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Family of Finite Population Mean Estimators with Auxiliary Information in the Presence of Measurement Error

Jiya Amir[*1] *and Mahnaz Makhdum*[2]

Abstract

For the unspecified population mean estimation, a family of exponential type estimators has been proposed when survey encounters measurement error under simple random sampling without replacement. Up to the first approximation degree, the mean square error and bias expressions of suggested estimators have been derived. Numerical and simulation studies were done to validate and compare the estimator with some other existing similar type estimators by using R.

Keywords

Measurement error, Mean square error (MSE), Bias, Auxiliary variable, Percent relative efficiency.

Mathematical Subject Classification: 62D05

1. Introduction

In the context of sample surveys, the assumption of accurate measurements is often challenged by real-world data errors stemming from various sources such as memory lapses, under-reporting, over-reporting, and prestige bias. These errors constitute a non-sampling error known as measurement error, signifying the deviation between the value that was recorded and the actual value of the surveyed variable. When measurement errors are not negligible, they can significantly compromise the validity and accuracy of statistical inferences, leading to unexpected and undesirable outcomes.

Several researchers have delved into the influence of measurement errors on different estimators of population parameters. Noteworthy contributors to this field include Cochran (1940), Cochran (1942), Cochran (1968), Wang (2002), Allen et al. (2003), Salas and Gregoire (2010), Shalabh and Tsai (2017) and Khalil et al. (2018).

This study focuses on estimation of the unknown population mean in the presence of measurement errors. A novel estimator is proposed to address this challenge, aiming to mitigate the effects of measurement errors on statistical inferences. Additionally, an R-program has been developed to validate and perform a comparative performance analysis against other widely used estimators. The objective is to assess the relative effectiveness

[*] Corresponding author
[1] Department of Statistics, Lahore College for Women University, Lahore-Pakistan.
Email: jiya.amir16@gmail.com
[2] National College of Business Administration and Economics (NCBA&E), Lahore, Pakistan.
Email: mahnaz.stats@gmail.com

of the newly suggested estimator in comparison to existing methods, providing valuable insights for researchers and practitioners grappling with the complexities of measurement error in survey data.

2. Notations

Using simple random sampling with no replacement, a sample of size n has been drawn from a finite population of size N. The measurement errors related to the variable of interest and auxiliary variable are as follows:

$$u_j = y_j - Y_j \tag{1}$$

$$v_j = x_j - X_j \tag{2}$$

Assuming that actual values of the study variable Y and auxiliary variable X are independent of measurement errors. Let $S_y^2 = \frac{\sum_{j=1}^{N}\left(y_j - \bar{Y}\right)^2}{N-1}$ and $S_x^2 = \frac{\sum_{j=1}^{N}\left(x_j - \bar{X}\right)^2}{N-1}$ be the population variances of Y and X respectively. The measurement errors of Y and X are uncorrelated with zero mean and with variances $S_u^2 = \frac{\sum_{j=1}^{N}\left(u_j - \bar{U}\right)^2}{N-1}$ and $S_v^2 = \frac{\sum_{j=1}^{N}\left(v_j - \bar{V}\right)^2}{N-1}$ respectively. Let $\rho_{xy} = \frac{S_{xy}}{S_x S_y}$ be the population correlation co-efficient between the variables Y and X.

Following are some important notations defined in order to obtain expressions for the mean square error and bias of the suggested family of estimators when a survey is faced with measurement errors.

$$w_y = \sum_{j=1}^{n}\left(y_j - \bar{Y}\right) \tag{3}$$

$$w_x = \sum_{j=1}^{n}\left(x_j - \bar{X}\right) \tag{4}$$

and

$$w_u = \sum_{j=1}^{n}\left(u_j - \bar{U}\right) \tag{5}$$

$$w_v = \sum_{j=1}^{n}\left(v_j - \bar{V}\right) \tag{6}$$

Also

$$\bar{y} = \bar{Y} + \frac{1}{n}\left(w_y + w_u\right) \tag{7}$$

$$\bar{x} = \bar{X} + \frac{1}{n}\left(w_x + w_v\right) \tag{8}$$

Furthermore,

$$\left.\begin{aligned} &E\left(\frac{w_y + w_u}{n}\right) = 0,\ E\left(\frac{w_x + w_v}{n}\right) = 0,\ E\left(\frac{w_y + w_u}{n}\right)^2 = \lambda_1\left(S_y^2 + S_u^2\right), \\ &E\left(\frac{w_x + w_v}{n}\right)^2 = \lambda_1\left(S_x^2 + S_v^2\right),\ E\left[\frac{1}{n}\left(w_y + w_u\right).\frac{1}{n}\left(w_x + w_v\right)\right] = \lambda_1 \rho_{xy} S_x S_y \end{aligned}\right\} \tag{9}$$

3. Existing estimators

The following is an overview of some of the existing mean estimators under measurement error:
The classical mean estimator is given as

$$\bar{y} = \frac{1}{n}\sum_{j=1}^{n} y_j \tag{10}$$

with variance expression

$$Var\left(\bar{y}\right) = \lambda_1\left(S_y^2 + S_u^2\right) \tag{11}$$

Cochran (1940) proposed a ratio estimator is given as:

$$t_{cr} = \left(\frac{\bar{X}}{\bar{x}}\right)\bar{y} \tag{12}$$

The expressions of bias and MSE of t_{cr} are given below:

$$Bias\left(t_{cr}\right) = \frac{1}{\bar{X}}\left[\lambda_1 R\left(S_x^2 + S_v^2\right) - \lambda_1\rho_{xy}S_xS_y\right] \tag{13}$$

$$MSE\left(t_{cr}\right) = \lambda_1\left(S_y^2 + S_u^2\right) + R^2\lambda_1\left(S_x^2 + S_v^2\right) - 2R\lambda_1\rho_{xy}S_xS_y \tag{14}$$

Bahl and Tuteja (1991) developed an exponential ratio type estimator given as

$$t_{bt} = \exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right)\bar{y} \tag{15}$$

with bias and mean square error expressions

$$Bias\left(t_{bt}\right) = \frac{1}{2\bar{X}}\left[\lambda_1 R\left(S_x^2 + S_v^2\right) - \lambda_1\rho_{xy}S_xS_y\right] \tag{16}$$

$$MSE\left(t_{bt}\right) = \lambda_1\left(S_y^2 + S_u^2\right) + \frac{1}{4}R^2\lambda_1\left(S_x^2 + S_v^2\right) - R\lambda_1\rho_{xy}S_xS_y \tag{17}$$

Regression estimator proposed by Watson (1937) is given as:

$$t_{reg} = b\left(\bar{X} - \bar{x}\right) + \bar{y} \tag{18}$$

with mean square error

$$MSE\left(t_{reg}\right) = \lambda_1\left(S_y^2 + S_u^2\right) + b_0^2\lambda_1\left(S_x^2 + S_v^2\right) - 2b_0\lambda_1\rho_{yx}S_yS_x \tag{19}$$

Singh et al. (2008) developed a ratio-product exponential type estimator given as

$$t_s = \left[\left(1-\mu\right)\exp\left(\frac{\bar{x} - \bar{X}}{\bar{X} + \bar{x}}\right) + \mu\exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right)\right]\bar{y} \tag{20}$$

With bias and mean square error

$$Bias(t_s) = \frac{(8\mu-1)}{32\bar{X}}\left[\lambda_1 R(S_x^2 + S_v^2)\right] - \frac{(2\mu-1)}{2\bar{X}}\left[\lambda_1\rho_{xy}S_xS_y\right] \tag{21}$$

$$MSE\left(t_s\right) = \lambda_1\left(S_y^2 + S_u^2\right) + \frac{\left(2\mu-1\right)^2}{4R^2}\left[\lambda_1\left(S_x^2 + S_v^2\right)\right] - \left(\frac{2\mu-1}{R}\right)\left[\lambda_1\rho_{xy}S_xS_y\right] \tag{22}$$

Singh et al. (2009) suggested a family of exponential type estimators by taking motivation from Khoshnevisan et al. (2007) given by

$$t_{ks} = \exp\left[\frac{\left(\beta_1 + \alpha_1\bar{X}\right) - \left(\beta_1 + \alpha_1\bar{x}\right)}{\left(\beta_1 + \alpha_1\bar{X}\right) + \left(\beta_1 + \alpha_1\bar{x}\right)}\right]\bar{y} \tag{23}$$

The bias and MSE of t_{ks} is given by

$$Bias\left(t_{ks}\right)=\frac{3\bar{Y}\phi_j^{\,2}}{8}\left[\lambda_1\left(S_x^2+S_v^2\right)\right]-\frac{\phi_j}{2}\lambda_1\rho_{xy}S_xS_y \tag{24}$$

$$MSE\left(t_{ks}\right)=\lambda_1\left(S_y^2+S_u^2\right)+\frac{\bar{Y}^2\phi_j^2}{4}\left[\lambda_1\left(S_x^2+S_v^2\right)\right]-\bar{Y}\phi_j\lambda_1\rho_{xy}S_xS_y \tag{25}$$

Upadhyaya et al. (2011) suggested a ratio exponential type estimator is given as:

$$t_{up}=\exp\left(1-\frac{2\bar{x}}{\bar{x}+\bar{X}}\right)\bar{y} \tag{26}$$

with expressions of bias and mean square error

$$Bias\left(t_{up}\right)=\frac{3R}{8\bar{X}}\left[\lambda_1\left(S_x^2+S_v^2\right)\right]-\frac{1}{2\bar{X}}\left[\lambda_1\rho_{xy}S_xS_y\right] \tag{27}$$

$$MSE\left(t_{up}\right)=\lambda_1\left(S_y^2+S_u^2\right)+\frac{1}{4R^2}\left[\lambda_1\left(S_x^2+S_v^2\right)\right]-\frac{1}{R}\left[\lambda_1\rho_{xy}S_xS_y\right] \tag{28}$$

By employing Singh et al. (2009), Yadav and Kadilar (2013) proposed an enhanced exponential family of estimators is given by

$$t_{yk}=K\exp\left[\frac{\left(\beta_1+\alpha_1\bar{X}\right)-\left(\beta_1+\alpha_1\bar{x}\right)}{\left(\beta_1+\alpha_1\bar{X}\right)+\left(\beta_1+\alpha_1\bar{x}\right)}\right]\bar{y} \tag{29}$$

with following bias and mean square expressions

$$Bias\left(t_{yk}\right)=\bar{Y}(K-1)+\frac{3\bar{Y}K\phi_j^2}{8}\left[\lambda_1\left(S_x^2+S_v^2\right)\right]-\frac{K\phi_j}{2}\left[\lambda_1\rho_{xy}S_xS_y\right] \tag{30}$$

$$MSE\left(t_{yk}\right)=\bar{Y}^2(K-1)^2+K^2\left\{\lambda_1\left(S_y^2+S_u^2\right)\right\}+\frac{\bar{Y}^2K\phi_j^2}{4}\left(4K-3\right)\left[\lambda_1\left(S_x^2+S_v^2\right)\right]-\bar{Y}K\phi_j\left(2K-1\right)\left[\lambda_1\rho_{xy}S_xS_y\right] \tag{31}$$

Kadilar (2016) developed a modified exponential type of estimator, which is given below:

$$t_k=\bar{y}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)\left(\frac{\bar{x}}{\bar{X}}\right)^{\delta} \tag{32}$$

The bias and MSE of t_k is given as:

$$Bias\left(t_k\right)=\frac{\left(8\delta-1\right)}{32\bar{X}}\left[\lambda_1R\left(S_x^2+S_v^2\right)\right]-\frac{\left(1-2\delta\right)}{2\bar{X}}\left[\lambda_1\rho_{xy}S_xS_y\right] \tag{33}$$

$$MSE\left(t_k\right)=\lambda_1\left(S_y^2+S_u^2\right)+\frac{\left(2\delta-1\right)^2}{4R^2}\left[\lambda_1\left(S_x^2+S_v^2\right)\right]-\frac{\left(1-2\delta\right)}{R}\left[\lambda_1\rho_{xy}S_xS_y\right] \tag{34}$$

4. Suggested generalized family of estimators

Following is the suggested family of exponential-type estimators when both study variable and auxiliary variable are impacted by measurement error.

$$t_J=t_1\bar{y}\left(\frac{\bar{X}^*}{\bar{x}^*}\right)+t_2\left(\bar{X}-\bar{x}\right)\exp\left(\frac{\bar{X}^*-\bar{x}^*}{\bar{X}^*+\bar{x}^*}\right) \tag{35}$$

Where $\bar{X}^*=\beta_1+\alpha_1\bar{X}$, $\bar{x}^*=\beta_1+\alpha_1\bar{x}$, t_1 and t_2 are optimize constants, $\alpha_1\neq 0$ and β_1 are either real numbers or functions of established parameters such as standard deviation $\left(S_x\right)$

, coefficient of variation $\left(C_x\right)$, coefficient of kurtosis $\left(\beta_{(2x)}\right)$, coefficient of skewness $\left(\beta_{(1x)}\right)$, or coefficient of correlation $\left(\rho_{xy}\right)$.

$$t_J = t_1\bar{y}\left(\frac{\beta_1+\alpha_1\bar{X}}{\beta_1+\alpha_1\bar{x}}\right)+t_2\left(\bar{X}-\bar{x}\right)\exp\left(\frac{\beta_1+\alpha_1\bar{X}-\beta_1-\alpha_1\bar{x}}{\beta_1+\alpha_1\bar{X}+\beta_1+\alpha_1\bar{x}}\right) \quad (36)$$

The bias and mean square error expressions are as follows, up to the first-order approximation:

$$Bias\left(t_J\right)=\bar{Y}\left(t_1-1\right)+t_1\bar{Y}\phi_j^2\lambda_1\left(S_x^2+S_v^2\right)-t_1\phi_j\lambda_1\rho_{xy}S_xS_y+t_2\frac{\phi_j}{2}\lambda_1\left(S_x^2+S_v^2\right) \quad (37)$$

$$MSE\left(t_J\right)=\bar{Y}^2\left[1+t_1{}^2A_{31}+t_2{}^2A_{32}-2t_1t_2A_{33}-2t_1A_{34}-t_2A_{35}\right] \quad (38)$$

The optimal values of t_1 and t_2 as below

$$t_{1,opt}=\frac{A_{36}}{2A_{37}},\quad t_{2,opt}=\frac{A_{38}}{2A_{37}}$$

and expression for optimum mean square error of suggested family of estimators is obtained as

$$MSE\left(t_J\right)_{opt}=\frac{\bar{Y}^2}{4A_{37}{}^2}\left[4A_{37}{}^2+A_{31}A_{36}{}^2+A_{32}A_{38}{}^2-2A_{33}A_{36}A_{38}-4A_{34}A_{36}A_{37}-2A_{35}A_{37}A_{38}\right] \quad (39)$$

Appendix A contains a list of important notations that were used in this article. Also substituting different values of α_1 and β_1 in Equation (36) to extract various estimators from the suggested family of estimators; Appendix B contains some of these values.

5. Theoretical efficiency comparison

This section compares the suggested estimator's efficiency with existing estimators of a similar type. Consequently, the following circumstances indicate when the recommended estimator is more effective.

(i) t_J vs. t_{cr}

$MSE\left(t_J\right)-MSE\left(t_{cr}\right)\le 0$ if

$$\rho_{xy}\le\frac{1}{2R\lambda_1S_xS_y}\left[\lambda_1\left(S_y^2+S_u^2\right)+R^2\lambda_1\left(S_x^2+S_v^2\right)-\bar{Y}^2\left(1+t_1{}^2a_3+t_2{}^2b_3-2t_1t_2c_3-2t_1d_3-t_2e_3\right)\right] \quad (40)$$

(ii) t_J vs. t_{bt}

$MSE\left(t_J\right)-MSE\left(t_{bt}\right)\le 0$ if

$$\rho_{xy}\le\frac{1}{R\lambda_1S_xS_y}\left[\begin{array}{l}\lambda_1\left(S_y^2+S_u^2\right)+\frac{1}{4}R^2\lambda_1\left(S_x^2+S_v^2\right)-\bar{Y}^2\left(1+t_1{}^2a_3+t_2{}^2b_3-2t_1t_2c_3-2t_1d_3\right.\\ \left.-t_2e_3\right)\end{array}\right] \quad (41)$$

(iii) t_J vs. t_{reg}

$MSE\left(t_J\right)-MSE\left(t_{reg}\right)\le 0$ if

$$\rho_{xy}\le\frac{1}{2b_0\lambda_1S_xS_y}\left[\lambda_1\left(S_y^2+S_u^2\right)+b_0^2\lambda_1\left(S_x^2+S_v^2\right)-\bar{Y}^2\left(1+t_1{}^2a_3+t_2{}^2b_3-2t_1t_2c_3-2t_1d_3-t_2e_3\right)\right] \quad (42)$$

(iv) t_J $vs.$ t_s

$MSE(t_J) - MSE(t_s) \le 0$ if

$$\rho_{xy} \le \frac{R}{(2\mu-1)\lambda_1 S_x S_y}\left[\begin{array}{l}\lambda_1\left(S_y^2+S_u^2\right)+\frac{(2\mu-1)^2}{4R^2}\left\{\lambda_1\left(S_x^2+S_v^2\right)\right\}-\bar{Y}^2\left(1+t_1^2a_3+t_2^2b_3-2t_1t_2c_3\right.\\ \left.-2t_1d_3-t_2e_3\right)\end{array}\right] \quad (43)$$

(v) t_J $vs.$ t_{ks}

$MSE(t_J) - MSE(t_{ks}) \le 0$ if

$$\rho_{xy} \le \frac{1}{\bar{Y}\phi_j\lambda_1 S_x S_y}\left[\begin{array}{l}\lambda_1\left(S_y^2+S_u^2\right)+\frac{\bar{Y}^2\phi_j^2}{4}\left\{\lambda_1\left(S_x^2+S_v^2\right)\right\}-\bar{Y}^2\left(1+t_1^2a_3+t_2^2b_3-2t_1t_2c_3\right.\\ \left.-2t_1d_3-t_2e_3\right)\end{array}\right] \quad (44)$$

(vi) t_J $vs.$ t_{up}

$MSE(t_J) - MSE(t_{up}) \le 0$ if

$$\rho_{xy} \le \frac{R}{\lambda_1 S_x S_y}\left[\begin{array}{l}\lambda_1\left(S_y^2+S_u^2\right)+\frac{1}{4R^2}\left\{\lambda_1\left(S_x^2+S_v^2\right)\right\}-\bar{Y}^2\left(1+t_1^2a_3+t_2^2b_3-2t_1t_2c_3-2t_1d_3\right.\\ \left.-t_2e_3\right)\end{array}\right] \quad (45)$$

(vii) t_J $vs.$ t_{yk}

$MSE(t_J) - MSE(t_{yk}) \le 0$ if

$$\rho_{xy} \le \frac{1}{\bar{Y}K\phi_j(2K-1)\lambda_1 S_x S_y}\left[\begin{array}{l}\bar{Y}^2(K-1)^2+K^2\left\{\lambda_1\left(S_y^2+S_u^2\right)\right\}+\frac{\bar{Y}^2K\phi_j^2}{4}(4K-3)\\ \left\{\lambda_1\left(S_x^2+S_v^2\right)\right\}-\bar{Y}^2\left(1+t_1^2a_3+t_2^2b_3-2t_1t_2c_3-2t_1d_3-t_2e_3\right)\end{array}\right] \quad (46)$$

(viii) t_J $vs.$ t_k

$MSE(t_J) - MSE(t_k) \le 0$ if

$$\rho_{xy} \le \frac{R}{(1-2\delta)\lambda_1 S_x S_y}\left[\begin{array}{l}\lambda_1\left(S_y^2+S_u^2\right)+\frac{(2\delta-1)^2}{4R^2}\left\{\lambda_1\left(S_x^2+S_v^2\right)\right\}-\bar{Y}^2\left(1+t_1^2a_3+t_2^2b_3-2t_1t_2c_3\right.\\ \left.-2t_1d_3-t_2e_3\right)\end{array}\right] \quad (47)$$

The expressions clearly demonstrate the suggested family of estimator's superiority over the other estimators taken into consideration.

6. Simulation study

In this section, the performance of the suggested family of estimators has been evaluated in relation to estimators proposed by Cochran (1940), Watson (1937), Bahl and Tuteja (1991), Singh et al. (2009), Upadhyaya et al. (2011), Kadilar (2016) and Yadav and Kadilar (2013) in terms of mean square error. The population has been simulated of size N from the bivariate normal distribution by considering the given parameters. A simple random sample of size n has been drawn without replacement from the simulated population using the simple random sampling technique. A simulation study was carried out using the R program and the results were averaged for 10,000 iterations. The percent relative efficiencies of the suggested and competent estimators in relation to the ordinary mean per unit estimator have been computed to evaluate the performance of the suggested estimators. Considering the three populations with following parameters to empirically validate the estimators and for simulation study.

Population 1 (Gupta et al., 2012)
$N = 1000$; $n = 100$; $\rho_{xy} = 0.31667$
$\mu = [2 \quad 2]$; $cov = \begin{bmatrix} 9 & 1.9 \\ 1.9 & 4 \end{bmatrix}$
Population 2 (Singh and Singh, 2014)
$N = 5000$; $n = 500$; $\rho_{xy} = 0.995059$
$\mu = [4.924306 \quad 4.927167]$; $cov = \begin{bmatrix} 101.4117 & 101.2066 \\ 101.2066 & 102.0075 \end{bmatrix}$
Population 3 (Azeem and Hanif, 2015)
$N = 5000$; $n = 500$; $\rho_{xy} = -0.18275$
$\mu = [9.953722 \quad 1.007391]$; $cov = \begin{bmatrix} 4.0629690 & -0.3740459 \\ -0.3740459 & 1.0310780 \end{bmatrix}$

Mean square errors (MSEs) and percent relative efficiencies (PREs) of competing estimators including classical mean $\bar{y}$, (Cochran, 1940) t_{cr}, (Bahl and Tuteja. 1991) t_{bt}, regression t_{reg}, (Singh et al., 2008) t_s, (Singh et al., 2009) t_{ks}, (Upadhyaya et al., 2011) t_{up}, (Kadilar, 2016) t_k and (Yadav and Kadilar, 2013) t_{yk} have been computed for both theoretical and simulated populations and shown in Table 1. Tables 2-3 show the MSEs and PREs of the suggested family of estimators for three aforementioned populations, where the following formula has been used to compute PREs with respect to the ordinary mean estimator $\left(\bar{y}\right)$:

$$PRE(.) = \frac{Var\left(\bar{y}\right)}{MSE(.)} \times 100 \tag{48}$$

Table 1: MSEs $\times$ 100 and PREs of existing estimators for Population 1, 2 and 3.

	Population 1		Population 2		Population 3	
Est.	MSE Theoretical (Simulated)	PRE	MSE Theoretical (Simulated)	PRE	MSE Theoretical (Simulated)	PRE
$\bar{y}$	4.5353 (4.4422)	100	18.2755 (18.3901)	100	0.3652 (0.3633)	100
t_{cr}	9.3778 (13.6331)	48.362	0.54633 (0.5478)	3345.154	0.3888 (0.3832)	93.930
t_{bt}	4.8672 (5.5616)	93.181	4.8381 (4.9538)	377.745	0.3752 (0.3711)	97.335
t_{reg}	4.1657 (4.1169)	108.872	0.5420 (0.5342)	3371.822	0.3609 (0.3585)	101.204
t_s	4.1657 (4.2485)	108.872	0.5420 (0.5773)	3371.822	0.3609 (0.3599)	101.204
t_{up}	5.0447 (5.5616)	89.903	4.8615 (4.9538)	375.921	0.3739 (0.3711)	97.683
t_k	4.1657 (4.2013)	108.872	0.5420 (0.5459)	3371.822	0.3609 (0.3599)	101.204

Table 2: MSEs × 100 and PREs of t_{ks}, t_{yk} and suggested estimators t_J for Population 1.

		t_{ks}		t_{yk}		t_J	
α_1	β_1	MSE Theoretical (Simulated)	PRE	MSE Theoretical (Simulated)	PRE	MSE Theoretical (Simulated)	PRE
1	0	5.0447 (5.6042)	89.903	4.9131 (5.4141)	92.309	4.0229 (4.8716)	112.737
$\beta_{1(2x)}$	$\beta_{1(1x)}$	4.2101 (4.2846)	107.723	4.1600 (4.1941)	109.021	4.1092 (4.2612)	110.368
S_x	$\beta_{1(1x)}$	5.0317 (5.5511)	90.134	4.9016 (5.3637)	92.527	4.0242 (4.8087)	112.699
C_x	$\beta_{1(1x)}$	4.9985 (5.5887)	90.732	4.8719 (5.4025)	93.091	4.0277 (5.0036)	112.602
1	$\beta_{1(1x)}$	4.9777 (5.6127)	91.112	4.8532 (5.4274)	93.449	4.0299 (4.8976)	112.542
ρ_{xy}	$\beta_{1(2x)}$	4.9374 (5.4692)	91.856	4.8171 (5.2916)	94.150	4.0340 (4.7789)	112.426
$\beta_{1(1x)}$	$\beta_{1(2x)}C_x$	4.4086 (4.7851)	102.873	4.3393 (4.6665)	104.515	4.0870 (4.5891)	110.968
S_x	$\beta_{1(1x)}$	5.0216 (5.5647)	90.316	4.8925 (5.3776)	92.699	4.0253 (4.9091)	112.669
C_x	ρ_{xy}	4.7747 (5.2206)	94.986	4.6708 (5.0628)	97.099	4.0506 (4.6814)	111.967
$\beta_{1(2x)}$	ρ_{xy}	4.3187 (4.3159)	105.016	4.2778 (4.2334)	106.020	4.1272 (4.1104)	109.887
1	ρ_{xy}	4.6794 (4.9986)	96.919	4.5848 (4.8543)	98.920	4.0601 (4.5638)	111.704

Table 3: MSEs × 100 and PREs of t_{ks}, t_{yk} and suggested estimators t_J for Population 2.

		t_{ks}		t_{yk}		t_J	
α_1	β_1	MSE Theoretical (Simulated)	PRE	MSE Theoretical (Simulated)	PRE	MSE Theoretical (Simulated)	PRE
1	0	4.8615 (4.8928)	375.921	4.8586 (4.8820)	376.152	0.5418 (0.5404)	3372.927
$\beta_{1(2x)}$	$\beta_{1(1x)}$	6.7351 (6.7207)	271.346	6.7288 (6.6991)	271.601	0.5418 (0.5256)	3373.365
S_x	$\beta_{1(1x)}$	4.8661 (4.8684)	375.565	4.8632 (4.8577)	375.796	0.5418 (0.5424)	3372.911
C_x	$\beta_{1(1x)}$	4.8884 (4.8218)	373.856	4.8854 (4.8111)	374.087	0.5418 (0.5354)	3372.835
1	$\beta_{1(1x)}$	4.9158 (4.8814)	371.771	4.9128 (4.8705)	372.002	0.5419 (0.5467)	3372.747
ρ_{xy}	$\beta_{1(2x)}$	4.9078 (4.9547)	372.376	4.9048 (4.9437)	372.607	0.5419 (0.5325)	3372.772
$\beta_{1(1x)}$	$\beta_{1(2x)}C_x$	7.4457 (7.3025)	245.450	7.4373 (7.2754)	245.728	0.5415 (0.5345)	3374.975

α_1	β_1	t_{ks} MSE Theoretical (Simulated)	t_{ks} PRE	t_{yk} MSE Theoretical (Simulated)	t_{yk} PRE	t_J MSE Theoretical (Simulated)	t_J PRE
S_x	$\beta_{1(1x)}$	4.8670 (4.9311)	375.501	4.8640 (4.9202)	375.732	0.5418 (0.5321)	3372.908
C_x	ρ_{xy}	5.6998 (5.6787)	320.634	5.6957 (5.6640)	320.866	0.5420 (0.5412)	3371.848
$\beta_{1(2x)}$	ρ_{xy}	16.2929 (16.0872)	112.169	16.1969 (15.8869)	112.834	0.5420 (0.5219)	3372.154
1	ρ_{xy}	6.4728 (6.5806)	282.344	6.4671 (6.5605)	282.592	0.5418 (0.5436)	3372.832

Table 4: MSEs × 100 and PREs of t_{ks}, t_{yk} and suggested estimators t_J for Population 3.

α_1	β_1	t_{ks} MSE Theoretical (Simulated)	t_{ks} PRE	t_{yk} MSE Theoretical (Simulated)	t_{yk} PRE	t_J MSE Theoretical (Simulated)	t_J PRE
1	0	0.3725 (0.3818)	97.623	0.3739 (0.3789)	98.054	0.3596 (0.3641)	101.578
$\beta_{1(2x)}$	$\beta_{1(1x)}$	0.3727 (0.3667)	97.622	0.3741 (0.3640)	97.994	0.3596 (0.3507)	101.578
S_x	$\beta_{1(1x)}$	0.3726 (0.3729)	97.660	0.3740 (0.3702)	98.031	0.3596 (0.3560)	101.578
C_x	$\beta_{1(1x)}$	0.3723 (0.3678)	97.728	0.3737 (0.3651)	98.099	0.3596 (0.3512)	101.577
1	$\beta_{1(1x)}$	0.3725 (0.3732)	97.692	0.3739 (0.3705)	98.063	0.3596 (0.3547)	101.578
ρ_{xy}	$\beta_{1(2x)}$	0.3716 (0.3785)	97.920	0.3730 (0.3757)	98.289	0.3596 (0.3617)	101.576
$\beta_{1(1x)}$	$\beta_{1(2x)}C_x$	0.3737 (0.3700)	97.362	0.3751 (0.3673)	97.736	0.3596 (0.3518)	101.581
S_x	$\beta_{1(1x)}$	0.3725 (0.3694)	97.688	0.3739 (0.3666)	98.058	0.3596 (0.3511)	101.578
C_x	ρ_{xy}	0.3735 (0.3785)	97.410	0.3750 (0.3757)	97.783	0.3596 (0.3576)	101.580
$\beta_{1(2x)}$	ρ_{xy}	0.3714 (0.3611)	97.964	0.3728 (0.3584)	98.333	0.3596 (0.3463)	101.575
1	ρ_{xy}	0.3727 (0.3713)	97.633	0.3741 (0.3685)	98.004	0.3596 (0.3540)	101.578

The graphically representation of the performance of suggested family of estimators for varying correlation coefficients $-0.9(0.2)0.9$ for various sample sizes such as $n = 100, 200, 300, 400, 500$ with the following values of parameters:

N	$\bar{X}$	$\bar{Y}$	S_x	S_y
1000	2	2	9	4

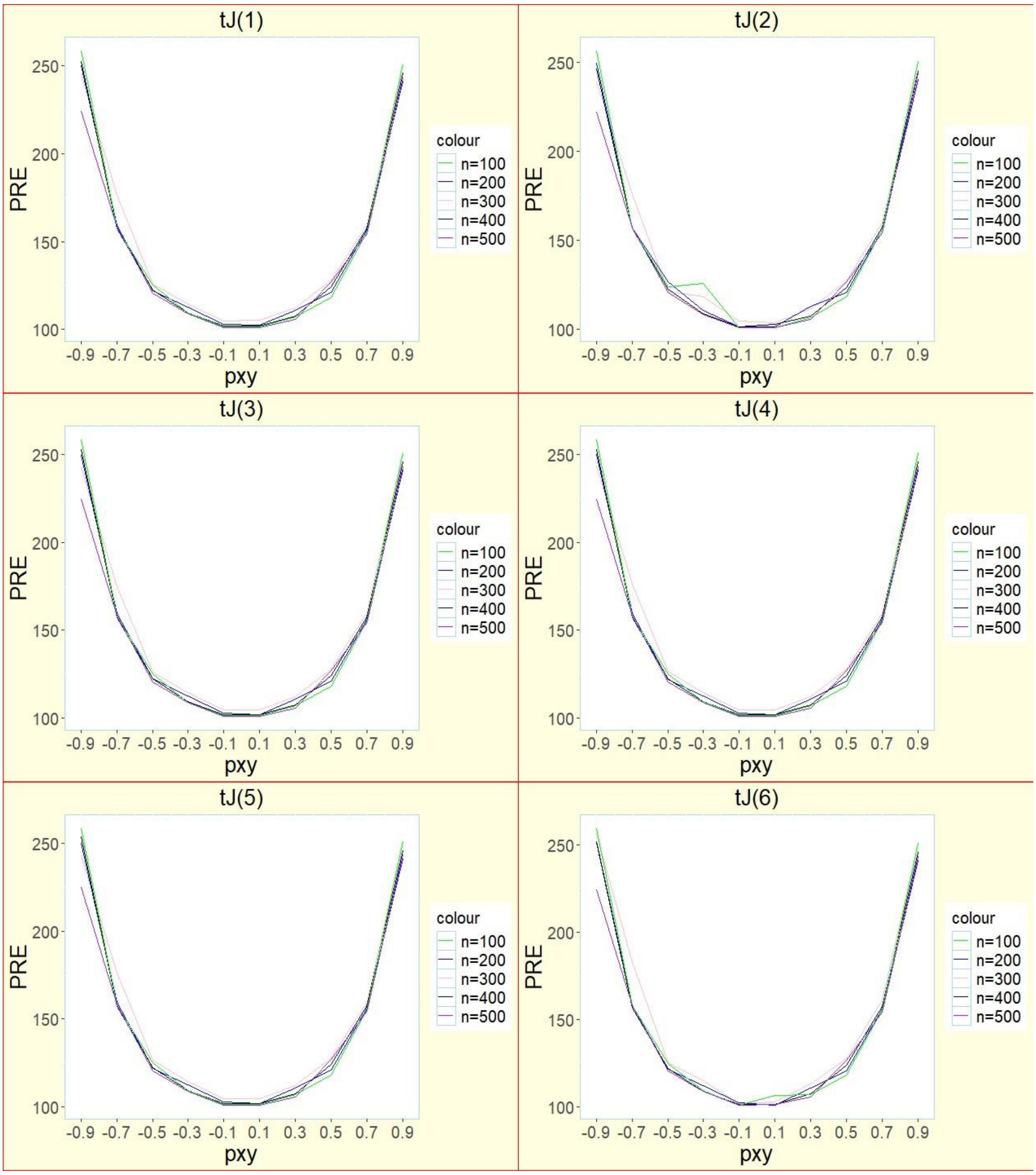

Figure 1: PREs of Suggested Family of Estimators ($t_{J(1)}$ to $t_{J(6)}$) with different correlation coefficients and sample size for different parameters of α_1 and β_1.

7. Numerical Study

The sources of the data for the numerical analysis were (Gujarati & Sangeetha, 2007) pg. 539, where
Y_i = true consumption expenditure, X_i = true income
y_i = measured consumption expenditure, x_i = measured income

N	n	$\bar{X}$	$\bar{Y}$	S_x	S_y	ρ_{xy}	S_v^2	S_u^2
70	10	1755.53	981.29	1406.13	613.66	0.778	36.00	36.00

where, $\beta_{1(1x)} = 0.0305$ and $\beta_{1(2x)} = 0.0569$.

Table 5 illustrates that the suggested estimators (t_J) outperform other existing estimators under measurement error when applied to real data, hence supporting the same conclusion.

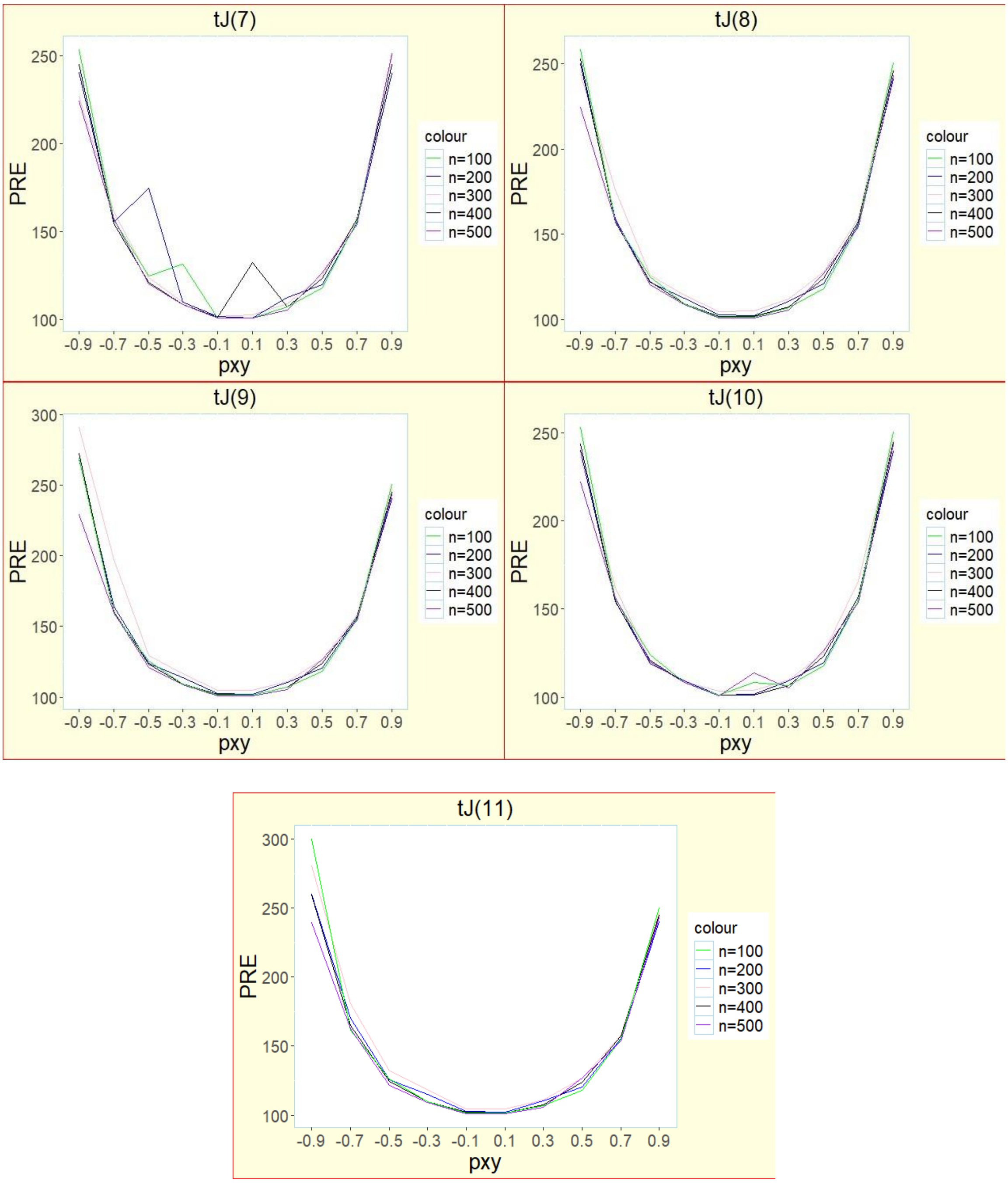

Figure 2: PREs of Suggested Family of Estimators ($t_{J(7)}$ to $t_{J(11)}$) with different correlation coefficients and sample size for different parameters of α_1 and β_1.

Table 5: MSEs of existing and suggested family of estimators for various values of α_1 and β_1

		Estimators									
α_1	β_1	$\bar{y}$	t_{cr}	t_{bt}	t_{reg}	t_s	t_{ks}	t_{up}	t_k	t_{yk}	t_J
1	0	32281.25	20905.45	13355.11	12744.15	12744.15	13355.11	64942.63	12744.15	13056.84	12199.63
$\beta_{1(2x)}$	$\beta_{1(1x)}$	32281.25	20905.45	13355.11	12744.15	12744.15	13358.35	64942.63	12744.15	13060.43	12200.60
S_x	$\beta_{1(2x)}$	32281.25	20905.45	13355.11	12744.15	12744.15	13355.11	64942.63	12744.15	13056.84	12199.63
C_x	$\beta_{1(1x)}$	32281.25	20905.45	13355.11	12744.15	12744.15	13355.23	64942.63	12744.15	13056.98	12199.66
1	$\beta_{1(1x)}$	32281.25	20905.45	13355.11	12744.15	12744.15	13355.21	64942.63	12744.15	13056.95	12199.66
ρ_{xy}	$\beta_{1(2x)}$	32281.25	20905.45	13355.11	12744.15	12744.15	13355.35	64942.63	12744.15	13057.11	12199.70
$\beta_{1(1x)}$	$\beta_{1(2x)}C_x$	32281.25	20905.45	13355.11	12744.15	12744.15	13359.96	64942.63	12744.15	13062.21	12201.08
S_x	$\beta_{1(1x)}$	32281.25	20905.45	13355.11	12744.15	12744.15	13355.11	64942.63	12744.15	13056.84	12199.63
C_x	ρ_{xy}	32281.25	20905.45	13355.11	12744.15	12744.15	13358.26	64942.63	12744.15	13060.33	12200.57
$\beta_{1(2x)}$	ρ_{xy}	32281.25	20905.45	13355.11	12744.15	12744.15	13399.86	64942.63	12744.15	13106.24	12212.69
1	ρ_{xy}	32281.25	20905.45	13355.11	12744.15	12744.15	13357.63	64942.63	12744.15	13059.64	12200.38

8. Conclusions

A generalized family of exponential-type estimators of the finite population mean have been proposed in this article, where measurement error is present on both study as well as auxiliary variables. Theoretical and simulation studies are conducted using real-life data sets in order to examine the performance of the estimators. The impact of measurement error on the generalised family of estimators is shown in Tables 1 – 4. The simulated and theoretical mean square errors are similar, as expected. When measurement error is taken into account, it becomes evident that the recommended estimator's efficiency decreases when compared to other existing estimators. Therefore, measurement errors must be carefully considered because, particularly in situations where they are not insignificant, they might have a significant impact on the estimator's efficiency. In all cases, the suggested family of exponential-type estimators performs the best as compared to similar existing estimators for all populations in the presence of measurement error. It is also observed that the suggested generalized estimators have the highest PREs for all pairs of α_1 and β_1 for population 2, for which auxiliary and study variables are strongly and positively correlated. The highest values of PREs are obtained for the pairs $(\alpha_1, \beta_1) = \left(\beta_{1(2x)}, \beta_{1(1x)}\right)$ and $\left(\beta_{1(1x)}, \beta_{1(2x)}C_x\right)$. From figure 1, it is also observed that the suggested estimators increase as the correlation coefficient increases and decreases as the correlation coefficient decreases for various sample sizes with different parameters of α_1 and β_1.

Appendix A

$$\phi_j = \frac{\alpha_1}{\beta_1 + \alpha_1 \bar{X}}, \; j = 1,2,\dots,11$$

$$\lambda_1 = {N-n}/{Nn}, \; R = {\bar{Y}}/{\bar{X}}, \; b = \frac{S_{xy}}{S_x^2}, \; b_0 = \frac{\lambda_1 \rho_{xy} S_x S_y}{\lambda_1\left(S_x^2 + S_v^2\right)} \text{ and } \mu = \frac{1}{2} + \frac{R\lambda_1 \rho_{xy} S_x S_y}{\lambda_1\left(S_x^2 + S_v^2\right)}$$

$$K = \frac{1 + \frac{3}{8}\phi_j^2 \lambda_1\left(S_x^2 + S_v^2\right) - \frac{\phi_j}{2\bar{Y}}\lambda_1 \rho_{xy} S_x S_y}{1 + \frac{\lambda_1\left(S_y^2 + S_u^2\right)}{\bar{Y}^2} + \phi_j^2 \lambda_1\left(S_x^2 + S_v^2\right) - \frac{2\phi_j \lambda_1 \rho_{xy} S_x S_y}{\bar{Y}}}, \; \delta = \frac{1}{2} - \frac{R\lambda_1 \rho_{xy} S_x S_y}{\lambda_1\left(S_x^2 + S_v^2\right)}$$

$$A_{31} = 1 + \frac{\lambda_1\left(S_y^2 + S_u^2\right)}{\bar{Y}^2} + 3\phi_j^2 \lambda_1\left(S_x^2 + S_v^2\right) - \frac{4\phi_j \lambda_1 \rho_{xy} S_x S_y}{\bar{Y}}$$

$$A_{32} = \frac{\lambda_1\left(S_x^2 + S_v^2\right)}{\bar{Y}^2}$$

$$A_{33} = \frac{\lambda_1 \rho_{xy} S_x S_y}{\bar{Y}^2} - \frac{3\phi_j \lambda_1\left(S_x^2 + S_v^2\right)}{2\bar{Y}}$$

$$A_{34} = 1 + \phi_j^2 \lambda_1\left(S_x^2 + S_v^2\right) - \frac{\phi_j \lambda_1 \rho_{xy} S_x S_y}{\bar{Y}}$$

$$A_{35} = \frac{\phi_j \lambda_1\left(S_x^2 + S_v^2\right)}{\bar{Y}}$$

$$A_{36} = 2A_{32}A_{34} + A_{33}A_{35}$$

$$A_{37} = A_{31}A_{32} - A_{33}^{\;2}$$

$$A_{38} = 2A_{33}A_{34} + A_{31}A_{35}$$

$$a_3 = 1 + \frac{1}{\bar{Y}^2}\left\{\lambda_1\left(S_y^2 + S_u^2\right)\right\} + 3\phi_j^2 \lambda_1\left(S_x^2 + S_v^2\right) - \frac{4\phi_j}{\bar{Y}}\left(\lambda_1 \rho_{xy} S_x S_y\right)$$

$$b_3 = \frac{1}{\bar{Y}^2}\left\{\lambda_1\left(S_y^2 + S_u^2\right)\right\}$$

$$c_3 = \frac{1}{\bar{Y}^2}\left(\lambda_1\rho_{xy}S_xS_y\right) - \frac{3\phi_j}{2\bar{Y}}\left\{\lambda_1\left(S_x^2 + S_v^2\right)\right\}$$

$$d_3 = 1 + \phi_j^2\lambda_1\left(S_x^2 + S_v^2\right) - \frac{\phi_j}{\bar{Y}}\left(\lambda_1\rho_{xy}S_xS_y\right)$$

$$e_3 = \frac{\phi_j}{\bar{Y}}\left\{\lambda_1\left(S_x^2 + S_v^2\right)\right\}$$

Appendix B

Some Members of t_J-Family of Estimators for Different Values of α_1 and β_1.

Estimators	α_1	β_1
$t_{J(1)} = t_1\bar{y}\left(\frac{\bar{X}^*}{\bar{x}^*}\right) + t_2\left(\bar{X} - \bar{x}\right)\exp\left(\frac{\bar{X}^* - \bar{x}^*}{\bar{X}^* + \bar{x}^*}\right)$	1	0
$t_{J(2)} = t_1\bar{y}\left(\frac{\beta_{1(2x)}\bar{X}^* + \beta_{1(1x)}}{\beta_{1(2x)}\bar{x}^* + \beta_{1(1x)}}\right) + t_2\left(\bar{X} - \bar{x}\right)\exp\left[\frac{\beta_{1(2x)}\left(\bar{X}^* - \bar{x}^*\right)}{\beta_{1(2x)}\left(\bar{X}^* + \bar{x}^*\right) + 2\beta_{1(1x)}}\right]$	$\beta_{1(2x)}$	$\beta_{1(1x)}$
$t_{J(3)} = t_1\bar{y}\left(\frac{S_x\bar{X}^* + \beta_{1(2x)}}{S_x\bar{x}^* + \beta_{1(2x)}}\right) + t_2\left(\bar{X} - \bar{x}\right)\exp\left[\frac{S_x\left(\bar{X}^* - \bar{x}^*\right)}{S_x\left(\bar{X}^* + \bar{x}^*\right) + 2\beta_{1(2x)}}\right]$	S_x	$\beta_{1(2x)}$
$t_{J(4)} = t_1\bar{y}\left(\frac{C_x\bar{X}^* + \beta_{1(1x)}}{C_x\bar{x}^* + \beta_{1(1x)}}\right) + t_2\left(\bar{X} - \bar{x}\right)\exp\left[\frac{C_x\left(\bar{X}^* - \bar{x}^*\right)}{C_x\left(\bar{X}^* + \bar{x}^*\right) + 2\beta_{1(1x)}}\right]$	C_x	$\beta_{1(1x)}$
$t_{J(5)} = t_1\bar{y}\left(\frac{\bar{X}^* + \beta_{1(1x)}}{\bar{x}^* + \beta_{1(1x)}}\right) + t_2\left(\bar{X} - \bar{x}\right)\exp\left[\frac{\left(\bar{X}^* - \bar{x}^*\right)}{\left(\bar{X}^* + \bar{x}^*\right) + 2\beta_{1(1x)}}\right]$	1	$\beta_{1(1x)}$
$t_{J(6)} = t_1\bar{y}\left(\frac{\rho_{yx}\bar{X}^* + \beta_{1(2x)}}{\rho_{yx}\bar{x}^* + \beta_{1(2x)}}\right) + t_2\left(\bar{X} - \bar{x}\right)\exp\left[\frac{\rho_{yx}\left(\bar{X}^* - \bar{x}^*\right)}{\rho_{yx}\left(\bar{X}^* + \bar{x}^*\right) + 2\beta_{1(2x)}}\right]$	ρ_{yx}	$\beta_{1(2x)}$
$t_{J(7)} = t_1\bar{y}\left(\frac{\beta_{1(1x)}\bar{X}^* + \beta_{1(2x)}C_x}{\beta_{1(1x)}\bar{x}^* + \beta_{1(2x)}C_x}\right) + t_2\left(\bar{X} - \bar{x}\right)\exp\left[\frac{\beta_{1(1x)}\left(\bar{X}^* - \bar{x}^*\right)}{\beta_{1(1x)}\left(\bar{X}^* + \bar{x}^*\right) + 2\beta_{1(2x)}C_x}\right]$	$\beta_{1(1x)}$	$\beta_{1(2x)}\ C_x$
$t_{J(8)} = t_1\bar{y}\left(\frac{S_x\bar{X}^* + \beta_{1(1x)}}{S_x\bar{x}^* + \beta_{1(1x)}}\right) + t_2\left(\bar{X} - \bar{x}\right)\exp\left[\frac{S_x\left(\bar{X}^* - \bar{x}^*\right)}{S_x\left(\bar{X}^* + \bar{x}^*\right) + 2\beta_{1(1x)}}\right]$	S_x	$\beta_{1(1x)}$
$t_{J(9)} = t_1\bar{y}\left(\frac{C_x\bar{X}^* + \rho_{yx}}{C_x\bar{x}^* + \rho_{yx}}\right) + t_2\left(\bar{X} - \bar{x}\right)\exp\left[\frac{C_x\left(\bar{X}^* - \bar{x}^*\right)}{C_x\left(\bar{X}^* + \bar{x}^*\right) + 2\rho_{yx}}\right]$	C_x	ρ_{yx}
$t_{J(10)} = t_1\bar{y}\left(\frac{\beta_{1(2x)}\bar{X}^* + \rho_{yx}}{\beta_{1(2x)}\bar{x}^* + \rho_{yx}}\right) + t_2\left(\bar{X} - \bar{x}\right)\exp\left[\frac{\beta_{1(2x)}\left(\bar{X}^* - \bar{x}^*\right)}{\beta_{1(2x)}\left(\bar{X}^* + \bar{x}^*\right) + 2\rho_{yx}}\right]$	$\beta_{1(2x)}$	ρ_{yx}
$t_{J(11)} = t_1\bar{y}\left(\frac{\bar{X}^* + \rho_{yx}}{\bar{x}^* + \rho_{yx}}\right) + t_2\left(\bar{X} - \bar{x}\right)\exp\left[\frac{\left(\bar{X}^* - \bar{x}^*\right)}{\left(\bar{X}^* + \bar{x}^*\right) + 2\rho_{yx}}\right]$	1	ρ_{yx}

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