigma}^{-1}(n K_1)^{½}\underline{Z}_1 + \underline{\Sigma}^{-1}(n K_2)^{½}\underline{Z}_2\}$$

$$-\frac{1}{2n} tr\{\underline{\Sigma}^{-1}(n K_1)^{½}\underline{Z}_1 + \underline{\Sigma}^{-1}(n K_2)^{½}\underline{Z}_2\}^2 + O_p(n^{-½})$$

$$= n \ln|\underline{\Sigma}| + (n K_1)^{½}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1) + (n K_2)^{½}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_2)$$

$$-\frac{1}{2n}\mathrm{tr}[\{\underline{\Sigma}^{-1}(n K_1)^{½}\underline{Z}_1\}^2 + \{\underline{\Sigma}^{-1}(n K_2)^{½}\underline{Z}_2\}^2$$

$$+ 2\underline{\Sigma}^{-1}(n K_1)^{½}\underline{Z}_1\underline{\Sigma}^{-1}(n K_2)^{½}\underline{Z}_2] + O_p(n^{-½})$$

$$= n \ln|\underline{\Sigma}| + (n K_1)^{½}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1) + (n K_2)^{½}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_2)$$

$$-\frac{1}{2n}\mathrm{tr}(\underline{\Sigma}^{-1}(n K_1)^{½}\underline{Z}_1)^2 - \frac{1}{2n}\mathrm{tr}(\underline{\Sigma}^{-1}(n K_2)^{½}\underline{Z}_2)^2$$

$$-\frac{1}{n}\mathrm{tr}\{\underline{\Sigma}^{-1}(n\,K_1)^{1/2}\underline{Z}_1\underline{\Sigma}^{-1}(n\,K_2)^{1/2}\underline{Z}_2\}+O_p(n^{-1/2})$$

$$=n\ \ln|\underline{\Sigma}|+(n\,K_1)^{1/2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1)+(n\,K_2)^{1/2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_2)$$

$$-\frac{1}{2n}(n\,K_1)\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1)^2-\frac{1}{2n}(n\,K_2)\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_2)^2$$

$$-\frac{1}{n}(n\,K_1)^{1/2}(n\,K_2)^{1/2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1\,\underline{\Sigma}^{-1}\underline{Z}_2)+O_p(n^{-1/2})$$

$$=n\ \ln|\underline{\Sigma}|+(n\,K_1)^{1/2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1)+(n\,K_2)^{1/2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_2)$$

$$-\tfrac{1}{2}K_1\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1)^2-\tfrac{1}{2}K_2\,\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_2)^2 \qquad 3.16$$

$$-(K_1K_2)^{1/2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1\,\underline{\Sigma}^{-1}\underline{Z}_2)+O_p(n^{-1/2})$$

From 3.11, 3.12 and 3.16 we are in a position to write 3.4 as:

$$-2\ln\lambda=n\ln|\underline{\Sigma}|+(n\ K_1)^{1/2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1)+(n\ K_2)^{1/2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_2)$$

$$-\tfrac{1}{2}K_1\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1)^2-\tfrac{1}{2}K_2\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_2)^2$$

$$-K_1K_2)^{1/2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1\underline{\Sigma}^{-1}\underline{Z}_2)+O_p(n^{-1/2})$$

$$-n\,K_1\ln|\underline{\Sigma}|-(n\,K_1)^{1/2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1)$$

$$+\tfrac{1}{2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1)^2-O_p(n^{-1/2})-n\,K_2\ln|\underline{\Sigma}|$$

$$-(nk_2)^{1/2}tr(\underline{\Sigma}^{-1}\underline{Z}_2)+\tfrac{1}{2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_2)^2-O_p(n^{1/2})$$

$$=n\ln|\underline{\Sigma}|-n\,K_1\ln|\underline{\Sigma}|-n\,K_2\ln|\underline{\Sigma}|-\tfrac{1}{2}K_1\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1)^2$$

$$-\tfrac{1}{2}K_2\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_2)^2+\tfrac{1}{2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1)^2+\tfrac{1}{2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_2)^2$$

$$-(K_1K_2)^{1/2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1\underline{\Sigma}^{-1}\underline{Z}_2)+O_p(n^{-1/2})$$

$$=\tfrac{1}{2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1)^2+\tfrac{1}{2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_2)^2-\tfrac{1}{2}K_1\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1)^2$$

3.17

$$-\tfrac{1}{2}K_2\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_2)^2-(K_1K_2)^{1/2}\,\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1\underline{\Sigma}^{-1}\underline{Z}_2)+O_p(n^{-1/2})$$

$$=\tfrac{1}{2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1)^2(1-K_1)+\tfrac{1}{2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_2)^2(1-K_2)$$

$$-(K_1K_2)^{1/2}\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1\underline{\Sigma}^{-1}\underline{Z}_2)+O_p(n^{-1/2})$$

$$=\tfrac{1}{2}K_2\mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_1)^2+\tfrac{1}{2}K_1\ \mathrm{tr}(\underline{\Sigma}^{-1}\underline{Z}_2)^2$$

$$-(K_1K_2)^{1/2}\,tr(\underline{\Sigma}^{-1}\underline{Z}_1\underline{\Sigma}^{-1}\underline{Z}_2)+O_p(n^{-1/2})$$

Mardia (1974) gave a generalized case for k populations to 3.17 i.e.

$$-2\ln\lambda = \tfrac{1}{2}[\sum_{i=1}^{k} tr((\underline{\Sigma}^{-1}\underline{Z}_i)^2 - tr\{\sum_{i=1}^{k} K_i^{1/2}\underline{\Sigma}^{-1}\underline{Z}_i\}^2] + O_p(n^{-1/2}) \qquad \textbf{3.18}$$

Following Mardia (1974) define;

$$\beta_2, p, i = E[(\underline{X}_i - \underline{\mu}_i)^T \underline{\Sigma}^{-1} (\underline{X}_i - \underline{\mu}_i)]^2$$

Where

$$\underline{\mu}_i = E(X_i)$$

Following Mardia (1974) define as a measure of multivariate kurtosis;

$$\gamma_{2.p.i} = \beta_{2.p.i} - p(p+2)$$

Assume

$$\gamma_2, p, i = \gamma_2, p, i$$

From Mardia (1974)

$$E[tr(\underline{\Sigma}^{-1}\underline{Z}_i)^2] = p(p+1)\{1+\frac{\gamma_2, p}{p(p+1)}\} \qquad \textbf{3.19}$$

$$E[tr\{\sum_{i=1}^{k} K_i^{1/2}\underline{Z}_i\underline{\Sigma}^{-1}\}^2] = \sum_{i=1}^{k} K_i\{p(p+1)+\gamma_2, \gamma_{2,p}\} \qquad \textbf{3.20}$$

Hence, ignoring the residual term of O_p ($n^{-1/2}$) from 3.18, 3.19 and 3.20;

$$E[-2\ln\lambda] = \tfrac{1}{2}(k-1)p(p+1)\{1+\frac{\gamma_2, p}{p(p+1)}\} \qquad \textbf{3.21}$$

For p=1, 3.21 is Box's result given in (2.8).

From 3.21 we can say that size of the likelihood ratio test statistic given in 3.1 is influenced by kurtosis when the parent distribution is non-normal, and similar to the univariate case the effect becomes larger as k (number of groups) increases. It is verified by the following Example. We look at the effect of increase in p in Example-3 as well.

3.4 Numerical Example

Example 3

The ε-contaminated normal distribution is considered. Let $\varepsilon = .10$ and $\sigma^2 = 9$. Thus $\kappa = 1.78$, and $\gamma_{2,p}$ for $p = 2,3,4$ and 5 is:

$\gamma_{2,2} = 14.24$
$\gamma_{2,3} = 26.70$
$\gamma_{2,4} = 42.72$
$\gamma_{2,5} = 62.30$
(of. Muirhead and Waternaux, 1980)
As an approximation suppose, following 3.21

$$-2\ln\lambda \rightarrow (1+\frac{\gamma_{2,p}}{p(p+1)})\chi^2(k-1)p(p+1)/2$$

$$p(\text{reject } H_{02}/H_{02} \text{ true}) = p[(1+\frac{\gamma_{2,p}}{p(p+1)})\chi^2(k-1)p(p+1)/2]$$
$$> \chi^2(k-1)p(p+1)/2, (0.05)$$

k=2 α=.05

For

$p = 2$ " $= p(3.37\chi_3^2 > 7.815)$
$= p(\chi_3^2 > 2.32) \doteq .51$

$p = 3$ " $= p(3.225\chi_6^2 > 12.5916)$
$= p(\chi_6^2 > 3.904) \doteq .69$

$p = 4$ " $= p(3.136\chi_{10}^2 > 18.307)$
$= p(\chi_{10}^2 > 5.838) \doteq .83$

$p = 5$ " $= p(3.08\chi_{15}^2 > 24.9958)$
$= p(\chi_{15}^2 > 8.12) \doteq .92$

k=3 α=.05

For

$p = 2$ " $= p(3.37\chi_6^2 > 12.5916)$
$= p(\chi_6^2 > 3.736) \doteq .71$

$$
\begin{aligned}
p = 3 \quad '' \quad & = p(3.225\chi^2_{12} > 21.0261) \\
& = p(\chi^2_{12} > 6.52) \doteq .89 \\
p = 4 \quad '' \quad & = p(3.136\chi^2_{20} > 31.4104) \\
& = p(\chi^2_{20} > 10.02) \doteq .968 \\
p = 5 \quad '' \quad & = p(3.08\chi^2_{30} > 43.773) \\
& = p(\chi^2_{30} > 14.212) \doteq .993
\end{aligned}
$$

REMARKS

For the ϵ-contaminated normal distribution the likelihood ratio test rejects the null hypothesis Ho_2 too frequently under Ho_2. The size of the test is 0.51 for $k = 2$ and 0.71 for $k = 3$ against $\alpha = 0.05$. The same happens with the increase in dimensions (p) of the covariance matrix.

4. SUMMARY AND CONCLUSIONS

The likelihood ratio test of testing the equality of k variances is sensitive to changes in kurtosis co-efficient from the normal theory value of zero. This sensitivity becomes greater when the number of variances to be compared exceeds than two. The size of test under Ho_1 is obtained in section 2. To find the numeric value of size the exponential distribution is considered for which (kurtosis co-efficient) $\gamma_2 = 6$. The probability of rejecting Ho_1 when it is true for $\alpha = 0.05$ and (number of groups) $k = 2$ is 0.33. There is a significant increase in sizes for $K = 4$ and 6. The value of sizes are 0.58 and 0.73; respectively. It becomes obvious that the test rejects the null hypothesis too frequently if the value of $\gamma_2 > 0$ and this effect increases with the increase in the number of groups. The second case is the uniform distribution for which $\gamma_2 < 0$; i.e. -1.2. The observed size for $k = 2$ is 0.002 which is very low as compared to $\alpha = 0.05$. Therefore, it becomes evident that test rejects null hypothesis too infrequently if the value of $\gamma_2 < 0$.

The likelihood ratio test of testing equality of k co-variance matrices is also sensitive for other values of kurtosis co-efficient than zero. The size of the test is obtained in section 3. For numerical verification ϵ-contaminated normal distribution is considered. The size of the test is much higher than 0.05 for $k = 2$ and 3. The respective values of the sizes are 0.51 and 0.92. It is confirmed that the test rejects the Ho_2 too frequently and even more for larger values of k. The effect on the size of the test for different dimensions (p) of the covariance matrix is also observed. For $k = 3$, and $p = 2, 3, 4$ and 5 the sizes are 0.71, 0.89, 0.968 and 0.993; respectively. The frequency of the rejection of Ho_2 when it is true increases with the increase in the dimension of the co-variance matrix.

It is concluded that:

Univariate Case

1. The likelihood ratio test for testing the equality of k variances is sensitive to the changes in the kurtosis co-efficient.
2. The size of the test increases against normal value of $\alpha = 0.05$ if $\gamma_2 > 0$ and decreases if $\gamma_2 < 0$.
3. The size of the test increases with the increase in the group size (K).

Multivariate Case

1. The likelihood ratio test for the testing the equality of k covariance matrices is also sensitive to the changes in the kurtosis co-efficient.
2. The size of the test increases against normal value of $\alpha = 0.5$ with the increase in the groups (k).
3. The size of the test increases against normal value of $\alpha = 0.05$ with the increase in the dimensions (p) of the matrix.
4.

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