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Improved Robust Estimation Techniques for the Scale Parameter of Birnbaum-Saunders Distribution

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Abstract

This study investigates the performance of improved estimation of robust estimators for the scale parameter of Birnbaum-Saunders distribution integrating sample information and unknown previous knowledge (non-sample information) by fixing the shape parameter. The three classes of point estimation techniques, linear shrinkage robust estimator, preliminary test robust estimator, and shrinkage preliminary test robust estimator are suggested for more efficient estimation. It is also recommended to use a Wald's test statistic to examine the non-sample data. The asymptotic theoretical properties of the recommended estimators are examined through simulation studies. The performance of the estimators is being evaluated based on simulated relative efficiency. Our simulation results decisively support asymptotic theory. A real data application is also carried out to demonstrate how effectively the suggested estimating methods work in practice.

Keywords

Asymptotic mean square error, Asymptotic bias, Birnbaum-Saunders distribution, Robust estimators, Shrinkage preliminary test estimator, Shrinkage estimator.

Mathematical Subject Classification: 62F10, 62F12.

1. Introduction

The two-parameter Birnbaum-Saunders (BS) distribution was initially developed by Birnbaum and Saunders (1969a, 1969b) as a failure time distribution. It is well known that the density function of the BS distribution is unimodal and positively skewed at the specific value of shape parameter (Lemonte et al., 2006). The BS distribution has gained attention because of its appealing characteristics, strong physical theoretical justifications, and connection to the normal model. The random variable $(y_1, y_2, y_3, \ldots, y_n)$ is Birnbaum-Saunders distributed with two parameters, denoted by $\mathcal{BS}(\alpha, \beta)$, if its probability density function is given by,

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$$f_Y(y;\alpha,\beta) = \frac{1}{2\alpha\beta\sqrt{2\pi}}\left[\left(\frac{\beta}{y}\right)^{\frac{1}{2}} + \left(\frac{\beta}{y}\right)^{\frac{3}{2}}\right]\exp\left\{-\frac{1}{2\alpha^2}\left(\frac{y}{\beta}+\frac{\beta}{y}-2\right)\right\}, \tag{1}$$

where, $y > 0$ and $\alpha,\beta > 0$. The α is shape parameter, the BS distribution approaches the normal distribution if $\alpha \to 0$. In this study we keep the shape parameter (α) fixed while we make improved estimation strategies for scale parameter (β) using sample and non-sample information simultaneously.

Over the past forty years, several papers addressing various inferential approaches for the parameters of the BS distribution and their properties have been published. The researchers are quite interested in estimating the BS parameters, and this topic has recently gained a lot of attention in the literature. Birnbaum and Saunders (1969b) discussed about the maximum likelihood estimator (MLE) of α and β. The asymptotic joint distribution of the MLEs was derived by Engelhardt *et al*. (1981). Based on a new parameterization, Ahmed *et al*. (2008) presented the parametric estimation for the BS lifetime distribution. Santos-Neto *et al*. (2014) suggested estimation and inference for the parameterization based on maximum likelihood (ML), moment, modified moment, and generalized moment methods. Wang et al. (2015) suggested a number of alternative estimators for the BS distribution and investigated asymptotic properties of their proposed estimators. Kazim and Salih (2022) compared the performance of maximum likelihood estimators with the Bayesian estimators on the basis of the mean square error. Nevertheless, various estimation aspects of the BS distribution have been developed by several authors using the sample information.

Even though the maximum likelihood estimators offer a few desirable qualities, they do not show explicit expressions, and are quite vulnerable to model departure, which frequently happens in real-world applications. Therefore, we need to replace ML estimators with some other alternative estimators in order to avoid nastiest estimation. The unrestricted robust estimator (URE) for the scale parameter of the BS distribution is suggested by Wang *et al*. (2015) is given below.

$$\hat{\beta}^{URE} = median(y_1, y_2, \dots, y_n), \tag{2}$$

where, $r = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $s = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{y_i}\right)^{-1}$ are the arithmetic mean and harmonic mean, respectively. They also provided asymptotic property of robust estimator $\hat{\beta}^{URE}$ as given below.

$$\sqrt{n}\left(\hat{\beta}^{URE} - \beta\right) \xrightarrow{D} \mathbb{N}\left(0, \frac{\pi(\alpha\beta)^2}{2}\right). \tag{3}$$

The significance of incorporating sample information (SI) and non-samples (NSI) has become more widely recognized in recent years. When combining data from multiple sources, the preliminary test and shrinkage estimation procedures are quite beneficial. Khan and Saleh (2001) proposed a study for determining the mean parameter for a single sample using both the SI and the NSI $(\theta = \theta_0)$ of normal distribution. Baklizi and Ahmed (2008) estimated the reliability parameter $R(t)$of Weibull distribution. Salman *et al*. (2014) suggested a single stage shrinkage estimator using non-sample uncertain prior knowledge as value of scale parameter for gamma distribution.

This study aims to improve the estimation of a robust estimator of scale parameter of the BS distribution. This study follows as in the next section, improved estimation methodologies are furnished. Section 3 is reserved for all the asymptotic theoretical results. All the simulation and numerical study including a real-life data set example can be found in Section 4. Section 5 concludes this study.

2. Estimation methodologies

In this section some improved estimation strategies are furnished as given below:

2.1 The linear shrinkage robust estimator (LSR)

The linear shrinkage (LSR) estimator is given below.

$$\hat{\beta}^{LSR} = \lambda\beta_0 + (1-\lambda)\hat{\beta}^{URE}, \tag{5}$$

where $0 \leq \lambda \leq 1$ is a shrinkage factor and β_0 is presumed non-sample uncertain prior knowledge. If non sample information is correct, we give more weight on λ that will approach to 1. If the non-sample information is faulty or not selected properly then the λ will be closer to zero.

2.2 Preliminary test robust estimator (PTR)

When the available non-sample uncertain prior knowledge is doubtful, it is preferable to construct a preliminary test robust (PTR) estimator by performing a pretest on $H_0: \beta = \beta_0$. The development of a pretest is mainly dependent on the distance of the test statistic from the unrestricted robust estimator ($\hat{\beta}^{URE}$) and β_0. The large sample Wald's type test statistic to test $\beta = \beta_0$ is

$$\mathcal{L}_n = \frac{\{\sqrt{n}(\hat{\beta}^{URE} - \beta_0)\}^2}{\mathbb{v}^*}, \tag{6}$$

where, $\mathbb{v}^*$is the asymptotic variance of the scale parameter mentioned in Equation (5). The sampling distribution of the $\mathcal{L}_n$converges to a central χ^2 distribution with 1 degree of freedom. Therefore, we may show preliminary test estimator (PTR) as:

$$\hat{\beta}^{PTR} = \hat{\beta}^{URE} - I(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*})\big(\hat{\beta}^{URE} - \beta_0\big), \tag{7}$$

where, $I(\cdot)$ is an indicator function. Using $\chi^2_{(1)}$distribution one can obtain critical values at level of significance α^*. The pretest estimator chooses β_0 or $\hat{\beta}^{URE}$ whether the null hypothesis is accepted or rejected by the test statistics.

2.3 The Shrinkage preliminary test robust estimator (SPTR)

The preliminary test estimator can be improved by replacing β_0 with $\hat{\beta}^{LSR}$in (7) to construct shrinkage preliminary (SPR) test estimator whose alternative form is

$$\hat{\beta}^{SPR} = \hat{\beta}^{URE} - \lambda I(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*})\big(\hat{\beta}^{URE} - \beta_0\big). \tag{8}$$

If we substitute $\lambda = 0$ the shrinkage preliminary test estimator becomes preliminary test robust estimator.

3. Asymptotic results

In this section we will furnish important lemmas, distributional results along and local alternatives to develop asymptotic theory of the suggested estimators. For a random sample of size n for BS distribution with parameter β, following asymptotic results hold.

Result 1: $\lim_{n\to\infty} X_{1,n} = \lim_{n\to\infty}\left[\sqrt{n}\left(\hat{\beta}^{URE} - \beta\right)\right] \xrightarrow{D} X_1 \sim N(0, \mathbb{V}^*)$,

Result 2: $\lim_{n\to\infty} X_{2,n} = \lim_{n\to\infty}\left[\sqrt{n}\left(\hat{\beta}^{URE} - \beta_0\right)\right] \xrightarrow{D} X_2 \sim N(\xi, \mathbb{V}^*)$,

Where $\xi = \sqrt{n}\left(\hat{\beta}^{URE} - \beta_0\right)$

Result 3: $\lim_{n\to\infty} X_{2,n}^* = \lim_{n\to\infty}\left[{\mathbb{V}^*}^{-1/2}\left[\sqrt{n}\left(\hat{\beta}^{URE} - \beta_0\right)\right] \xrightarrow{D} X_2^* \sim N(\xi^*, 1)\right]$,

where $\xrightarrow{D}$ is convergence in distribution and $\xi^* = \mathbb{V}_1^{1/2}\xi$.

3.1 Lemmas

Lemma 1: Let Z is a random variable with mean ξ and variance 1 i.e., $Z \sim \mathcal{N}(\xi, 1)$, then the following results hold

$$E[ZI(0 < Z^2 < z)] = \xi P\left(\chi^2_{\nu,\Delta} < z\right)$$

$$E[Z^2 I(0 < Z^2 < z)] = P\left(\chi^2_{3,\Delta} < z\right) + \xi^2 P\left(\chi^2_{5,\Delta} < z\right)$$

where,$\chi^2_{\nu,\Delta}$ is chi-square random variable with ν degree of freedom, and $\Delta = \frac{\xi^2}{\mathbb{V}^*}$ is non-centrality parameter.

3.2 Local alternatives

The following sequence of local alternatives $\{\mathcal{K}_n\}$ can be considered (Baklizi and Ahmed 2008).

$$\mathcal{K}_n : \beta = \beta_{0_n} \text{, where } \beta_{0_n} = \beta_0 + \frac{\xi}{\sqrt{n}}$$
$$\beta = \beta_0 + \frac{\xi}{\sqrt{n}}. \quad (9)$$

Using theses distributional results, lemmas along with local alternatives, we can state the asymptotic properties of the suggested estimators.

3.3 Biases of the estimators

Form above discussion, based on results, the biases of the traditional estimator with proposed estimators can be presented in following forms.

$$\mathscr{b}\left(\hat{\beta}^{URE}\right) = 0,$$

$$\mathscr{b}\left(\hat{\beta}^{LSR}\right) = -\lambda\xi, \tag{10}$$

$$\mathscr{b}\left(\hat{\beta}^{PTR}\right) = -\xi\mathcal{H}_3\left(\chi^2_{1,\alpha^*};\Delta\right),$$

with $\boldsymbol{\mathcal{H}_v}\left(x,\ \Delta = \frac{\xi^2}{\mathbb{V}^*}\right)$ stands for the non-central chi-square distribution function with v degrees of freedom and the non-centrality parameter Δ with α^* level of significance.

$$\mathscr{b}\left(\hat{\beta}^{SPR}\right) = -\lambda\xi\mathcal{H}_3\left(\chi^2_{1,\alpha^*};\Delta\right) \tag{11}$$

Using the sequences of local alternatives in relation (9), derivation of these results is straightforward.

3.4 Asymptotic mean square error

The asymptotic mean square error of the URE, LSR and SPTR are

$$AMSE\left(\hat{\beta}^{URE}\right) = \mathbb{V}^*,$$

$$AMSE\left(\hat{\beta}^{LSR}\right) = \lambda^2\mathbb{V}^*\Delta + \mathbb{V}^*(1+\lambda^2-2\lambda), \tag{12}$$

$$AMSE\left(\hat{\beta}^{SPR}\right) = \mathbb{V}^* - (\lambda)\mathbb{V}^*\mathcal{H}_3\left(\chi^2_{1,\alpha^*};\Delta\right)[2-\lambda] + \lambda\Delta\mathbb{V}^*\left[2\mathcal{H}_3\left(\chi^2_{1,\alpha^*};\Delta\right)\right] - \{2-\lambda\}\mathcal{H}_5\left(\chi^2_{1,\alpha^*};\Delta\right) \tag{13}$$

The expressions for mean square error for $\hat{\beta}^{URE}$ and $\hat{\beta}^{LSR}$ can be determined easily, while mean square error $\hat{\beta}^{SPR}$ can be obtained by doing a straightforward mathematical calculation. As a special case, we can get the asymptotic mean square error of the preliminary test estimator by plugging the value $\lambda = 1$ in asymptotic mean square error expression of the $\widehat{\beta}^{SPR}$. The derivations of AMSE of these estimators are furnished in Baklizi and Ahmed (2008) after changing some notations and symbols.

4. Simulation and numerical study

The effectiveness of all the aforementioned estimators is evaluated on the basis of simulated relative efficiency for various sample sizes, parameter values, shrinkage intensity, and degree of significances α^*= 0.01, 0.05, 0.10, 0.30. The entire simulation research is carried out in *R language*, and the function '*rbisa*' from the *VGAM* package is used to generate random numbers from the BS distribution by setting the shape parameter to $\alpha = 1$ and $\beta = 1$ (Lemonte *et al*., 2006). A random sample with *n* = 30, 50, and 100 from the BS distribution is drawn. The entire simulation is run and repeated 10,000 times. The ratio of simulated mean square errors (SMSE), which is the evaluation criterion for the suggested estimators, is used to simulate relative efficiency.

$$RE\left(\hat{\beta}^{URE}:\hat{\beta}^*\right) = \frac{SMSE\left(\hat{\beta}^{URE}\right)}{SMSE\left(\hat{\beta}^*\right)},$$

where, $\widehat{\beta}^{*}$is any proposed estimator defined in Section 2. The suggested estimator is more efficient than the benchmark if the RE is greater than 1. We introduced an amount of shift $\Psi^{*} = (\beta - \beta_0)$in the sample data defined as to assess how far we deviate from the uncertain prior information (*UPI*) to closely observe the behaviour of the estimators. The simulation studies are performed with different combinations of the configurations such that sample sizes, shrinkage intensity $\lambda = 0.2$, 0.50, 0.70 and 0.95, with different levels of significance $\alpha^{*} = 0.01$, 0.05, 0.10, 0.30. The graphic representations of the performances of the estimators are provided. To save some space, we have limited the results of this study to the few combinations that are displayed in Tables 1 and 2. A visual representation of the numerical outcomes of the simulation experiments presented in tables is shown in Figures 1 and 2. The following conclusion may be drawn from the simulation studies:

i. For all the cases of sample sizes, at $\Psi^{*} = 0$, $\hat{\beta}^{PTR}$ beats its competitive estimator as it has higher simulated relative efficiency at level of significance 1%. Until a particular point in the simulation, the relative efficiency of $\hat{\beta}^{PTR}$ declines and after that it approaches to 1 with increasing amount of Ψ^{*}.

ii. As the sample size is increased, the simulated relative efficiency for $\hat{\beta}^{PTR}$ also increases, specifically at $\Psi^{*} = 0$, at level of significance 1%. As we increase level of significance the efficiency of the said estimator decreases.

iii. When we start to depart from a parameter's true value, $\hat{\beta}^{PTR}$ becomes more efficient than any other recommended estimators, but as we continue to increase the degree of deviation, it gradually loses efficiency and becomes less efficient than $\hat{\beta}^{URE}$.

iv. The preliminary test estimator performs better than all other estimators in terms of the maximum SRE for small region of Ψ^{*}, but for slightly large values of the shift parameter, it loses effectiveness and $\hat{\beta}^{SPR}$ takes control of the scenario.

v. Overall, the best estimator is $\hat{\beta}^{PTR}$ as compared to all others its competitive estimators when there is no shift introduced for data generation.

vi. If the shift amount is being increased $\Psi^{*} \geq 0.20$ in case of sample n = 50, $\hat{\beta}^{SPR}$ defeats all the other estimators of its class and remained best estimator as compared to $\hat{\beta}^{PTR}$regardless the value of Ψ^{*}.

vii. The efficiency of $\hat{\beta}^{LSR}$ decreases continuously, it does not tend to towards the 1 as shown from figures as well.

Table 1: Simulated Relative Efficiencies of the estimators at $\Psi^* = 0.50$, and $n = 30, 50, 100$, $\lambda = 0.5$.

$\boldsymbol{n}$	$\boldsymbol{\Psi^*}$	$\widehat{\boldsymbol{\beta}}^{LSR}$	$\widehat{\boldsymbol{\beta}}^{PTR}$		$\widehat{\boldsymbol{\beta}}^{SPTR}$	
			$\alpha^* = 0.01$	$\alpha^* = 0.05$	$\alpha^* = 0.01$	$\alpha^* = 0.05$
30	0.00	4.0000	16.643	5.2983	3.3891	2.5539
	0.10	3.8199	5.8841	2.8137	3.6200	2.3713
	0.20	2.9327	1.8483	1.2790	2.8216	1.7477
	0.30	2.3095	0.9939	0.7643	2.1601	1.2524
	0.50	1.4508	0.4568	0.4433	1.2131	0.7917
	0.75	1.0115	0.2826	0.3906	0.7208	0.6540
	1.00	0.7880	0.2306	0.4577	0.5428	0.6869
	2.00	0.4744	0.4655	0.9484	0.6897	0.9762
	3.00	0.3803	0.9276	1.0000	0.9674	1.0000
	4.00	0.3327	0.9980	1.0000	0.9991	1.0000
50	0.00	4.0000	14.3594	4.6202	3.3087	2.4252
	0.10	3.4120	3.2984	1.917	3.0438	1.9548
	0.20	2.3248	1.0599	0.8289	2.0223	1.2593
	0.30	1.6691	0.5484	0.5172	1.2851	0.8806
	0.50	0.9745	0.2857	0.3960	0.6680	0.6577
	0.75	0.6544	0.2477	0.5099	0.5078	0.7217
	1.00	0.4775	0.3238	0.7424	0.5629	0.8706
	2.00	0.2824	0.9859	1.0000	0.9937	1.0000
	3.00	0.2294	1.0000	1.0000	1.0000	1.0000
	4.00	0.1934	1.0000	1.0000	1.0000	1.0000
100	0.00	4.0000	12.5076	3.9816	3.2262	2.2812
	0.10	2.8014	1.6369	1.1909	2.3336	1.5299
	0.20	1.5494	0.5159	0.5233	1.1112	0.8588
	0.30	0.9407	0.2934	0.3936	0.6547	0.6576
	0.50	0.5236	0.2703	0.5538	0.5127	0.7516
	0.75	0.3344	0.5693	0.9151	0.7646	0.9606
	1.00	0.2491	0.9247	0.9937	0.9659	0.9971
	2.00	0.1376	1.0000	1.0000	1.0000	1.0000
	3.00	0.1154	1.0000	1.0000	1.0000	1.0000
	4.00	0.0980	1.0000	1.0000	1.0000	1.0000

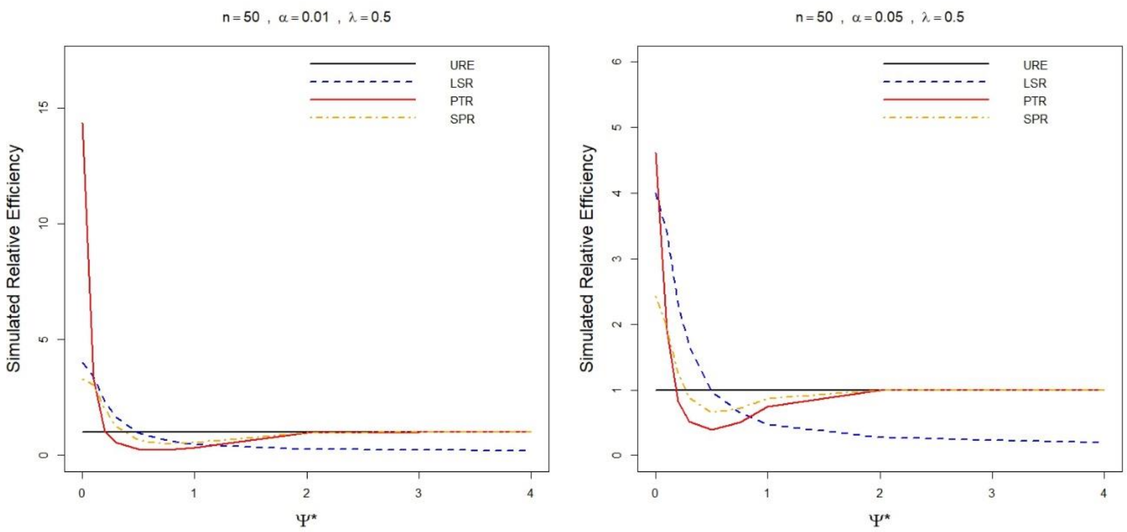

Figure 1: Simulated relative efficiency of the estimators n= 50, Ψ^* =0.5, $\alpha^* = 0.01, 0.05$.

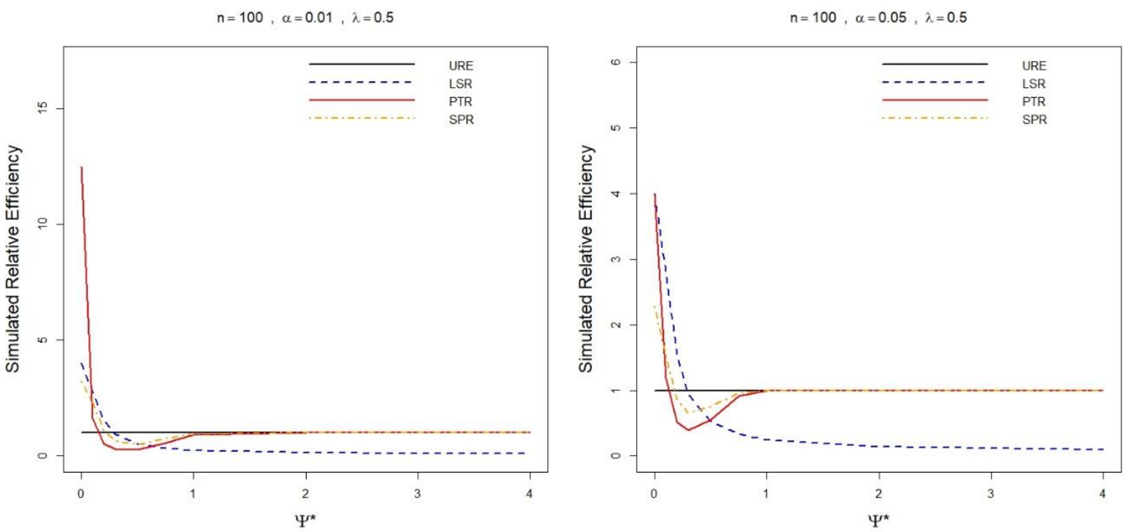

Figure 2: Simulated relative efficiency of the estimators n= 100, Ψ^* =0.5, $\alpha^* = 0.01, 0.05$.

5. Real data application

In this section, we will furnish real data application to evaluate the performance of the suggested estimators.

5.1 Ball size for electronic industry

This data is being taken from the book (Leiva, 2015, p.96). This data is being analyzed for improved estimation of BS distribution considering shape parameter. The ball size (millimetres) for electronic industry data set is consistent of $n = 100$.The bootstrap methodology was used to assess how well the proposed estimators performed. The shrinkage intensity was taken at ξ =0.5. The summary statistic is provided in Table 2 as given below.

Table 2: Summary statistics for ball size of electronic industry.

n	r	s	$\hat{\beta}^{URE}$
100	3.0359	2.1483	2.77

Table 3: Simulated Relative Efficiencies based on bootstrap samples.

b_0	$\hat{\beta}^{LSR}$	$\hat{\beta}^{PTR}$				$\hat{\beta}^{SPR}$			
		α^*				α^*			
		0.01	0.05	0.10	0.30	0.01	0.05	0.10	0.30
2.77	4.00	16.69	6.77	4.23	1.68	3.39	2.77	2.34	1.44
4.00	0.38	0.18	0.30	0.42	0.76	0.42	0.56	0.66	0.88

i. The preliminary test estimator outperforms all its rival estimators and produces the highest simulated relative efficiency. Although the estimator's performance is declining as the level of significance rises, it still outperforms the shrinkage preliminary test estimator in terms of efficiency when $b_0 = 2.77$ as its true parametric value.
ii. When *UPI* is used $\hat{\beta}^{SPR}$ is most efficient estimator in terms of efficiency due to deviation from the parametric true value and apply NSI ($b_0 = 4.00$)

6. Discussion and conclusion

In this present study, the efficiency of the shrinkage preliminary and preliminary test estimators was studied when the sample and non-sample information are integrated. We compared the asymptotic properties of the three suggested estimators with the benchmark unrestricted robust estimator. A graphic illustration of the estimator's efficiency as shown by simulated relative efficiency was also provided. In order to decide whether the null hypothesis should be accepted or rejected when taking non sample information into account, a large sample Wald’s test statistic is also constructed. The theoretical support for each finding is provided by the simulation experiments performed in this work. The preliminary test estimator has been found to perform better than both the linear shrinkage estimator and the shrinkage preliminary test estimator in several areas of the parametric space. Regardless of sample sizes and levels of significance, the shrinkage preliminary test estimator performs best when a little amount of variation from the true value is noticed. But it is clear from this study that in the aforementioned circumstances, a preliminary test estimator is recommended. The shrinkage preliminary test estimator is advised as a last resort in the event of divergence from the true value.

Appendix

In this appendix, we will give mathematical derivations of results in Sections 3.3 and 3.4

Proof of Equation (10)

$$\mathcal{b}\left(\hat{\beta}^{LSR}\right) = \lim_{n\to\infty} E\left[\sqrt{n}\left(\hat{\beta}^{LSR} - \beta\right)\right] = \lim_{n\to\infty} E\left[\sqrt{n}\left\{\left(\lambda\beta_0 + (1-\lambda)\hat{\beta}^{URE} - \beta\right)\right\}\right]$$
$$= \lim_{n\to\infty} E\left[\sqrt{n}\left\{\left(\lambda\beta_0 + \hat{\beta}^{URE} - \lambda\hat{\beta}^{URE} - \beta\right)\right\}\right] = \lim_{n\to\infty} E\left[\sqrt{n}\left\{\left(\hat{\beta}^{URE} - \beta\right) + \left(\lambda\beta - \lambda\hat{\beta}^{URE}\right)\right\}\right]$$
$$= \lim_{n\to\infty} E\left[\sqrt{n}\left(\hat{\beta}^{URE} - \beta\right) - \lambda\sqrt{n}\left(\hat{\beta}^{URE} - \beta_0\right)\right] = \lim_{n\to\infty} E\left[\sqrt{n}\left(\hat{\beta}^{URE} - \beta\right)\right] - \lim_{n\to\infty} E\left[\lambda\sqrt{n}\left(\hat{\beta}^{URE} - \beta_0\right)\right]$$

$= \lim_{n\to\infty} E[X_{1,n}] - \lim_{n\to\infty} E[\lambda X_{2,n}] = E[X_1] - \lambda E[X_2]$

By using the expectation of X_1 and X_2

$$= 0 - \lambda\xi = -\lambda\xi$$

Proof of Equation (11)

$$\mathscr{b}\left(\hat{\beta}^{SPR}\right) = \lim_{n\to\infty} E\left[\sqrt{n}\left(\hat{\beta}^{SPR} - \beta\right)\right]$$

$$= \lim_{n\to\infty} E\left[\sqrt{n}\left(\hat{\beta}^{URE} - \lambda I\left(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*}\right)\left(\hat{\beta}^{URE} - \beta_0\right) - \beta\right)\right]$$

$$= \lim_{n\to\infty} E\left[\sqrt{n}\left(\hat{\beta}^{URE} - \beta\right)\right] - \lambda \lim_{n\to\infty} E\left[\sqrt{n}\left(\hat{\beta}^{URE} - \beta_0\right) I\left(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*}\right)\right]$$

Using Results 1, 2 and 3 and Lemmas 1 and Equation (9) in Section 3, we may proceed as

$$= E[X_{1,n}] - \lambda E\left[\lambda_{2,n} I\left(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*}\right)\right] = -\lambda E\left[X_{2,n} I\left(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*}\right)\right] = -\lambda E\left[X_2 I\left(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*}\right)\right]$$

$$= -\lambda\left[\mathbb{V}^{*\frac{1}{2}} \mathbb{V}^{*\frac{-1}{2}} X_2 I\left(\chi^2_{1,\Delta} < \chi^2_{1,\alpha^*}\right)\right] = -\lambda \mathbb{V}^{*\frac{1}{2}}\left[\mathbb{V}^{*\frac{-1}{2}} X_2 I\left(\chi^2_{1,\Delta} < \chi^2_{1,\alpha^*}\right)\right]$$

As $\mathbb{V}^{*\frac{-1}{2}} X_2 = X_2^*$ we may get

$= -\lambda \mathbb{V}^{*\frac{1}{2}}\left[X_2^* I\left(\chi^2_{1,\Delta} < \chi^2_{1,\alpha^*}\right)\right]$ and $X_2^* \sim N\left(\mathbb{V}^{\frac{1}{2}}\xi, \mathbb{V}\right)$, therefore we have

$$= -\lambda \mathbb{V}^{\frac{1}{2}} \mathbb{V}^{\frac{-1}{2}} \xi P\left(\chi^2_{3,\Delta} < \chi^2_{1,\alpha^*}\right) = -\lambda \mathbb{V}^{\frac{1}{2}} \mathbb{V}^{\frac{-1}{2}} \xi P\left(\chi^2_{3,\Delta} < \chi^2_{1,\alpha^*}\right) = -\lambda\xi \mathcal{H}_3\left(\chi^2_{1,\alpha^*}; \Delta\right)$$

After inserting the λ= 1 we get bias of $\hat{\alpha}^{PTR}$

Proof of Equation (12)

$$AMSE\left(\hat{\beta}^{LSR}\right) = \lim_{n\to\infty} E\left[\sqrt{n}\left(\hat{\beta}^{LSR} - \beta\right)\right]^2 = \lim_{n\to\infty} E\left[\sqrt{n}\left(\hat{\beta}^{LSR} - \beta_0\right) - (\beta - \beta_0)\right]^2$$

$$= \lim_{n\to\infty} E\left[\sqrt{n}\left(\hat{\beta}^{LSR} - \beta_0\right)\right]^2 + \lim_{n\to\infty} E\left[\sqrt{n}(\beta - \beta_0)\right]^2 - 2\lim_{n\to\infty} E\left[\sqrt{n}\left(\hat{\beta}^{LSR} - \beta_0\right)\sqrt{n}(\beta - \beta_0)\right]$$

As $\sqrt{n}(\beta - \beta_0) = \xi$ and $\sqrt{n}\left(\hat{\beta}^{LSE} - \beta\right) = X_{2,n} - \lambda X_{2,n}$

$$= \lim_{n\to\infty} E\left[(X_{2,n} - \lambda X_{2,n})\right]^2 + \lim_{n\to\infty} E\,[\xi]^2 - 2\lim_{n\to\infty} E\left[(X_{2,n} - \xi X_{2,n})(\xi)\right]$$

$$= E\left[X_2^2 + \lambda^2 X_2^2 - 2\lambda X_2^2\right] + \xi^2 - 2E[X_2 - \lambda X_2]\,\xi$$

$$= E[X_2^2] + \lambda^2 E[X_2^2] - 2\lambda E[X_2^2] + \xi^2 - 2E[X_2]\xi + 2E[\lambda X_2]\,\xi$$

$$= \left(\mathbb{V}^* + \xi^2\right) + \lambda^2\left(\mathbb{V}^* + \xi^2\right) - 2\lambda\left(\mathbb{V}^* + \xi^2\right) - 2\xi^2 + 2\lambda\xi^2$$

As, $\Delta = \frac{\xi^2}{\mathbb{V}^*}$ then, $\mathbb{V}^*\Delta = \xi^2$

$$= \mathbb{V}^* + \xi^2 + \lambda^2\mathbb{V}^* + \lambda^2\xi^2 - 2\lambda\mathbb{V}^* - 2\lambda\xi^2 + \xi^2 - 2\xi^2 + 2\lambda\xi^2$$

$$= \mathbb{V}^* + \lambda^2\mathbb{V}^* + \lambda^2\mathbb{V}^*\Delta - 2\lambda\mathbb{V}^* - 2\lambda\mathbb{V}^*\Delta + 2\lambda\mathbb{V}^*\Delta$$

$$= \lambda^2\mathbb{V}^*\Delta + \mathbb{V}^*(1 + \lambda^2 - 2\lambda)$$

Proof of Equation (13)

$$AMSE\left(\hat{\beta}^{SPR}\right) = \lim_{n\to\infty} E\left[\sqrt{n}\left(\hat{\beta}^{SPR} - \beta\right)\right]^2$$

$$= \lim_{n\to\infty} E\left[\sqrt{n}\left\{\left(\hat{\beta}^{URE} - \beta\right) - \lambda\left(\hat{\beta}^{URE} - \beta_0\right) I\left(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*}\right)\right\}\right]^2$$

$$= \lim_{n\to\infty} E\left[\begin{array}{c}\left\{\sqrt{n}\left(\hat{\beta}^{URE} - \beta\right)\right\}^2 + \lambda^2 I^2\left(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*}\right)\left\{\sqrt{n}\left(\hat{\beta}^{URE} - \beta_0\right)\right\}^2 - 2\lambda\sqrt{n}\left(\hat{\beta}^{URE} - \beta\right) \\ I\left(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*}\right)\sqrt{n}\left(\hat{\beta}^{URE} - \beta_0\right)\end{array}\right]$$

$$= E\left[\lim_{n\to\infty}(X_{1,n}^2) + \lambda^2 \lim_{n\to\infty}\{(X_{2,n}^2)I^2(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*})\} - 2\lambda \lim_{n\to\infty}\{\sqrt{n}(\hat{\beta}^{URE} - \beta)I(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*})\sqrt{n}(\hat{\beta}^{URE} - \beta_0)\}\right]$$

$$= E[X_1^2] + \lambda^2 E\{(X_2^2)I^2(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*})\} - 2\lambda E\left[\lim_{n\to\infty}\{\sqrt{n}(\hat{\beta}^{URE} - \beta)I(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*})\sqrt{n}(\hat{\beta}^{URE} - \beta_0)\}\right] =$$

$$\mathbb{V}^* + \lambda^2\mathbb{V}^* + E\{X_2^{*2}I^2(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*})\} - 2\lambda E\left[\lim_{n\to\infty}\{\sqrt{n}(\hat{\beta}^{URE} - \beta)I(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*})\sqrt{n}(\hat{\beta}^{URE} - \beta_0)\}\right] \quad (A.1)$$

Now considering the third term from Equation (A.1)

$$= -2\lambda E\left[\lim_{n\to\infty}\{\sqrt{n}(\hat{\beta}^{URE} - \beta)I(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*})\sqrt{n}(\hat{\beta}^{UML} - \beta_0)\}\right]$$

$$= -2\lambda E\left[\lim_{n\to\infty}\{\sqrt{n}(\hat{\beta}^{URE} - \beta_0)\}^2 I(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*})\right] + 2\lambda E\left[\lim_{n\to\infty}\{\sqrt{n}(\hat{\beta}^{URE} - \beta_0)\sqrt{n}(\beta - \beta_0)I(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*})\}\right]$$

$$= -2\lambda E\left[\lim_{n\to\infty}(X_{2,n}^2)I(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*})\right] + 2\lambda E\left[\lim_{n\to\infty}\{\xi X_{2,n}^2 I(\mathcal{L}_n < \mathcal{L}_{n,\alpha^*})\}\right]$$

$$= -2\lambda E[(X_2^2)I(\chi_{1,\Delta}^2 < \chi_{1,\alpha^*}^2)] + 2\xi\lambda E[X_2 I(\chi_{1,\Delta}^2 < \chi_{1,\alpha^*}^2)]$$

$$= -2\lambda E\left[\left(\mathbb{V}^{*\frac{1}{2}}\mathbb{V}^{*\frac{-1}{2}}X_2\right)^2 I(\chi_{1,\Delta}^2 < \chi_{1,\alpha^*}^2)\right] + 2\xi\lambda E\left[\left(\mathbb{V}^{*\frac{1}{2}}\mathbb{V}^{*\frac{-1}{2}}X_2\right)^2 I(\chi_{1,\Delta}^2 < \chi_{1,\alpha^*}^2)\right]$$

$$= -2\lambda\mathbb{V}^* E\left[\left(\mathbb{V}^{*\frac{-1}{2}}X_2\right)^2 I(\chi_{1,\Delta}^2 < \chi_{1,\alpha^*}^2)\right] + 2\xi\mathbb{V}^{*\frac{1}{2}}E\left[\mathbb{V}^{*\frac{-1}{2}}X_2 I(\chi_{1,\Delta}^2 < \chi_{1,\alpha^*}^2)\right] \quad (A.2)$$

Putting (A.2) term into Equation (A.1)

$$= \mathbb{V}^* + \lambda^2\mathbb{V}^*[P(\chi_{3,\Delta}^2 < \chi_{1,\alpha^*}^2) + \mathcal{V}^{-1/2}\xi^2 P(\chi_{5,\Delta}^2 < \chi_{1,\alpha^*}^2)] - 2\mathbb{V}^*\lambda E[X_2^{*2}I(\chi_{1,\Delta}^2 < \chi_{1,\alpha^*}^2)] + 2\lambda\mathcal{V}^{\frac{1}{2}}\xi E[X_2^* I(\chi_{1,\Delta}^2 < \chi_{1,\alpha^*}^2)]$$

$$= \mathbb{V}^* + \lambda^2\mathbb{V}^*\mathcal{H}_3(\chi_{1,\alpha^*}^2;\Delta) + \lambda^2\xi^2\mathcal{H}_5(\chi_{1,\alpha^*}^2;\Delta) - 2\mathbb{V}^*\lambda\mathcal{H}_3(\chi_{1,\alpha^*}^2;\Delta) - 2\lambda\xi^2\mathcal{H}_5(\chi_{1,\alpha^*}^2;\Delta) + 2\lambda\xi^2\mathcal{H}_3(\chi_{1,\alpha^*}^2;\Delta)$$

$$= \mathbb{V}^* - \lambda\mathbb{V}^*\mathcal{H}_3(\chi_{1,\alpha^*}^2;\Delta)[2-\lambda] + \lambda\Delta\mathbb{V}^*[2\mathcal{H}_3(\chi_{1,\alpha^*}^2;\Delta)] - \{2-\lambda\}\mathcal{H}_5(\chi_{1,\alpha^*}^2;\Delta)$$

Putting the value of $\lambda = 1$ we will get $AMSE\ (\hat{\alpha}^{PTR})$.

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