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More Efficient Estimation Strategy for (*k-d)* Class Estimator in Existence of Multicollinearity and Heteroscedasticity: Some Monte Carlo Simulation Evidence

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Abstract

The typical linear regression model does this to have some sort of heteroscedasticity in the error terms and linear correlation in the regressors. The ordinary least squares estimates are significantly impacted by each of these issues. When these assumptions violated in any multiple linear regression model then ordinary least square estimator happen to unstable and no longer remain best linear unbiased estimator. Therefore, in attempt to tackle the issue of Multicollinearity the rigid, Liu and (*k*-*d*) regression exist and easily accessible in literature. The adaptive estimator was recommended to obtain an efficient estimator in comparison to the conventional least square estimator to address the problem of heteroscedasticity. This current work suggests the improved method of adaptation for (*k*-*d*) class estimator to get more efficient results when dealing with multicollinearity and heteroscedasticity occur at same time. All the numerical work is done by using simulation scheme Monte Carlo, with different degrees of collinearity, severity (existence) of heteroscedasticity, and sample size to assess the performance of the suggested estimator. The simulation results provide best performance of adaptive (*k*-*d*) class estimator which is our proposed estimator.

Keywords

Multicollinearity, Heteroscedasticity, Kernel Estimator, Estimation, Regression.

1. Introduction and preliminaries

Regression analysis is one of the most important fields of study because it is used in almost every aspect of life. When dealing with real data, regression modelling has a greater impact in business, economics, medicine, and agriculture. The term multicollinearity was firstly introduced by Frisch (Frisch, 1934). The collinearity is severe issue for multiple linear regression models while the predictors are correlate with each other. The effected estimators and variance-covariance matrix exhibit misleading inferences as matrix of $X'X$ is ill-conditioned as well as determinant of this matrix occasionally singular $\left(|X'X| = 0\right)$. This issue too disturbs the property of estimator called best linear unbiased estimator (BLUE). Other important assumptions which may be violated is "Homoscedasticity". Its means error term does not constant from one observation to another. Due to violation of this assumption of CLRM heteroscedasticity occurs in the data which has very serious

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impacts on the inference about coefficients of regression model. The traditional ordinary least square estimator (OLSE) not remain BLUE when heteroscedasticity is present, which is the key consequences. When the existence of heteroscedasticity prevails the estimation method ordinary least square (OLS) leads to incorrect inferences about regression coefficient as well as (OLSE) becomes inefficient. Thus, the OLSE is not the best solution to estimate the coefficients of a regression line. The interested reader can get more detailed discussion about the effects of heteroscedasticity (Aslam, 2006; Ahmed *et al*, 2011; Aslam *et al*, 2013; Aslam, 2014). In literature, different tests are suggested to detect the presence of heteroscedasticity (Ahmed *et al*, 2011; Aslam, 2014; Park, 1966).

In attempt to combat with multicollinearity (Hoerl and Kennard, 1970; Liu, 1993; Kaciranlar, 2003; Kaciranlar and Sakallioglu, 2007) gave the estimation methods in existence of multicollinearity in order estimate the regression model coefficients. The first time (White, 1980) overcame the challenge of heteroscedasticity by developing the heteroscedasticity consistence covariance matrix estimator (HCCME), which gave break through and made possible to deal with heteroscedasticity and offer asymptotically valid inferences. If the heteroscedasticity error variances ($\hat{\sigma}^2$) are identified, the weighted least squares (WLS) method can satisfy the BLUE property. To estimate variance function using non-parametric and parameter approaches is covered in detail in the literature (see, Bickel, 1978; Box and Hill, 1974; Carrol, 1982; Fuller and Rao, 1978; Rupert, 1994). The weighting strategy fails because heteroscedasticity is rarely discovered in real-world performance. As a result, we must discover estimators of this type that perform equally well even when the unknown form functional of heteroscedasticity may be regarded as estimator is the adaptive estimator (Aslam *et al*., 2013; Carrol, 1982). The more work done by Aslam (2014) who used Carroll's adaptation for regression (Carrol, 1982) and proposed an estimator called Adaptive Rigid Regression (ARRE), which is more efficient than Ordinary Rigid Regression Estimator (ORRE) and OLSE. In our current study, we follow a similar adaptation for the (*k*-*d*) regression proposed by (Sadaullah, 2006). We use (*k*-*d*) regression using variable estimation of the unknown error to construct the equation. Therefore, we present the adaptive (*k*-*d*) estimator (AKDE). Thus, we proposed an adaptive (*k*-*d*) estimator (AKDE), in order to get more efficient estimates for regression coefficients.

2. Existing estimation strategies

In this section, we highlight the main estimation strategies to tackle the issue of multicollinearity and heteroscedasticity.

2.1 Ordinary least square (OLS)

If we consider following multiple linear regression model,

$$y = X\theta + \epsilon_i \tag{1}$$

where y is an $n \times 1$ column vector of observations of response variable, X is an $n \times k$ design matrix of rank k, θ is an $k \times 1$ vector of unknown regression coefficients, ϵ_i is $n \times 1$vector of random errors with zero mean and variance $\Omega = \sigma^2 I_n$, where I_n is an identity matrix of order n. In order to estimate the model (1), there is the most suitable choice to use OLSE which may be defined as

$\hat{\theta}_{OLS} = (X'X)^{-1}(X'y)$ (2)

$\hat{\theta}_{OLS}$ satisfies the BLUE property of linear regression model.

2.2 The ordinary rigid regression estimator (ORRE)

When the assumption of multicollinearity is violated then it is impossible to estimates by equation (2). It paves the way to very grave impacts of the OLSE that may become not BLUE and unstable (Gujarati, 2003). Hoerl and Kennard (1970) proposed ORRE which overcome the issue after a positive constant is added. (*k*) into the design matrix's diagonal values $(X'X)$ is ill-conditioned matrix. ORRE can be shown as:

$$\hat{\theta}_{ORR} = (X'X + kI)^{-1}(X'y) \quad (3)$$

here $k \geq 0$ is biased ridge and shrinkage parameter. This shrinkage parameter *k* indulges the value exact 0 then the ORRE transform into OLSE. The choice of *k* rigid parameter is available in literature (Aslam, 2014; Hoerl *et al*., 1975; Khalaf and Shukur, 2005 and Kiberia, 2003). We select three rigid parameters from available literature Khalaf and Shukur, 2005 and Kiberia, 2003) also same parameters followed by Aslam (2014).

2.3 Liu estimator (LE)

Liu estimator (LE) is proposed (Liu, 1993) to tackle the problem of collinearity between regressors after adding an identity matrix into $(X'X)$ biased and shrinkage constant "*d*" is also added into $(X'y)$ and joining of OLSE. Thus, LE can obtained as

$$\hat{\theta}_d = (X'X + I)^{-1}(X'y + dI)\hat{\theta}_{OLS} \quad (4)$$

LE is the shrinkage estimator of OLSE $\widehat{\theta_d} = \hat{\theta}_{OLS}$ when $d = 1$

2.4 The (*k*-*d*) estimator (KD)

Sadullah *et al*. (2008) suggested another biased and shrinkage estimator namely "(*k*-*d*) estimator".

$$\hat{\theta}_{kd} = (X'X + kI)^{-1}(X'y + d_{op}\hat{\theta}_{ORR}) \quad (5)$$

Where $k \geq 0$ and $(-\infty < d < +\infty)$. (*k*-*d*) class estimator is shrinkage estimator towards OLSE and ORRE. We choose the optimum value to calculate d as follows.

$$\hat{d}_{op} = \frac{\sum_{i=1}^{p} \frac{\lambda_i\left(\hat{\alpha}_i - \hat{\sigma}^2\right)}{(\lambda_i + 1)^2(\lambda_i + k)}}{\sum_{i=1}^{p} \frac{\lambda_i\left(\lambda_i\hat{\alpha}_i^{\,2} + \hat{\sigma}^2\right)}{(\lambda_i + 1)^2(\lambda_i + k)^2}} \quad (6)$$

3. Adaptation

If the variance functional form of error term is unidentified, Carrol (1982) gave an adaptation for heteroscedasticity when regression model is linear using the core adaptive estimator as defined by Bickel (1978). In order to estimate the variance function, Carroll (1982) put its name of this adaptive estimator as "kernel estimator". Carroll supposed the error variability to be a smooth function of average value as:

$$\hat{\sigma}_i{}^2 = H\left(t_i, \hat{\theta}\right) = g(t_i),$$

Here "g" is unidentified, and "t_i" can be found as:

$t_i = x_i{}' \hat{\beta}_{OLS}$

$\hat{\sigma}_i{}^2$Could estimate with the help of kernel estimator, as the computational form of $\hat{\sigma}_i{}^2$ is presented by Nadaraya (Nadaraya, 1964) as given below.

$$\hat{\sigma}_i{}^2 = \frac{\sum_{i=1}^{n} K\left(\frac{t_j - t_i}{q}\right)\mu_i{}^2}{\sum_{i=1}^{n} K\left(\frac{t_j - t_i}{q}\right)}$$

Where $\hat{\mu}_i{}^2$ are residuals attained after estimating OLS coefficients, K(.) is defined to be kernel function with q as smoothing parameter. So, the proposed adaptive estimator can be shown as:

$$\hat{\theta}_{ALS} = \left(X'\widehat{W}X\right)^{-1}(X'\widehat{W}y) \quad (7)$$

$\widehat{W}$is regarded as adaptive estimator of W.

Aslam (2014) also used this adaptive estimation procedure to fit ridge regression (RR) in attempt to get more efficient estimator as "adaptive ridge regression estimator" (ARRE). The proposed estimator by [8] is shown below.

$$\hat{\theta}_{ARR} = \left(X'\widehat{W}_{ARR}X + kI\right)^{-1}(X'\widehat{W}_{ARR}y) \quad (8)$$

where, $\widehat{W}_{ARR}$ are weights assigned into diagonal matrix and called it ARR.

$$\widehat{W}_{ARR} = diag\left(\frac{1}{\hat{\sigma}^2{}_{ARR1}}, \frac{1}{\hat{\sigma}^2{}_{ARR2}}, \dots, \frac{1}{\hat{\sigma}^2{}_{ARRn}}\right)$$

In order to derive the adaptive (*k*-*d*) Class Estimator (AKDCE), we extended work of Aslam *et al*. (2013) work by replacing, $\hat{\theta}_{ORR}$in Equation (5) with $\hat{\theta}_{ARR}$ which is given in Equation (8). Resultantly, we get the adaptive (*k*-*d*) estimator (AKDE) as given below:

$$\hat{\theta}_{AKD} = \left(X'\widehat{W}_{AKD}X\right)^{-1}(X'\widehat{W}_{AKD}y + d_{op}\hat{\theta}_{ARR}) \quad (9)$$

where,

$$\widehat{W}_{AKD} = diag(\frac{1}{\hat{\sigma}^2_{AKD1}}, \frac{1}{\hat{\sigma}^2_{AKD2}}, \ldots, \frac{1}{\hat{\sigma}^2_{AKDn}}).$$

In equation (9), $\widehat{W}_{AKD}$is the matrix of diagonal form called adaptive (*k-d*) regression weights and shrinkage estimator "d_c" is calculated with the help of estimator given in Equation (6). Resultantly, in Equation (9) the given estimator is suggested estimator to deal with the problem of collinearity and heteroscedasticity both occur at same time. The estimator proposed in Equation (9) is more efficient because estimated mean square error (EMSE) is less than its traditional (*k-d*) estimator.

4. Evaluation criterion

The performances of the above stated estimators, the EMSE criterion is used as used by many researchers, the interested readers can get more detail in Alheety *et al*. (2009), Aslam (2014), Liu (1993), Liu (2003), Manson *et al*. (2010). On behalf of the specific estimator $\hat{\theta}_i$ of θ, The EMSE can be numerically find by given mathematical formula in simulation.

$$EMSE(\hat{\theta}) = \sum_{i=1}^{R}[(\hat{\theta}_i - \theta)'(\hat{\theta}_i - \theta)] \big/ R$$

Where $\hat{\theta}_i$ values which are estimated from θ in *ith* replication, where R is the cumulative sum of all simulation replications. Furthermore, the desirable performance of proposed estimator AKDE is assessed in form of relative efficiency, $EMSE(\hat{\theta}_i)/\ EMSE(\hat{\theta}_{AKD})$. Mean Square Error (MSE) is calculated here as:

$$MSE = E(\hat{\theta}_i - \hat{\theta})^2$$

4.1 Estimating the biasing rigid parameters

Now we finalized the rigid parameter *k*, Hoerl and Kennard (1970) suggested an estimator to compute biased rigid parameter *k* which is recognized as the "HKB estimator" and is presented as:

$$\hat{k}_{HKB} = \frac{r\hat{\sigma}^2}{\hat{\theta}_{OLS}'\hat{\theta}_{OLS}}$$

where $\hat{\sigma}^2$ is the mean square error of residuals.

Another researcher (Kiberia, 2003) suggested a different estimator of ridge parameter *k*

$$\hat{k}_{GK} = \frac{\hat{\sigma}^2}{(\prod_{j=1}^{r} \hat{\theta}^2{}_{OLS})^{1/r}}$$

The third biased rigid parameter proposed by (Khalaf and Shukur, 2005) for selecting the appropriate value of shrinkage ridig parameter *k*. Therefore, the estimator is shown below:

$$\hat{k}_{KS} = \frac{\lambda_{max}\hat{\sigma}^2}{(n-r)\hat{\sigma}^2 + \lambda_{max\hat{\theta}^2{}_{OLS}max}}$$

$\hat{k}_{KS}$ the above stated estimator, where $\lambda_{max}$is maximum eigen value of $X'X$ matrix and $\hat{\theta}^2{}_{OLS}max$ is the highest value of $\hat{\theta}_{OLS}$. For our analysis, we used these three *k's* for choosing ridge parameter.

5. Simulation scheme

The Monte Carlo simulation scheme which we used in order to get empirical results of our suggested AKDE is being discussed. Many authors used this simulation scheme can be found in the literature, e.g., Aslam (2014), Newhouse *et al*. (1971), Manson *et al*. (2010).

5.1 The Monte Carlo scheme

Three parametric cases are covered by the simulation process here. The columns of predictors are calculated in accordance with (Aslam, 2014) and the reference cited in

$$x_{ij} = (1-\rho^2)^{0.5} w_{ij} + \rho w_{i4} \qquad i = 1,2,3,\dots,n \quad j = 1,2,3$$

Here the setup is that the ρ is degree of collinearity between any two regressors is represented by ρ^2 and w_{i1}, w_{i2}, w_{i3}, and w_{i4} are independent standard normal N (0, 1) pseudo-random numbers, we used $\rho = 0.80, 0.90, 0.95\ and\ 0.99$
The array of value in matrix of the response variable can be determined as under

$$yi = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} + u_i; \qquad i = 1,2,\dots,n,$$

where $u_i = \sigma_i \varepsilon_i$ and ε_i'sare independent standardized normally distributed N (0,1) and β_0 is assuming 0. For the first case $\rho = 0.80,$ we keep fix the regression coefficient values $(\beta_1, \beta_2, \beta_3)$ as 0.5881, 0.5581, 0.5854, respectively [8]. For $\rho = 0.90,$ we fix the regression coefficient values $(\beta_1, \beta_2, \beta_3)$ as 0.5796, 0.5706 and 0.5818, respectively. For $\rho = 0.95.$ we fixed the regression coefficients at. $(\beta_1, \beta_2, \beta_3)$ as 0.5775, 0.5749 and 0.5797, respectively [8]. Finally, at a high amount of collinearity ($\rho = 0.99$) we keep fix the regression coefficient values $(\beta_1, \beta_2, \beta_3)$ as 0.5773, 0.5772 and 0.5776, respectively. These constants' values must always satisfy the requirement that$\beta' \beta = 1$. These values are selected so that the eigenvector for $X'X$ matrix the next largest Eigen value is normalized (Aslam, 2014; McDonald *et al*, 1975 and Newhouse and Oman 1971) for more details.

Since the performance of error terms, σ_i is heteroscedastic in nature. Subsequently, u_i are independent normally distributed with (0, $\upsilon_i{}^2$). As a result of (Aslam, 2014) and the references listed therein, the error variance is obtained as shown below.

$$\upsilon_i{}^2 = exp(g_1 x_{1i} + g_2 x_{2i} + g_3 x_{3i}),$$

where, g_i is being actual scalar for j=1,2,3. Here $g_1 = g_2 = g_3$. In lieu of $\xi = 0$, It turns into a homoscedastic error term. You can gauge the level of heteroscedasticity by using

$$\Delta = \max(\upsilon_i{}^2)/\min(\upsilon_i{}^2)$$

for every given value of ρ, The method described above was selected so that we have Δ = 1, 4, 36 and 100. When there is mild to severe heteroscedasticity, knowing the degree of it will help the estimators perform more effectively. Our study concentrated on Δ =36, where Δ = 1 causes the error term to become homoscedastic. In this simulation study we used sample size (n) is set at 50 or 100. To maintain the same level of multicollinearity and heteroscedasticity across all initial sample sizes, the 50 observations for each explanatory variable are created at random, reproduced twice, three times, and four times, and then replicated. The repetition number of Monte Carlo simulation is fixed to be R = 5000. The Programming routines in well-known R language are used to carry out all the computation required for the study's estimators by using "*np*" package. We concentrate on how well various estimators perform such as OLSE, ORRE, KDE ARRE and AKDE. We used rigid parameters suggested by three authors mention above and some constant values of $k = 0, 0.01, 0.0$.

The estimated MSE for each estimator is presented in Tables 1 to 3. The estimators on which we have concentrated our research. We started with a sample size of 50 and then raised it to 100 for comparison. To compare the different additional metrics and assess how well the estimators performed, they are also shown in Tables 1 to 3 (OLSE, RRE, KDE and AKDE). As there is moderate level to severe level of heteroscedasticity as the amount $\Delta = 4, 36, 100$ is generated in the data set with dissimilar level of multicollinearity (ρ =0.80, 0.90, 0.95, 0.99) in the predictors. It is evident that AKDE has a lower MSE as compared with all the other estimators for each estimator of rigid parameter (k = 0, 0.01, 0.05, HK, GK, KS).

When the severity of the existence of heteroscedasticity for example $\Delta = 4$, there is moderate level of existence of heteroscedasticity. For all the levels of collinearity the AKDE is more efficient as compared to other estimators. With the increase in severity level of existence of heteroscedasticity $\Delta = 36, 100$, with all rigid parameters values (k) and increasing the sample AKDE own lower MSE. Resultantly, it is established that the suggested estimator is more efficient when the problem of multicollinearity and heteroscedasticity occur at same time.

Table 1: Estimated Mean Square Error (EMSE) and $\Delta = 4$.

	n	**50**					**100**				
ρ	**K**	**OLSE**	**RRE**	**ARRE**	**KDE**	**AKDE**	**OLSE**	**RRE**	**ARRE**	**KDE**	**AKDE**
0.80	**0**	0.1778	0.1778	0.1665	0.1727	**0.1617**	0.0872	0.0872	0.0782	0.0865	**0.0776**
	0.001	0.1750	0.1742	0.1621	0.1696	**0.1578**	0.0881	0.0876	0.0783	0.0871	**0.0778**
	0.005	0.1757	0.1717	0.1578	0.1690	**0.1553**	0.0901	0.088	0.0784	0.0883	**0.0786**
	HK	0.1743	0.1346	0.1243	0.1497	**0.1376**	0.0879	0.0754	0.0673	0.0801	**0.0714**
	GK	0.1738	0.1088	0.1026	0.1321	**0.1207**	0.0873	0.0697	0.0623	0.0763	**0.0678**
	KS	0.1748	0.1605	0.1498	0.1634	**0.1524**	0.0883	0.0842	0.0747	0.0854	**0.0758**
0.90	**K=0**	0.1990	0.199	0.1917	0.1896	**0.1827**	0.1001	0.1001	0.0952	0.0986	**0.0938**
	0.001	0.2012	0.1992	0.1914	0.1908	**0.1836**	0.0983	0.0973	0.0923	0.0962	**0.0913**
	0.005	0.1955	0.1861	0.1812	0.1816	**0.1768**	0.101	0.096	0.0913	0.0963	**0.0915**
	HKB	0.1989	0.1366	0.1367	0.1549	**0.1438**	0.098	0.0778	0.0752	0.0827	**0.0797**
	GK	0.2000	0.0973	0.0997	0.1248	**0.1101**	0.0979	0.0691	0.0682	0.0761	**0.0746**
	KS	0.1993	0.1720	0.1677	0.1760	**0.1514**	0.1002	0.0917	0.0866	0.0931	**0.0878**
0.95	**K=0**	0.4597	0.4597	0.4546	0.4282	**0.4131**	0.2378	0.2378	0.2313	0.2311	**0.2247**
	0.001	0.4651	0.4549	0.4525	0.4315	**0.4087**	0.2410	0.2357	0.2301	0.2318	**0.2262**
	0.005	0.4672	0.4195	0.4166	0.4231	**0.4096**	0.2233	0.2004	0.1968	0.2055	**0.2015**
	HKB	0.4759	0.2671	0.2682	0.3660	**0.3522**	0.2373	0.1591	0.1546	0.1859	**0.1803**
	GK	0.4663	0.1352	0.1416	0.2486	**0.2256**	0.2312	0.0993	0.0993	0.1365	**0.1351**
	KS	0.4607	0.3614	0.3585	0.3973	**0.3835**	0.2321	0.1986	0.1923	0.2074	**0.2007**
0.99	**K=0**	3.0095	3.0095	2.9533	2.8590	**2.7025**	1.5090	1.5090	1.4341	1.4484	**1.3756**
	0.001	3.1105	2.7962	2.7175	2.9592	**2.8735**	1.5223	1.3684	1.2993	1.4496	**1.3744**
	0.005	3.0992	1.9125	1.8772	2.9288	**2.7675**	1.5536	0.9598	0.9069	1.4387	**1.3565**
	HKB	3.0653	1.066	1.0265	2.8446	**2.0491**	1.4780	0.5532	0.5128	1.2479	**1.1779**
	GK	3.0143	0.6108	0.6035	2.6815	**2.5051**	1.5398	0.3142	0.2921	1.1113	**1.0290**
	KS	3.0137	1.7189	1.6350	2.8343	**2.7164**	1.5088	0.9692	0.9026	1.3878	**1.2973**

Table 2: Estimated Mean Square Error (EMSE) and $\Delta = 36$.

	n	50					100				
ρ	K	OLSE	RRE	ARRE	KDE	AKDE	OLSE	RRE	ARRE	KDE	AKDE
0.80	0	0.3490	0.3490	0.1929	0.3367	**0.1867**	0.1755	0.1755	0.0777	0.1735	**0.0769**
	0.001	0.3516	0.3500	0.1880	0.3387	**0.1826**	0.1760	0.1752	0.0789	0.1737	**0.0783**
	0.005	0.3489	0.3411	0.1832	0.3340	**0.1802**	0.1767	0.1728	0.0777	0.1730	**0.0779**
	HK	0.3525	0.2536	0.1292	0.3021	**0.1498**	0.1755	0.1424	0.06115	0.1571	**0.0661**
	GK	0.3477	0.1850	0.1071	0.2540	**0.1171**	0.1756	0.1181	0.0550	0.1422	**0.0592**
	KS	0.3531	0.3214	0.1685	0.3300	**0.1736**	0.1774	0.1681	0.0746	0.1712	**0.0760**
0.90	K=0	0.2628	0.2628	0.1540	0.2471	**0.1464**	0.1301	0.1301	0.0699	0.1276	**0.0687**
	0.001	0.2654	0.2629	0.1532	0.2486	**0.1460**	0.1284	0.1272	0.0691	0.1251	**0.0682**
	0.005	0.2608	0.2489	0.1465	0.2405	**0.1426**	0.1298	0.1239	0.0679	0.1236	**0.0678**
	HKB	0.2616	0.1652	0.1038	0.1949	**0.1183**	0.1307	0.0975	0.0554	0.1057	**0.0593**
	GK	0.2653	0.1216	0.0908	0.1541	**0.1018**	0.1321	0.082	0.0509	0.0936	**0.0559**
	KS	0.2618	0.2216	0.1327	0.2275	**0.1363**	0.1322	0.1197	0.0660	0.1217	**0.0670**
0.95	K=0	0.6161	0.6161	0.4730	0.5777	**0.4429**	0.3077	0.3077	0.2199	0.2986	**0.2132**
	0.001	0.5993	0.5861	0.4532	0.5580	**0.4308**	0.3075	0.3008	0.2150	0.2958	**0.2111**
	0.005	0.6086	0.5462	0.4186	0.5574	**0.4259**	0.2998	0.2691	0.1915	0.2779	**0.1972**
	HKB	0.6045	0.3268	0.2437	0.4731	**0.3486**	0.3059	0.1958	0.1352	0.2406	**0.1644**
	GK	0.5942	0.1706	0.1322	0.3309	**0.2443**	0.3060	0.1127	0.0811	0.1725	**0.1190**
	KS	0.6335	0.4956	0.3688	0.5595	**0.4155**	0.3050	0.2582	0.1841	0.2744	**0.1950**
0.99	K=0	4.3014	4.3014	2.9177	4.1411	**2.8074**	2.0688	2.0688	1.2534	1.9929	**1.2047**
	0.001	4.2617	3.8352	2.5285	4.0976	**2.699**	2.1646	1.9486	1.1678	2.0863	**1.2481**
	0.005	4.1753	2.5911	1.6800	3.9932	**2.5861**	2.1227	1.3166	0.7772	2.0077	**1.1778**
	HKB	4.1349	1.3891	0.8399	3.8991	**3.5825**	2.0613	0.7580	0.4352	1.8424	**1.2298**
	GK	4.1435	0.8245	0.4886	3.7967	**2.7962**	2.1095	0.4481	0.2436	1.5926	**1.4161**
	KS	4.1923	2.3315	1.4485	4.0086	**2.5888**	2.0739	1.2997	0.7543	1.9516	**1.1740**

Table 3: Estimated Mean Square Error (EMSE) and $\Delta = 100$.

	n	50					100				
ρ	K	OLSE	RRE	ARRE	KDE	AKDE	OLSE	RRE	ARRE	KDE	AKDE
0.80	0	0.5402	0.5402	0.2466	0.5196	**0.2388**	0.2659	0.2659	0.0823	0.2622	**0.0816**
	0.001	0.5178	0.5155	0.2363	0.4973	**0.2296**	0.2645	0.2633	0.0799	0.2603	**0.0793**
	0.005	0.5469	0.5349	0.2332	0.5237	**0.2303**	0.2696	0.2637	0.0801	0.2638	**0.0804**
	HK	0.5481	0.3766	0.1433	0.4744	**0.1701**	0.2619	0.2025	0.0605	0.2322	**0.0661**
	GK	0.5291	0.2656	0.1360	0.3955	**0.1325**	0.2602	0.1559	0.0570	0.2019	**0.0564**
	KS	0.5275	0.4776	0.2030	0.4947	**0.2123**	0.2582	0.2437	0.0757	0.2490	**0.0774**
0.90	K=0	0.3490	0.3490	0.1437	0.3250	**0.1368**	0.1708	0.1708	0.0614	0.1666	**0.0604**
	0.001	0.3526	0.3494	0.1419	0.3274	**0.1353**	0.1735	0.1719	0.0611	0.1683	**0.0602**
	0.005	0.3451	0.3300	0.1387	0.3159	**0.1349**	0.1726	0.1651	0.0601	0.1642	**0.06007**
	HKB	0.3500	0.2082	0.0957	0.2568	**0.1089**	0.1806	0.1265	0.0484	0.1416	**0.0520**
	GK	0.3473	0.1529	0.0916	0.1954	**0.0884**	0.1748	0.0989	0.0447	0.1139	**0.0456**
	KS	0.3392	0.2827	0.1184	0.2937	**0.1229**	0.1740	0.1561	0.0569	0.1593	**0.0580**
0.95	K=0	0.7317	0.7317	0.4661	0.6871	**0.4373**	0.3644	0.3644	0.2072	0.3533	**0.2005**
	0.001	0.7399	0.7237	0.4545	0.6909	**0.4336**	0.3743	0.3362	0.1833	0.3493	**0.1894**
	0.005	0.7255	0.6515	0.4074	0.6683	**0.4176**	0.3729	0.2293	0.1226	0.2946	**0.1548**
	HKB	0.7499	0.3913	0.2325	0.5998	**0.3491**	0.3710	0.1281	0.0749	0.2096	**0.1159**
	GK	0.7561	0.2137	0.1348	0.4525	**0.2720**	0.3633	0.3047	0.1698	0.3275	**0.1817**
	KS	0.7347	0.5647	0.3384	0.6536	**0.3896**	0.3613	0.3028	0.1674	0.3255	**0.1794**
0.99	K=0	5.1473	5.1473	2.7678	4.9735	**2.6777**	2.6188	2.6188	1.1615	2.5462	**1.1281**
	0.001	5.0817	4.5760	2.4464	4.8998	**2.6247**	2.5304	2.2786	1.0011	2.4510	**1.0749**
	0.005	5.1189	3.1837	1.5520	4.9325	**2.4096**	2.5426	1.5819	0.6774	2.4362	**1.0373**
	HKB	5.2175	1.7662	0.8074	5.1500	**2.1350**	2.5618	0.9339	0.3759	2.3641	**1.3214**
	GK	5.1331	1.0012	0.4604	4.9553	**2.3965**	2.5952	0.5514	0.2131	2.0829	**1.8561**
	KS	5.2056	2.8799	1.3657	5.0375	**2.5557**	2.5619	1.5776	0.6756	2.4404	**1.1070**

5.2 Graphical presentation

The performance of the estimators can also be presented graphically as shown in Figure 1 and Figure 2. In both figures the level of heteroscedasticity is kept Δ 100 and we fix rigid biased parameter at k = 0.001, with higher degree of multicollinearity e.g. (0.95) portray almost the same picture as given in Fig.1. In graphical presentation exhibits the line of MSE as increased sample size from 50 to 200 the MSE decreasing and lower line goes parallel with ARRE line of MSE. MSE's of the ARRE and AKDE estimators fall very below than rest of the estimators as shown in the above stated tables. In few cases, the ARRE and AKDE seem to be identical in their performance but generally, the AKDE becomes more efficient for severe heteroscedasticity and serious degree of collinearity.

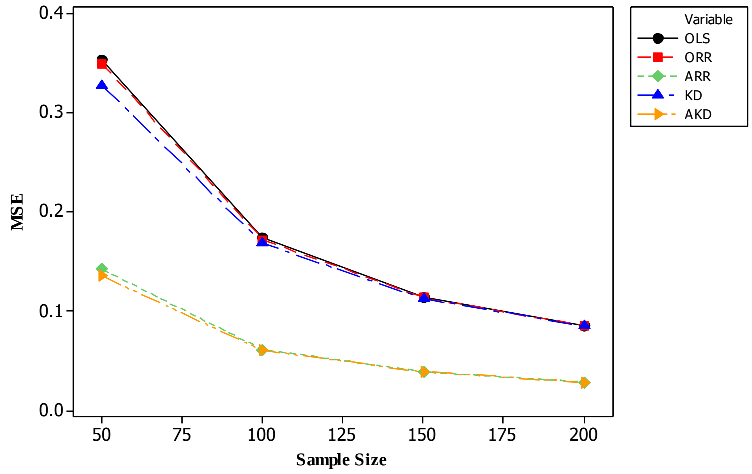

Figure 1: Comparing MSE with different sample sizes at ρ = .90 k = 0.001, for Δ=100.

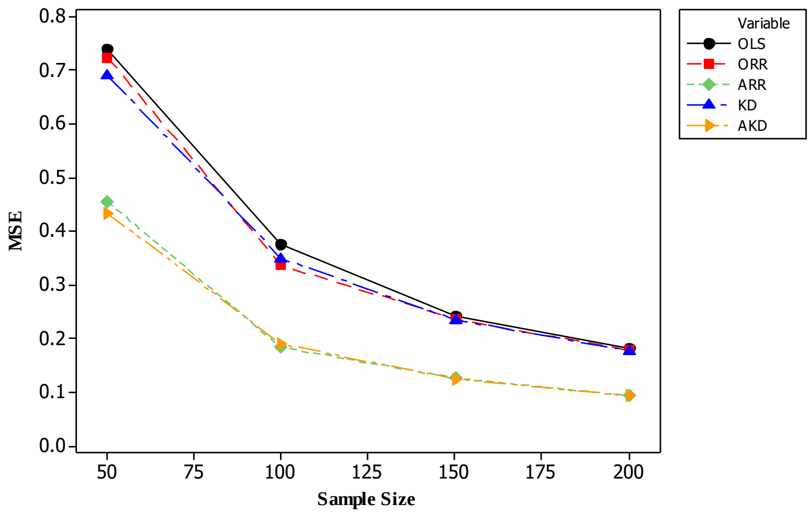

Figure 2: Comparing MSE with different sample sizes at ρ = .95 k = 0.001, for Δ=100.

6. Discussion and conclusion

In this research study, we suggested efficient estimator to solve problem of multicollinearity and heteroscedasticity concerns at same time, we examine the performance of the suggested estimator (AKDE) and compare it with other existing estimators in this work. We took different level of collinearity of predictor variables and different level of severity of existence of heteroscedasticity with sample size 50, 100, 150, 200. All the numerical work is done following Aslam and Pasha (2009), we used the Monte Carlo simulation method in this study. In the preceding section, we stated that our proposed estimator is more effective than other available estimators. As a result, when assumptions linear regression model (multicollinearity and heteroscedasticity) are being violated the AKDE class estimator is the best option over OLSE. When the linear regression model suffers from both multicollinearity and heteroscedasticity simultaneously, our suggested estimator is an appropriate choice.

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