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# MOMENTS OF A LINEAR COMBINATION OF TWO LOGISTIC ORDER STATISTICS

BY

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1. **INTRODUCTION:** In literature, contributions appear on logistic order statistics from Birbaum and Dudman, Gupta and Shah, Shah, Tarter and Clark. Malik and Abraham also investigate the exact distribution of quasimedians. The expressions developed by them for these distributions appear to have an error [+], and even when corrected the consequent formulas remain fairly complicated for numerical studies. For this purpose as well, in this paper we determine moments of

$$Z = aY_i + bY_j, \qquad 1 \le i < j \le n \quad (1.1)$$

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[+] The variable W introduced by Malik and Abraham in their paper for the integral in Eq. (2.4) is assigned the limits form 0 to 1, whereas it cannot be independent of U.

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$$= m(r-1)!\sum_{j=m-1}^{r-1}(j!)^{-1}(A^{(j)})_{m-1} \quad 1 < m \le r \qquad (2.1)$$

$$= 0 \qquad\qquad m > r$$

where

$$(A^{(r)})_\ell = \frac{d^\ell}{dt^\ell} A^{(r)}(t)|_{t=0}$$

**Proof.** Expressing $A^{(r)}(t)$ *as* $\sum_{i=1}^{r} a_{i,r}\, t^i$, we can get the following relation among the coefficients of $t^i$ in the two polynomials $A^{(r-1)}(t)$ *and* $A^{(r)}(t)$, on multiplying the former by (t+r-1).

$$a_{i,r} = a_{i-1,r-1} + (r-1)a_{i,r-1} \qquad (2.2)$$

For any $A^{(j)}(t)$, the coefficient $a_{i,j} > 0$ when $1 \le i \le j$, and is zero otherwise. In fact

$$a_{i,j} = (i!)^{-1}(A^{(j)})_i$$

So that the result for the case m=1 is now obvious. When m>1, the proof follows by the repeated use of Eq. (2.2).

**3. MOMENTS OF Z:** Let us first determine the moment generating function of Z.

where $Y_1, Y_2, \ldots\ldots, Y_n$ are the order statistics based on a random sample of n observations from the logistic distribution defined in its reduced form by the equation

$$f(x) = e^{-x} \Big/ (1+e^{-x})^2, \qquad -\infty < x < \infty \quad (1.2)$$

which has the mean zero and variance $\pi^2/3$. The constants a,b may be assigned relevant numbers to provide for an order statistic, sum of two order statistics, sample median, quasimedian etc.

We obtain a general result for $E(Z^K)$, k=1,2,...... The first four moments of Z are expressed in terms of $\psi^i(x)$ functions ++ , used by Shah, the evaluation of which is not difficult. These moments are derived for particular statistics such as the order statistics $Y_i$, sample medians and quasimedians by substituting the relevant values of a and b in (1.1) and the main result. The information on these moments can be useful in investigating the nature of their distributions and assessing their departures from normality.

**2. LEMMA:** Let $A^{(r)}(t)$ denote the polynomial (t+r-1)(t+r-2) ...(t+1)t in t of degree r. Then

$$(A^{(r)})_m = (r-1)! \qquad 1 = m \leq r$$

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++ $\psi^i(x) = \frac{d^{i+1}}{dt^{i+1}}\left[\log(\Gamma(x+t))\right] \Big|_{t=0}$. For i=0,1,2,...... the corresponding functions are expressed as series in x in Eq. (3.5) of this paper.

where

$$A^{(r)}(at) = (at+r-1)(at+r-2)\ldots\ldots(at+1)(at),$$

and

$$\beta_r(t) = \beta_r^{*}(t)\beta_r^{**}(t)$$

with

$$\beta^{*}(t) = \beta\,(at+r+i,\, j-i) \tag{3.2}$$

$$\beta^{**}(t) = \beta\,(\overline{a+b}t+r+j,\, n-j-bt+1)$$

as the beta functions.

The $k$th moment of Z about zero is

$$\mu_k' = \frac{d^k}{dt^k} M(t) \quad |_{t=0}$$

On carrying out differentiation $\ell$ times on $A^{(r)}(at)\ \beta_r(t)$, and writing

$$\frac{d^{\ell} A^{(r)}(at)}{dt^{\ell}}|_{t=0} = a^{\ell}\,\frac{d^{\ell} A^{(r)}(t)}{dt^{\ell}}|_{t=0}\ for\ a^{\ell} A^{(r)}$$

and

$$\frac{d^{\ell}\beta_r(t)}{dt^{\ell}}|_{t=0}\ for\ (\beta_r)_{\ell}, \tag{3.3}$$

it is seen that

On applying the usual formula for the order statistics $Y_i, Y_j$ when the parent distribution is based on Eq. (1.2), we have the probability density $g(Y_i, Y_j)$ given as

$$\beta_0^{-1}(e^{-y_i} - e^{-y_j})^{j-i-1} e^{-y_i-(n-j+1)y_j}(1+e^{-y_i})^{-j}(1+e^{-y_j})^{n-j+1}$$

$$-\infty < Y_i < Y_j < \infty$$

where $\beta_0$ is the product $\beta(i, j-i).\beta(j, n-j+1)$ of two beta coefficients. The moment generating function of Z, given by

$$M(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\exp(\text{at } y_i + \text{bt } y_j)]\, g(y_i, y_j)\, dy_i\, dy_j$$

is reduced to the integral

$$\beta_0^{-1}\int_0^1\int_0^v u^{at+i-1} v^{bt}(1-u)^{-at}(1-v)^{n-bt-j}(v-u)^{j-i-1}\, du\, dv$$

on using the transformation

$$u = 1\big/(1+e^{-y_i}),\; v = 1\big/(1+e^{-y_j}).$$

Following now a similar approach as in Shah [5], that is, expanding $(1-u)^{-at}$ and integrating w.r.t. u and then v, we get

$$M(t) = \beta_o^{-1}\sum_{r=0}^{\infty}\frac{1}{r!}A^{(r)}(at)\,\beta_r(t), \tag{3.1}$$

$$\psi^{o}(z) = -r + \sum_{\eta=1}^{\infty} \frac{z-1}{\eta(\eta+z-1)}$$

$$\psi^{\ell-1}(z) = (-1)^{\ell}\Gamma(1)\sum_{\eta=1}^{\infty} \frac{z-1}{(\eta+z-1)^{\ell}}$$

With

$\lambda_1 = a, \lambda_2 = r+i, \lambda_3 = 0, \lambda_4 = j-i$, from above for $\ell \geq 1$

$$(\log \beta_r^{*})_{\ell} = a^{\ell}[\psi^{\ell-1}(r+i) - \psi^{\ell-1}(r+j)] \tag{3.6}$$

Similarly,

$$(\log \beta_r^{**})_{\ell} = (a+b)^{\ell}\psi^{\ell-1}(r+j) + (-b)^{\ell}\psi^{\ell-1}(n-j+1)$$

$$-a^{\ell}\psi^{\ell-1}(n+r+1) \tag{3.7}$$

In the summation (3.4), $(A^{r})_{o}$, which is $A^{r}$, has obviously the value zero for $r \geq 1$ and one for r=0, and $(\beta_r)_o$ is $\beta_r$. It may also be noted that the notation $\beta_o$ adopted earlier in $g(y_i, y_j)$ is in fact the value of $\beta_r$ for r=0.

We can summarize the foregoing results and state the following theorem.

THEOREM: Let $Z = aY_i + bY_j$ where $Y_i < Y_j$ are the order statistics of a random sample of size n from the logistic distribution defined by Eq.(1.2). The *k*th moment of Z is given as

$$\frac{d^k}{dt^k}[A^{(r)}(at), \beta_r(t)]\big|_{t=0} = \sum_{\ell=0}^{k}\binom{k}{\ell} a^{\ell} (A^{(r)})_{\ell} (\beta_r)_{k-\ell} \quad (3.4)$$

The *k*th moment of Z about zero includes the above terms of all values of r from 0 to ∞. So, the next step is evaluation of the quantities involved in the above summation. As regards $(A^{(r)})_{\ell}$, this is readily available from Lemma 1. For the purpose of calculation of $(\beta_r)_{k-\ell}$ in (3.4), which is turn depends upon $\beta^{*}$ and $\beta^{**}$ defined in (3.2), we introduce a beta function $\beta(\lambda_1 t + \lambda_2, \lambda_3 t + \lambda_4)$ with $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ as constants independent of t. Expressing it in gamma functions and differentiating $\ell$ times with respect to t, it is not difficult to see that

$$\frac{d^{\ell}}{dt^{\ell}} \log \beta(\lambda_1 t + \lambda_2, \lambda_3 t + \lambda_4) \big|_{t=0},$$

which may be written as $(\log \beta)_{\ell}$, comes to

$$\lambda_1^{\ell} \psi^{\ell-1}(\lambda_2) + \lambda_3 \psi^{\ell-1}(\lambda_4) - (\lambda_1 + \lambda_2)^{\ell} \psi^{\ell-1}(\lambda_2 + \lambda_4)' \quad (3.5)$$

where

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$\log \Gamma(t)$ can be expressed as $-r(t-1) + \sum_{\eta=1}^{\infty}\left[\frac{t-1}{\eta} - \log\left(\frac{\eta + t - 1}{\eta}\right)\right]$,

where r is Euler's constant with its value 0.5772157.. Writing

$\psi^{i-1}(z)$ *for* $\frac{d^i}{dt^i}\log(\Gamma(t+z))\big|_{t=0}$, it is now easy to get Eq. (3.5)

$$(\beta_r^{**})_1 = \beta_r^{**}[(a+b)\psi^o(r+j) - b\psi^o(n-j+1) - a\psi^o(n+r+1)]$$

Thus from (4.2) and (4.1)

$$(\beta_r)_1 = \beta_r k_r(0) \tag{4.4}$$

So that the first moment is

$$\mu_1' = k_o(0) + a\sum_{r=1}^{\infty} \beta_r \big/ r\beta_o ,$$

that is,

$$k_o(o) + a\sum_{r=1}^{\infty}[\beta\,(r+i, n-i+1) \,/\, r\beta\,(i, n-i+1)] \tag{4.5}$$

4.2 SECOND MOMENT. This moment requires the value $(A_r)_1$ , $(A_r)_2$ , $(B_r)_1$ , $(B_r)_2$ as is evident when k=2 in (3.8). The first and third values are given by (4.2). By Lemma 1, for $r \geq 2$.

$$(A_r)_2 = 2(r-1)!\sum_{p=1}^{r-1} 1\big/ p \tag{4.6}$$

which by the argument[1] as given at the footnote

$$= 2(r-1)!\,[\psi^0(r) - \psi^0(1)] \tag{4.7}$$

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[1] $\psi^o(z) = -\gamma + \sum_{\eta=1}^{\infty} (z-1) \big/ \eta\,(z+\eta-1) = -\gamma + \sum_{\eta=1}^{\infty}\left(1\big/\eta - 1\big/(z+\eta-1)\right)$

which, when z is an integer >1, can be expressed in the form $-\gamma + \sum_{\eta=1}^{z-1} 1\big/\eta_i$ . If z=1, we have $\psi^0(1) = -\gamma$ .

$$\mu_k' = \beta_o^{-1}\left[(\beta_o)_k + \sum_{r=1}^{\infty}\sum_{\ell=1}^{k}\binom{k}{\ell} a^{\ell}(r!)^{-1}(A_r)_{\ell}(\beta_r)_{k-\ell}\right] \quad (3.8)$$

$(A_r)_\ell$ and $\beta_r$ being the quantities evaluated in Eqs. (2.1), (3.2)-(3.7).

**4. CALCULATION OF MOMENTS:** Although the method of finding moments proposed in the previous section is straight forward, their numerical evaluation is not simple. We consider the first four moments of Z using the following notations.

$$k_r(w) = a^{w+1}[\psi^{w}(r+i) - \psi^{w}(r+j)] + (a+b)^{w+1}\psi^{w}(r+j)$$

$$+(-b)^{w+1}\psi^{w}(n-j+1) - a^{w+1}\psi^{w}(n+r+1). \quad (4.1)$$

4.1 FIRST MOMENT. On taking k=1 in Eq. (3.8), the first moment of Z comes to

$$\mu_1' = \beta_o^{-1}[\beta_o)_1 + \sum_{r=1}^{\infty} a(A_r)_1\beta_r]/r!].$$

We now evaluate the quantities involved. From (2.1) and (3.2),

$$(A_r)_1 = (r-1)! \quad , \quad \beta_r = \beta_r^{*}\beta_r^{**},$$

$$(\beta_r)_1 = (\beta_r^{*})_1\beta_r^{**} + \beta_r^{*}(\beta_r^{**})_1. \quad (4.2)$$

Using (3.6) and (3.7) it is easy to see that

$$(\beta_r^{*})_1 = a\beta_r^{*}[\psi^{o}(r+i) - \psi^{o}(r+j)] \quad (4.3)$$

4.3 THIRD MOMENT. Calculation of this moment depends upon $(A_r)_1, (A_r)_2, (A_r)_3, (\beta_r)_1, (\beta_r)_2, (\beta_r)_3$. So the additional quantities needed for calculation are $(A_r)_3$ and $(\beta_r)_3$. From Lemma 1

$$(A_r)_3 = 3(r-1)! \sum_{p=2}^{r-1} (A_p)_2 / p! \qquad r \geq 3$$

which can be expressed as

$$3!(r-1)!\,\delta_{r3} \sum_{p=2}^{r-1} (\psi^o(p) - \psi^o(1)) / p! \quad r \geq 1 \tag{4.11}$$

where $\delta_{r3}$ is introduced to assume a value 1 if $r \geq 3$, and zero other-wise. On the lines as for $(\beta_r)_1, (\beta_r)_2$ it can be shows that

$$(\beta_r)_3 = \beta_r[k_r(2) + 3k_r(1)\,k_r(0) + k_r^3(0)]. \tag{4.12}$$

The third moment of Z is now completely given by

$$\mu_3' = \frac{(\beta_o)_3}{\beta_o} + 3a\sum_{r=1}^{\infty} \frac{\beta_r}{r\beta_o}\left[\frac{(B_r)_2}{\beta_r} + 2a\frac{(\beta_r)_1}{\beta_r}(\psi^o(r) - \psi^o(1)) + 2a^2\delta_{r3}\sum_{p=2}^{r-1}(\psi^o(p) - \psi^o(1)) / p\right] \tag{4.13}$$

4.4 FOURTH MOMENT. Following similar arguments as for the first three moments it can be shown that

$$\mu_4' = \frac{(\beta_o)_4}{\beta_o} + 4a\sum_{r=1}^{\infty} \frac{\beta_r}{r\beta_o}\left[\frac{(\beta_r)_3}{\beta_r} + 3a\frac{(\beta_r)_2}{\beta_r}(\psi^o(r) - \psi^o(1))\right.$$

$(A_r)_2$ vanishes for r=1 by Lemma 1, and since it is the case with (4.7) as well, we can consider the applicability of (4.7) for $r \geq 1$. From (3.2),

$$(\beta_r)_2 = (\beta_r^*)_2 \beta_r^{**} + 2(B_r^*)_1(\beta_r^{**})_1 + B_r^*(B_r^{**})_2$$

On using (3.6), (3.7) and (4.3) the second order derivatives of $\beta_r^*, B_r^{**}$ are

$$(B_r^*)_2 = a^2 \beta^* [\{\psi^o(r+i) - \psi^0(r+j)\} + \psi^{-1}(r+i) - \psi^1(r+j)]$$

$$(B_r^{**})_2 = \beta_r^{**}[\{(a+b)\psi^o(r+j) - b\psi^o(n-j+1) - a\psi^o(n+r+1)\}^2$$

$$+(a+b)^2\psi^1(r+j) + b^2\psi^1(n-j+1) - a^2\psi^1(n+j+1)]$$

So $$(\beta_r)_2 = \beta_r[k_r^2(0) + k_r(1)] \tag{4.8}$$

Now using (4.2), the second moment which arises from (3.8) with k=2, comes to

$$\frac{(\beta_o)_2}{\beta_o} + 2a\sum_{r=1}^{\infty}\frac{\beta_r}{r\beta_o}\left[\frac{(\beta_r)_1}{(\beta_r)} + a(\psi^o(r) - \psi^o(1)\right] \tag{4.9}$$

and when further simplified after substituting (4.4) and (4.8), it is

$$\mu_2' = k_o^2(0) + k_o(1) + 2a\sum_{r=1}^{\infty}\frac{\beta(r+i, n-i+1)}{r\beta(i, n-i+1)}[k_r(0) + a(\psi^o(r) - \psi^o(1)] \tag{4.10}$$

To prove (iii) for instance, from (4.12) the third moment comes to $\beta_o^{-1}(\beta_o)_3 = k_o(2) + 3k_o(0)k_o(1) + k_o^3(0)$

which by (4.1),

$$= \psi^2(j) - \psi^2(n-j+1) + 3[\psi^o(j) - \psi^o(n-j+1)][\psi^1(j) + \psi^1(n-j+1)] + [\psi^o(j) - \psi^o(n-j+1)]^3$$

And now the use of the last three moments leads to the required moment of Y about its mean.

5.2 SAMPLE MEDIAN. We consider separate cases for even and odd sample sizes.

**Case:** (odd n). When a=0, b=1, j=(n+1)/2, it is seen from above that the statistic $Y_{med}$ which is sample median, has

$$E(Y_{med}) = 0$$

$$E[(Y_{med})^2] = 2\psi^1\left(\frac{n+1}{2}\right)$$

$$E[(Y_{med})^3] = 0$$

$$E[(Y_{med})^4] = 2\psi^3\left(\frac{n+1}{2}\right) + 12\psi^1\left(\frac{n+1}{2}\right)$$

It is easy to show that our expression for variance coincides with that given in Tarter and Clark [6], which is

$$\frac{\pi^2}{3} - 2\sum_{i=1}^{\frac{n-1}{2}}\left(\frac{1}{i}\right)^2$$

$$+6a^2\delta_{r3}\frac{(\beta_r)_1}{\beta_r}[\sum_{p=2}^{r-1}(\psi^o(p)-\psi^o(1))/p]$$

$$+6a^3\delta_{r4}[\sum_{q=3}^{r-1}\sum_{p=2}^{q-1}(\psi^o(p)-\psi^o(1))/pq]], \quad (4.14)$$

where

$$(\beta_r)_4=\beta_r[k_r(3)+4k_r(0)k_r(2)+6k_r^2(0)k_r(1)+3k_r^2(1)+k_r^4(0)] \quad (4.15)$$

and $\delta_{\gamma 4}=1$ *if* $r\geq 4$, and zero otherwise.

**5. PARTICULAR CASE:** We consider in this section moments of particular statistics yielded by Z through relevant substitutions.

5.1 ORDER STATISTIC $Y_J$. With a=0. b=1 we obtain the following results.

(i) Its first moment is $\psi^o(j)-\psi^o(n-j+1)$.

(ii) Its variance is $\psi^1(j)+\psi^1(n-j+1)$.

(iii) The third moment about its mean is $\psi^2(j)-\psi^2(n-j+1)$

(iv) The fourthmoments about its mean is:

$$\psi^3(j)+\psi^3(n-j+1)+3[\psi^1(j)+\psi^1(n-j+1)]^2$$

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**Case:** (Even n). For this case the moments are given on taking $i=n/2$, $j=(n+2)/2$, $a=b=1/2$ in the main results.

5.3 QUASIMEDIANS. For Z to be the *r*th quasimedian, let $a=b=1/2$, $i=m-r+1$, $j=m+r+1$ if $n=2m+1$ (odd), and $i=m-r$, $j=m+r+1$ if $n=2m$; $m \geq r = 1.2$ ............

It is obvious from (4.5) that a quasimedian has zero mean. The other moments about mean follow form (4.10), (4.13), (4.14).