Journal of Statistics
Volume 27, 2023, pp. 103-115
ISSN-P: 1684-8403
ISSN-E: 2957-8981

Modified-Weibull Distribution with Applications on Real Life Data Sets

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Abstract

Among the most commonly used distributions for analysing lifetime data are exponential distributions, Rayleigh distributions, linear failure rate distributions, and Weibull distributions. There are several desirable properties and pleasant physical interpretation properties to these distributions. This paper introduces a new more flexible model for lifetime data and named as Generalized Reverse Exponential Transformed Weibull (GRETW) Distribution. We have discussed the properties of the GRETW. The order statistics of the proposed distribution have also been studied. Moreover, maximum likelihood estimator of unknown parameters has been obtained. Different data sets are analysed and observe that in comparison to other distributions, this distribution can provide a better fit.

Keywords

Exponential distribution, Rayleigh distribution, Generalized Reverse Exponential Transformed Weibull, Failure rate, Maximum likelihood estimation.

1. Introduction

Weibull distribution is the best lifetime distribution because of its flexibility. The Weibull distribution can absorb the characteristics of some types of distributions such as exponential and Rayleigh distribution. The Weibull distribution is flexible and applicable in a variety of fields, but it is not applicable in some situations especially when the hazard rate functions are monotone, bathtub or bimodal shapes. When $\beta=1$ Weibull distribution becomes Rayleigh and when $\alpha=1$ it becomes exponential distribution.

For analyzing lifetime data, several models have been proposed in the literature including additive Weibull (Xie and Lai, 1995), exponentiated Weibull (Mudholkar *et al*., 1995; Mudholkar and Huston, 1996), extended Weibull (Xie *et al*., 2002), modified Weibull (Lai *et al*., 2003), beta modified Weibull (Silva *et al*., 2010), Kumaraswamy Weibull (Cordeiro *et al*., 2010), transmuted Weibull (Aryal and Tsokos, 2011), Kumaraswamy modified Weibull (Cordeiro *et al*., 2014), transmuted complementary Weibull geometric (Afify *et al*., 2014), Marshall-Olkin additive Weibull (Afify *et al*., 2016) and Kumaraswamy transmuted exponentiated additive Weibull (Nofal *et al*., 2016) distributions.

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The objective of this research paper is to introduce and explore the properties of a new probability model. We introduce a three-parameter distribution called as Generalized Reverse Exponential Transformed Weibull (GRETW) distribution with three parameters (α, β, θ). A decreasing or unimodal PDF function is observed for the GRETW distribution. We also derive the properties of the GRETW Distribution. Characteristics such as survival function, hazard function, moments, moment generation function, quantile function, entropy. Maximum likelihood estimation (MLE) of unknown parameters of GRETW distribution are derived. Lastly different lifetime datasets are used to analyze and to observe that GRETW distribution is a better fit than the existing ones.

According to the paper's structure, the GRETW distribution is presented in Section 2 and its properties are also discussed in Section 2. Section 3 discusses the MLE. In Section 4 we applied the GRETW distribution to real life data sets, and in Section 5, we concluded the paper followed by recommendations in Section 6.

2. Generalized Reverse Exponential Transformed Weibull (GRETW) distribution

Several models have been introduced in literature for examining lifetime data, including the Weibull distribution which has been adapted by numerous authors to increase its versatility and practicality. Kumar *et al.* (2015) proposed a modified version of the Weibull distribution known as the DUS transformation to yield a new distribution.

$$F(x) = \frac{e^{G(x)}-1}{e-1}, \tag{1}$$

In this paper we modified the DUS generator by subtracting it from total probability in equation. Afterwards, we generalized and transformed the CDF by adding shape parameter say θ>0, for more flexibility. Thus, the modified generator named as Generalized Reverse Exponential Transformed generator that is given below:

$$F(x) = 1 - \frac{e^{1-[G(x)]^{\theta}}-1}{e-1}, \tag{2}$$

2.1 Cumulative distribution function of GRETW Distribution

By the definition of CDF, we can write our obtained result of GRETW distribution as:

$$F(x;\alpha,\beta,\theta) = \{\frac{e-e^{1-\left[1-e^{-\alpha x^{\beta}}\right]^{\theta}}}{e-1}; \quad x>0 \quad \text{and} \quad 0; elsewhere \tag{3}$$

where, α is scale and β, θ are shape parameters of GRETW distribution and $\alpha > 0$, $\beta > 0$, $\theta > 0$.

The CDF of the GRETW distribution is shown in Figure 1. The probability value is zero (0) for minimum value of random variable X and for maximum value of random variable X the probability value approaches to one. The middle and lower tail of the curve tells the effect of various parametric values of the CDF. The CDF approaches to one with the increasing value of RV. For data set α=1, β=1, θ=1 the curve of CDF approaches to one exponentially whereas for data set α=0.5, β=5, θ=1as we increased the value of X the probability of CDF approaches towards one. For other parametric values its probability also increases with less or more variability respectively.

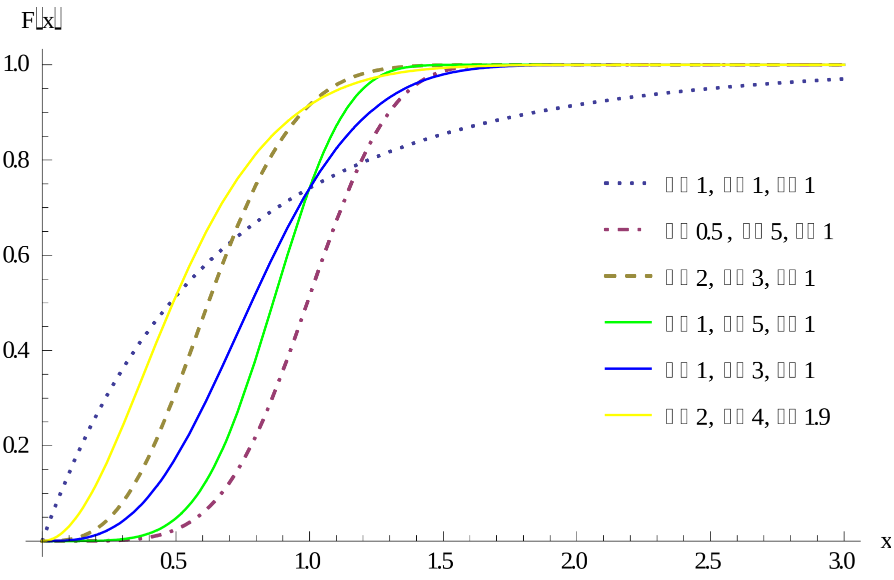

Figure 1: Plots of CDF of GRETW Distribution.

2.2 Probability density function of GRETW distribution

The PDF of GRETW distribution is utilized to find out the probability at a specific interval. The PDF of GRETW distribution is as likely in following axiom.

If X $\sim GRETW(x;\alpha,\beta,\theta)$, where $\alpha > 0, \beta > 0, \theta > 0$ then the PDF of GRETW distribution is given as:

$$f(\alpha,\beta,\theta) = \begin{cases} \frac{\theta}{e-1}\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\left(1-e^{-\alpha x^{\beta}}\right)^{\theta-1} e^{1-\left[1-e^{-\alpha x^{\beta}}\right]^{\theta}}, & x > 0 \\ 0, & elsewhere \end{cases} \quad (4)$$

The plot of GRETW distribution shows different positive, negative, bathtub, exponential and symmetrical shapes. Figure 2 shows that the GRETW distribution is more flexible as it can be formed from exponential type phenomenon and has a good room for symmetrical phenomenon as well. It can be used when survival and hazard functions have bathtub shapes on lifetime data analysis.

For data set α =0.8, β =0.7, θ =2 and for data set α =0.8, β =1.3, θ =0.9 the distribution behaves positively skewed with the difference of more leptokurtic and less peaked respectively. Exponential and symmetrical shapes are seen at α =0.5, β =1.6, θ =2.8, α =0.9, β =1.5, θ =4.8, α =0.4, β =1.4, θ =2.9 and α =1.5, β =0.9, θ =7.1 respectively.

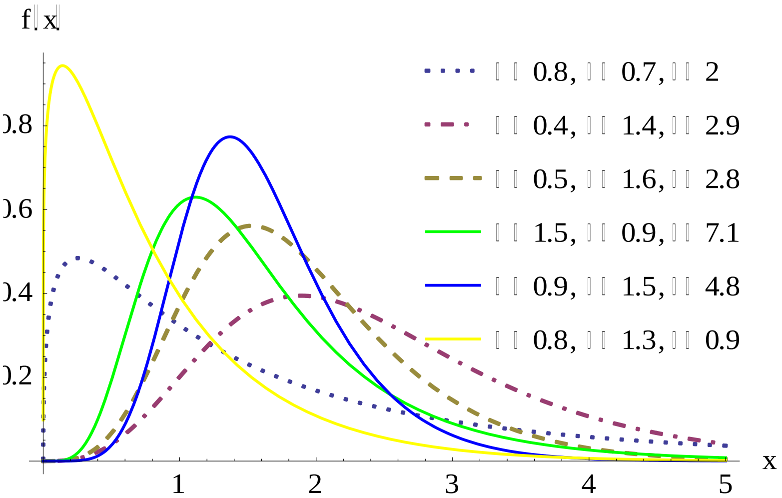

Figure 2: PDFs of GRETW distribution for different parameter values.

2.3 Survival function of GRETW distribution

The survival function is a key concept in survival analysis that characterizes the likelihood that a specific event has not happened up to a given point in time. If X ~ $GRETW(x; \alpha, \beta, \theta)$, then survival function is given as:

$$s(x; \alpha, \beta, \theta) = e^{-\alpha x^{\beta}} \tag{5}$$

The survival function of GRETW distribution can be seen in Figure 3. As the value of the random variable X increases, the survival function gradually converges towards zero which means that the chances of survival decrease.

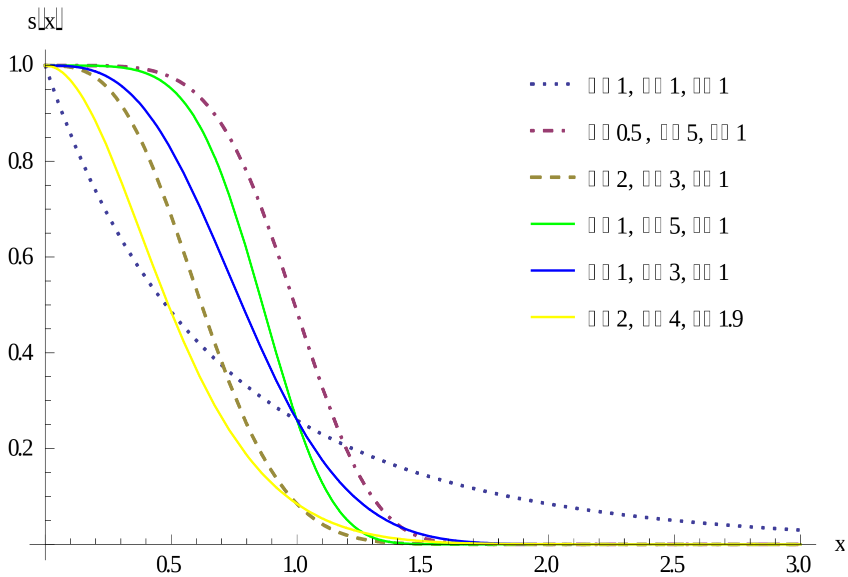

Figure 3: Survival function of GRETW distribution for different parameter values.

2.4 Hazard function of GRETW Distribution

The hazard rate, which is also referred to as the failure rate, denotes the frequency at which a specific event takes place over a specific time period, provided that the event has not yet occurred up to a certain point in time. It signifies the momentary rate of incidence of the event and is significantly associated with the hazard function in survival analysis. If $X \sim GRETW(\alpha, \beta, \theta), where\ \alpha > 0,\ \beta > 0,\ \theta > 0,$ then hazard function of GRETW distribution is given as:

$$h(x, \alpha, \beta, \theta) = \{\frac{\theta\alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}} (1 - e^{-\alpha x^{\beta}})^{\theta-1} e^{1-[1-e^{-\alpha x^{\beta}}]^{\theta}}}{e^{1-[1-e^{-\alpha x^{\beta}}]^{\theta}} - 1}, \qquad x > 0 \tag{6}$$

The hazard rates of GRETW distribution have different shapes i.e. increasing, decreasing, constants, strictly increasing, reverse bathtub, exponential and curvilinear shapes. The distribution having much more hazard shapes is considered as best distribution. By fixing the value of two parameters and then varying third parameter different shapes of hazard function are obtained as given below.

For parameter values α =0.5, 1<β<3, θ =1.5, Figure 4a shows downward increasing trend. Figure 4b shows upward increasing trend for parameter values1<α<3, β=1.5, θ =1.5. For the Figure 4c, the set of parameter values 1<α<3, β=1.5, θ =1.5 shows that with increase in *x*, there is a parallel increase in hazard rates. In Figures 4c and 4d, for the parameter values 1<α<3, β=3.1, θ =1.1, 1<α<3, β=5.1, θ =1.1, hazard function has reverse bathtub shapes. In Figures 4e-4g the curves show curvilinear pattern with less or more variation respectively.

2.5 Moments of GRETW distribution

If $X\sim GRETW(x; \alpha, \beta, \theta)$ then the r^{th} moment about origin is given as:

$$\mu_r' = E() = \frac{\theta e^{2} (\alpha)^{-\frac{r}{\beta}} (-1)^{-j}}{e-1} \left[\left(\frac{r}{\beta} + \theta + \theta j \right)^{-1} + \sum_{i=0}^{\infty} \quad P_i \left(\frac{r}{\beta} \right) \left[\left(\frac{r}{\beta} + i + \theta j + \theta + 1 \right)^{-1} \right] \right] \tag{7}$$

2.6 Moment generating function of the GRETW distribution

If X ~ $GRETW(x; \alpha, \beta, \theta)$ then the MGF of GRETW distribution is as follows:

$$M_X(t) = \frac{e\theta(\alpha)^{-\frac{i}{\beta}}}{e-1} \sum_{k=0}^{\infty} \quad (-1)^k . \frac{1}{k!} \left[B\left(\frac{i}{\beta} + \theta k + \theta, 1 \right) \right] + \sum_{j=0}^{\infty} \quad P_j \left(\frac{i}{\beta} \right) \left[B\left(j + \left(\frac{i}{\beta} + \theta k + \theta + 1, 1 \right) \right) \right] \tag{8}$$

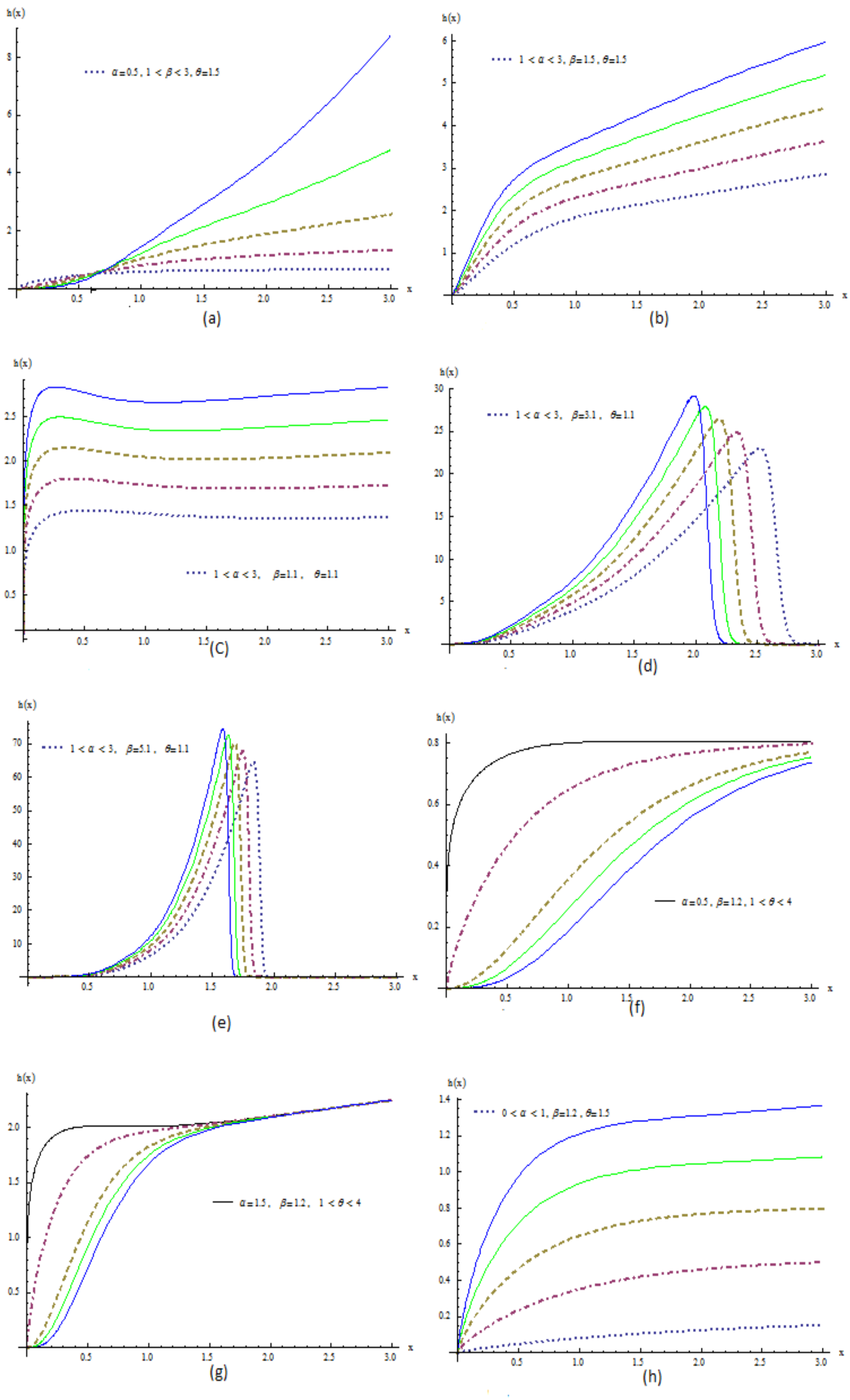

Figure 4: Plots of hazard function of GRETW distribution at different values of parameters.

2.7 Quantile function of GRETW distribution

If X ~ $GRETW(x;\alpha,\beta,\theta)$ then the quantile function is

$$x_q = \left(\frac{ln ln\,(q)^{\frac{1}{\theta}}}{\alpha}\right)^{\frac{1}{\beta}}, \quad 0 < q < 1 \tag{9}$$

2.8 Entropy of GRETW Distribution

Entropy refers to a gauge of the level of uncertainty or randomness present in a random variable. It measures the quantity of information embedded in a message or signal or the extent of chaos in a system. If X ~ GRETW$(x;\alpha,\beta,\theta)$ then Verma Entropy of order θ and type δ is given as:

$$H_\delta^\lambda = \left\{\begin{matrix} \frac{1}{\beta(\lambda-\delta)} log(\frac{\theta\alpha\beta}{e-1})^{\delta+\lambda-1}\left[\sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i\right]^{\delta+\lambda-1} e^{(\delta+\lambda-1)-\sum_{j=0}^{\infty}\binom{\theta(\delta+\lambda-1)}{j}(-1)^j} \\ \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{1}{(\theta-i)(\delta+\lambda-1)+\theta k(\delta+\lambda-1-j)}\right]^{\frac{(\beta-1)(\delta+\lambda-1)+1}{\beta}} \end{matrix}\right\} \tag{10}$$

2.9 Order statistics of GRETW distribution

If X~GRETW$(x;\alpha,\beta,\theta)$, then the order statistic of i^{th} order is as follows:

$$f\left(x_{(i)}\right) = \frac{n!}{(i-1)!\,(n-1)!}\theta\alpha\beta x^{\beta-1}e^{-\alpha x^\beta}\left(1-e^{-\alpha x^\beta}\right)^{\theta-1} e^{1-(1-e^{-\alpha x^\beta})^\theta} \left(\frac{1}{e-1}\right)^n\left(e^{1-(1-e^{-\alpha x^\beta})^\theta}-1\right)^{n-i} \tag{11}$$

3. Maximum likelihood estimation of GRETW distribution

The method of maximum likelihood estimation (MLE) is the most commonly used approach to deduce estimators of unknown quantities. Essentially, the MLE method seeks to determine the values of parameters that optimize the likelihood function, which denotes the probability of observing the data as a function of the parameters.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size *n* drawn from GRETW distribution taking α, β and θ as unknown parameters. The Likelihood Function (LF) of GRETW distribution is given by:

$$L(x;\alpha,\beta,\theta) = \prod_{i=0}^{n}\left[\frac{\theta}{e-1}\alpha\beta x^{\beta-1}e^{-\alpha x^\beta}\left(1-e^{-\alpha x^\beta}\right)^{\theta-1} e^{1-\left[1-e^{-\alpha x^\beta}\right]^\theta}\right] \tag{12}$$

$$\frac{\partial \ln L}{\partial(\alpha)} = \frac{n}{\alpha} - \sum_{i=1}^{n} x_i\beta + (\theta-1)\sum_{i=1}^{n}\left(\frac{e^{-\alpha x^\beta}x_i^\beta}{1-e^{-\alpha x^\beta}}\right) - \sum_{i=1}^{n}\left(\theta(1-e^{-\alpha x^\beta})^{\theta-1}e^{-\alpha x^\beta}x_i^\beta\right) \tag{13}$$

$$\frac{\partial \ln L}{\partial(\beta)} = \frac{n}{\beta} + \sum_{i=1}^{n} \log x_i - \alpha\beta \sum_{i=1}^{n} {x_i}^{\beta-1} + (\theta-1)\sum_{i=1}^{n}\left(\frac{\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}}{1-e^{-\alpha x^{\beta}}}\right) - \theta \sum_{i=1}^{n}(1-e^{-\alpha x^{\beta}})^{\theta-1}\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}} \quad (14)$$

$$\frac{\partial \ln L}{\partial(\theta)} = \frac{n}{\theta} + \sum_{i=1}^{n} \log(1-e^{-\alpha x^{\beta}}) + (1-e^{-\alpha x^{\beta}})^{\theta} \log(1-e^{-\alpha x^{\beta}}) \quad (15)$$

The equations (13)-(15) are nonlinear equations, and therefore, the maximum likelihood estimates are obtained numerically. These estimates are subsequently utilized to determine the MLEs of population parameters α, β and θ that are unknown.

4. Applications of GRETW distribution

This section delves into the practical applications of the GRETW distribution model on two real-life datasets. Additionally, a comparison has been made between the GRETW distribution and both the Weibull and Rayleigh distributions. Based on the graphical analysis using methods such as histograms and testing the goodness of fit, it has been observed that the proposed model outperforms the other two distributions in terms of -2 log likelihood (-2LL) and probability values. The goodness of fit test indicates that a model with smaller -2 log likelihood values and larger probability values is a better fit when compared to other models under consideration.

4.1 Failure times of the air conditioning system of an airplane

GRETW distribution is fitted along with Weibull distribution and Rayleigh distribution. The first data set from Linhart and Zucchini (1986) on the failure times of the air conditioning system of an airplane. The following dataset displays the failure times of 34 air conditioning systems installed in an airplane.

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2

The MLE's of unknown parameters of GRETW, Weibull and Rayleigh distribution, -2 log-likelihood (-2LL) and p-value of goodness of fit test are provided in Table 1.

Table 1: Maximum likelihood estimates and goodness of fit.

Model	α	β	θ	-2LL	Anderson Darling (p-value)	Cramer-Von mises (p-value)
GRETW	0.8794	0.7273	2.1017	109.867	0.9901	0.9752
Weibull	0.5262	1.0102	-	110.899	0.9498	0.9162
Rayleigh	0.1383	-	-	149.183	0.000018	0.00014

Since the -2LL value of the GRETW distribution is lower than that of the Weibull and Rayleigh distributions, and the p-value of Anderson Darling test for the GRETW distribution is higher than that of the other distributions, it can be concluded that the GRETW distribution is a better fit for the failure time data of the air conditioning system of an airplane when compared to the Weibull and Rayleigh distributions. The empirical relationship between GRETW, Weibull and Rayleigh distribution are shown in Figures 5-6. From Figure 8, we can see that GRETW is best fitted distribution than the Weibull and Rayleigh distribution.

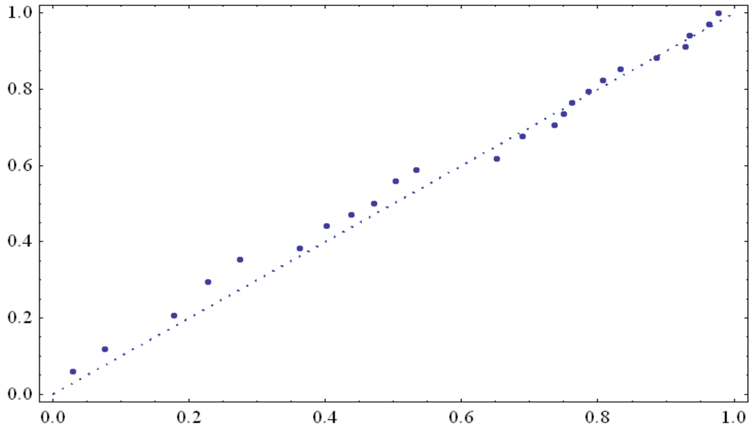

Figure 5: Empirical distribution of GRETW.

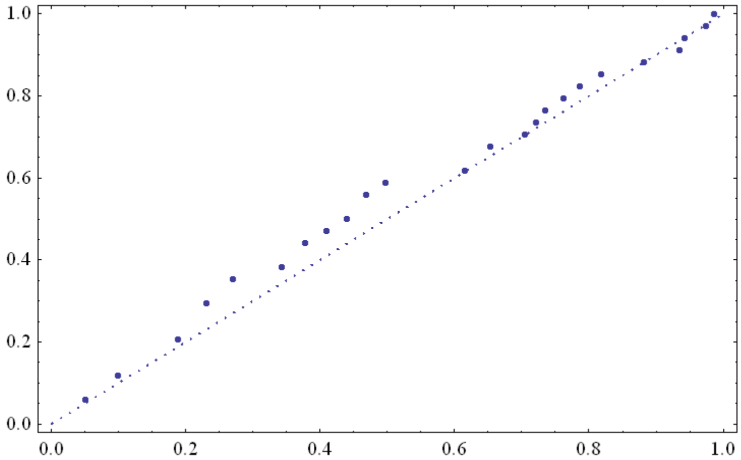

Figure 6: Empirical distribution of Weibull.

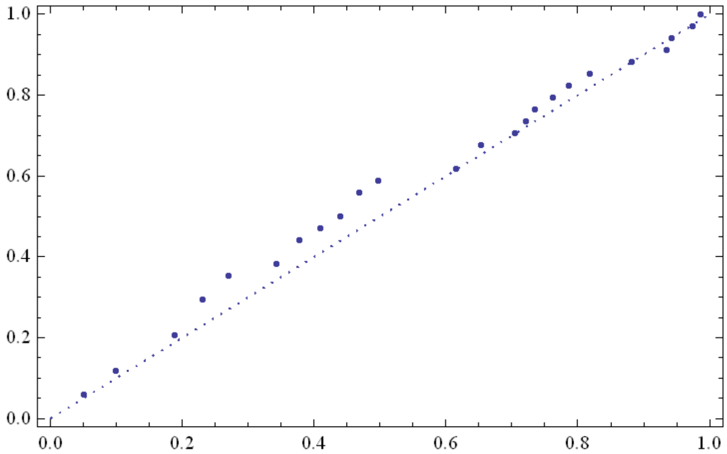

Figure 7: Empirical distribution of Rayleigh.

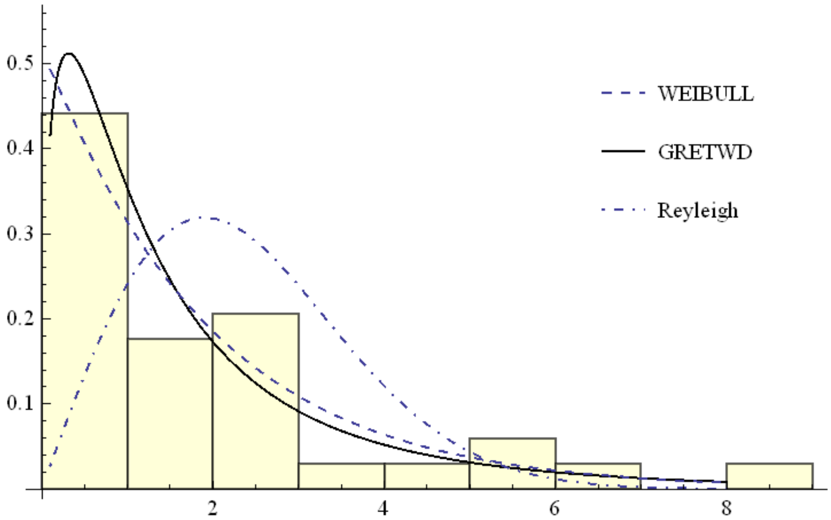

Figure 8: Fitted distributions for failure time of the air conditioning system of airplane.

4.2 Failure times of particular windshield model

The application of proposed GRETW model with Weibull and Rayleigh distribution are fitted on the failure times of windshield model. This data set is previously studied by Murthy *et al*. (2004). The data of failure times of 85 particular wind shield Model is as follows:

0.040 ,1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.820, 3.000, 4.035, 1.281, 2.085, 2.890,4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

The MLE's of unknown parameters of GRETW distribution, Weibull and Rayleigh distributions, -2LL and p-value of goodness of fit test are provided in Table 2. As the value of -2LL of GRETW distribution is smaller than the value of Weibull and Rayleigh distribution also the p-values of Anderson Darling, and Pearson of GRETW distribution is greater than the p-value of Weibull and Rayleigh distribution so we can say that GRETW distribution is better fitted than Weibull and Rayleigh distribution on the data of failure of 85 Particular Windshield Model. The empirical relationship between GRETW, Weibull and Rayleigh distribution is given in Figures 9-11.

Table 2: Maximum likelihood estimates and goodness of fit.

Model	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\theta}$	**-2LL**	**Anderson Darling (p-value)**	**Cramer Von Mises (p-value)**	**Pearson χ^2 (p-value)**
GRETW	0.2143	1.5329	1.6807	288.360	0.7447	0.7605	0.8326
Weibull	0.1406	1.7989	-	291.226	0.1251	0.2086	0.0884
Rayleigh	0.1063	-	-	293.416	0.2337	0.3735	0.0751

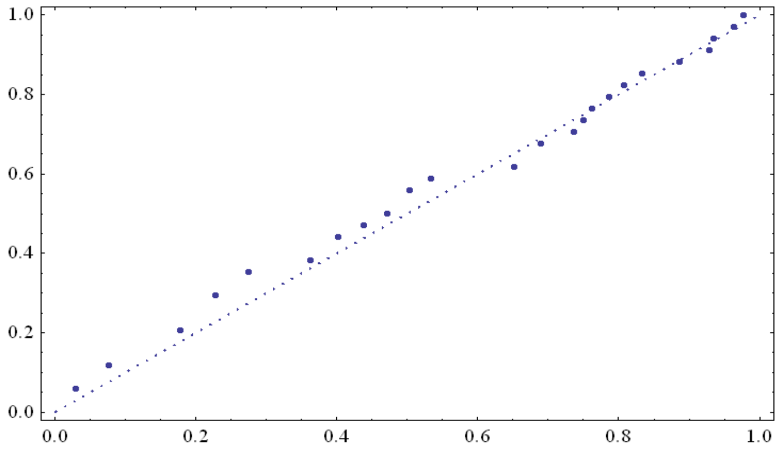

Figure 9: Empirical distribution of GRETW.

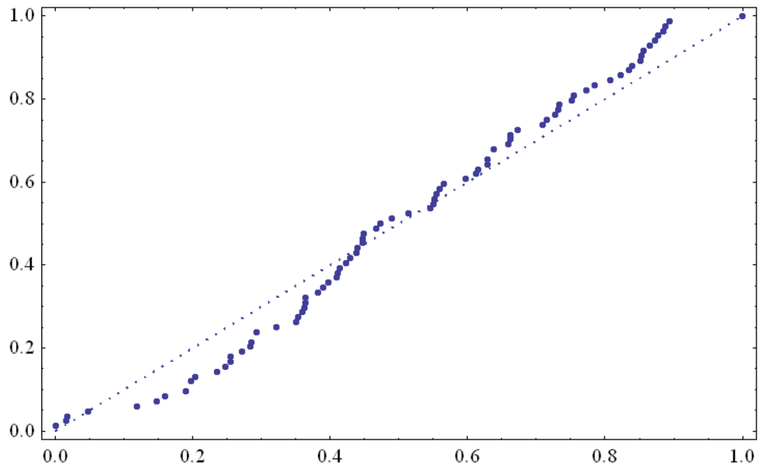

Figure 10: Empirical distribution of Weibull.

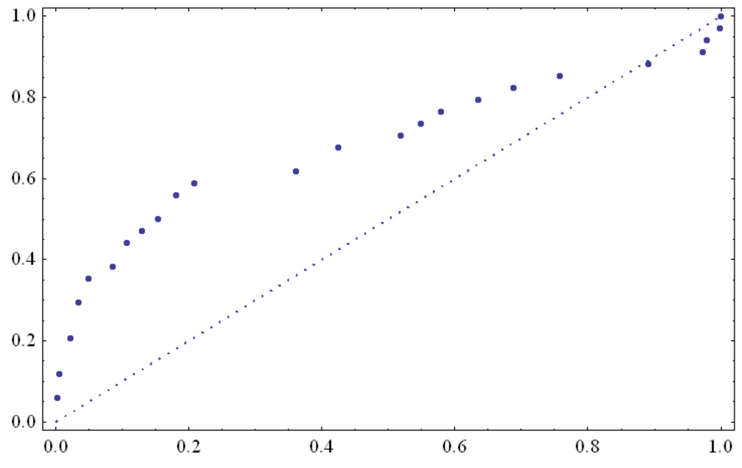

Figure 11: Empirical distribution of Rayleigh.

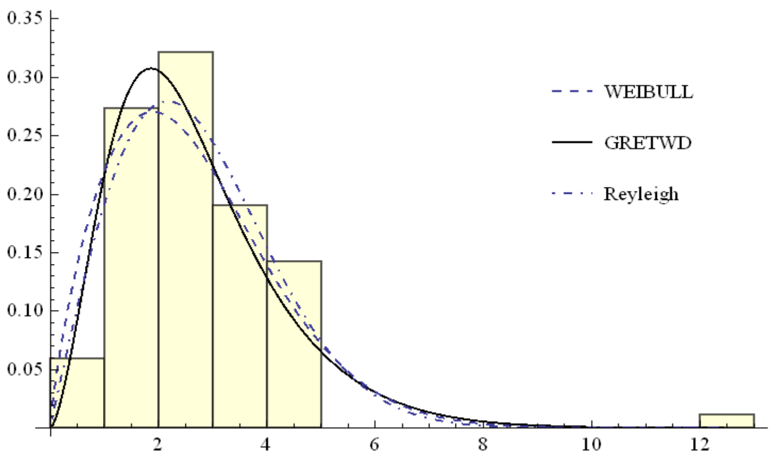

Figure 12: Fitted distributions for Failure times of particular windshield model.

It is evident from Figure 12 that GRETW is best fitted distribution than the Weibull and Rayleigh distributions for failure times of particular windshield model.

5. Conclusion

In this paper we have proposed a new distribution named as Generalized Reverse Exponential Transformed Weibull distribution. The proposed distribution has three parameters(α, β, θ). α is scale and β, θ are shape parameters. We have derived many properties and characteristics of the GRETW distribution i.e., CDF, PDF, hazard function, survival function, quantile function, median, moments, entropy, and estimation etc. The GRETW distribution is more flexible as it may take different shapes depending on the values of the parameters. It has a hazard function decreasing after increasing initially. To know the uncertainty of GRETW distribution entropy has also been derived. By using maximum likelihood estimates, unknown parameters are derived. Through the analysis of various datasets, we have demonstrated the usefulness of the GRETW in modelling different types of data and have shown its ability to capture a range of hazard rate shapes, from monotonic to bathtub shape. Our findings suggest that the GRETW is a versatile and reliable model that can be used in a variety of fields, including medicine, engineering, and economics. We can see after describing all the characteristics of the distribution that this distribution can be used in place of Weibull distribution to get more flexible results.

6. Recommendations

Folks who are inquisitive about growing a greater flexible lifetime distribution(s) they will use some other lifetime distribution in place of Weibull distribution. Some different existing distributions may be used to increase a new version by the usage of a few other appropriate statistical method including mixture and many others.

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