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An Exponentially Ratio Type Estimator under Ranked Set Sampling (RSS) and its Efficiency

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Abstract

This study demonstrated the efficiency of an exponential-type estimator under Ranked Set Sampling (RSS) design. For the mathematical expressions, the MSE and Bias had been derived up to 1st and 2nd degree approximation respectively. Family of this proposed estimator has derived also and for efficiency comparison, mathematical conditions have been derived by comparing various existing estimators under RSS design. For the efficiency, a numerical comparison was also done by taking 3 real life population data for high negative correlation, moderate and high positive correlation. On the basis of this proof, it is revealed that exponentially ratio type estimator is most efficient than all other existing estimators.

Keywords

Ranked set sampling, Simple random sampling, Exponential ratio type estimator, Relative efficiency.

1. Introduction

The theory of ranked set sampling is a current development that empowers one to provide more structure to the collected sample units, while in the name is a little inaccuracy as it is not as much a sampling method as it is a data measurement procedure. Rank set sampling needs the judgment of ordered elements to obtain an estimate of the population true value. The method works best when measuring or quantifying a particular element is challenging, although drawing and ranking the members of a set of a certain size is simple and can be done with some success. Only one element is quantified in each set, but all components are ranked. A certain amount of quantified items and a mean for each rank are produced after processing enough sets. Despite ranking mistakes, the average of these means provides a fair assessment of the population mean. The technique of ranked set sampling (RSS) was first suggested by McIntyre (1958) as a cost efficient different from simple random sampling (SRS) for those conditions where measurements are difficult or costly to find but (judgment) ranking of units according to the variable of the study say Y, is quite easy and inexpensive. It is acknowledged that the estimation of the population means using RSS is more efficient than that the one attained using SRS.

McIntyre (1958) and Takahasi and Wakimoto (1968) anticipated a perfect ranking of the elements, that is, there are no inaccuracies in ranking the elements. However, in most

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circumstances, the ranking could not be done perfectly. Dell and Clutter (1972) demonstrated that the sample mean under the RSS is an unbiased estimator of the population mean, although there are errors in ranking or not it remains an unbiased estimator**.** Stokes measured the situation where the ranking is done based on a concomitant (auxiliary) variable X as an alternative visual judgment of expertise. For example, at a risky unwanted, or waste location, it can be possible to rank pollutant levels according to the degree of waste. Also, it is used in estimating the size of trees in a forest by ranking their widths. In these examples, the variable of interest (pollutant level or size of the three) would be highly associated with the associated (auxiliary) variable.

2. The procedure of the ranked set sampling

To generate ranked sets, we must divide the nominated first phase sample into equal-sized sets. To develop an RSS design, we must therefore choose a set size that is obviously small, about three or four, in order to minimize ranking error. Call this set size as m, where m is the number of sample units assigned to each set. Then, we proceed as follows.

Step 1: Casually select n=m*r sampling items from the population.
Step 2: Assign the n selected items as randomly as likely into m sets, each of size m.
Step 3: Up till now without knowing any values for the variable of concern, rank the units within each set based on an observation of comparative values for this variable. This may be done through personal judgement or by measuring a covariate that is associated with the variable of interest.
Step 4: choose a sample for authentic study by taking the smallest ranked unit in the first set, and then the second smallest ranked unit in the second set, and so on until the largest ranked unit is chosen in the last set.
Step 5: Repeat steps 1 to 4 for r cycles until the required sample size, n=m*r, is obtained for analysis. Consider the set size m=3 with r=4 cycles as an example.

3. Background theory of estimators

Let $Y_{(i)j}$ be the independent random variables all having the same cumulative distribution function F(x). The estimated mean of the RSS of the population mean $\bar{Y}$ is

$$\bar{y}_{RSS} = \frac{1}{m*r}\sum_{j=1}^{r}\sum_{i=1}^{m} Y_{(i)j} = l_{11}. \tag{1}$$

According to Chen, Z. (2001), $\bar{y}_{RSS}$ is an unbiased estimator.
The traditional Classical Ratio estimator (2009) for the population mean μ under simple random sampling is:

$$\bar{y}_{SRS} = \bar{r}\bar{X} = l_{12} \tag{2}$$

where, $\bar{y}_{SRS} = \frac{\bar{y}}{\bar{x}}\bar{X}$.

The Singh and Tailor (2003) type ratio estimators of population mean μ that is converted in RSS expressions by Khan *et al*. (2016) using the unbiased type of Hartley and Ross (HR) in (1954).

$$\bar{y}_{Sin(RSS)} = \bar{r}_{(i)}\,\bar{X} = l_{13} \tag{3}$$

The Sisodia and Dwivedi (1981) estimator existing in Simple Random Sampling is converted in RSS.

$$\bar{y}_{SD(RSS)} = r_1^s \bar{X}' = l_{14} \quad (4)$$

The existing estimator of Kadilar and Cingi (2009) proposed by Khan *et al* (2016) in unbiased Hartley and Ross (HR) type form under RSS is

$$\bar{y}_{kc(RSS)} = r_2^s \bar{X}'' = l_{15} \quad (5)$$

The Singh and Taylor (1999) ratio type estimator existing in Simple Random Sampling (SRS) and it is converted in Ranked Set Sampling (RSS).

$$\bar{y}_{ST(RSS)} = \frac{\bar{y}_{(i)}}{(\bar{x}_{(i)}+\rho)}(\bar{X}+\rho) = l_{16} \quad (6)$$

The Bhal and Tuteja (1991) exponential ratio type estimator converted into RSS.

$$\bar{y}_{BT(RSS)} = \bar{y}_{[i]} \exp\left[\frac{(\bar{X}-\bar{x}_{(i)})}{(\bar{X}+\bar{x}_{(i)})}\right] = l_{17}{}^{e} \quad (7)$$

The Kadilar and Cingi (2009) estimator is converted in RSS for making comparison with the proposed exponential ratio type estimator.

$$\bar{y}_{KC(RSS)} = \bar{y}_{[i]} \exp\left[\frac{(a\,\bar{X}+b)-(a\bar{x}_{(i)}+b)}{(a\,\bar{X}+b)+(a\bar{x}_{(i)}+b)}\right] = l_{18}{}^{e} \quad (8)$$

3.1 Some useful notations and expectations

The unbiased estimator of the population mean μ is given by,

$$\bar{y}_{RSS} = \frac{\sum_{j=1}^{r}\sum_{i=1}^{m} Y_{(i)j}}{mr} \quad (9)$$

$$\bar{x}_{RSS} = \frac{1}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m} x_{(i)j} \quad (10)$$

$$\bar{r}_{(i)} = \frac{1}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m} r_{(i)j} \quad (11)$$

$$r_1^s = \frac{\bar{y}_{[i]}}{\bar{x}'_{(i)}} \quad (12)$$

where, $\bar{x}'_{(i)} = (\bar{x}_{(i)} + C_x)$, $\bar{X}' = \bar{X} + C_x$

$$r_2^s = \frac{\bar{y}_{[i]}}{\bar{x}''_{(i)}} \quad (13)$$

where, $\bar{x}''_{(i)} = (\bar{x}_{(i)}C_x + \rho)$ and $\bar{X}'' = \bar{X}C_x + \rho$.

The weighted terms occurred due to ranking the objects and because of the ordered distribution in the ranked set sampling are given by

(i) $w_{y(i)}^2 = \frac{1}{(m^2 r)\bar{Y}^2} \sum_{i=1}^{m} \left(\mu_{y[i]} - \bar{Y}\right)^2$

(ii) $w_{x(i)}^2 = \frac{1}{(m^2 r) * \bar{X}^2} \sum_{i=1}^{m} (\mu_{x(i)} - \bar{X})^2$

(iii) $w_{yx(i)} = \frac{\sum_{i=1}^{m} (\mu_{y[i]} - \bar{Y})(\mu_{x(i)} - \bar{X})}{(m^2 r)\ \bar{Y}\bar{X}}$

To obtain Bias and MSE of the estimators, the useful expectations under RSS are:

$\bar{y}_{[i]} = \bar{y}(1 + e_0), \bar{x}_{(i)} = \bar{X}(1 + e_1)$.

E (e_0) = E (e_1) = 0, E (e_0^2) = γc_y^2-$w_{y(i)}^2$, E (e_1^2) = γc_x^2-$w_{x(i)}^2$.

E $(e_0 e_1)$ =$\gamma\ c_{yx} - \ w$, where $\gamma = \frac{1}{m * r}$.

In the next section the bias and mean square errors of the above estimators are discussed.

3.2 Bias and mean square error expressions

The variance of l_{11} by using the above expectations under RSS is obtained as:

$$V\ (l_{11}) = \bar{Y}^2 \left(\gamma c_y^2 - w_{y[i]}^2\right) \tag{14}$$

The mean square error (MSE) and bias of l_{12} up to 1st degree and 2nd degree approximation respectively are given as:

$$MSE_{\bar{y}_{SRS}} = \alpha \bar{Y}^2 \{c_y^2 + c_x^2 - 2c_{yx}\} \tag{15}$$

where, $\alpha = 1 - \frac{f}{n}$.

$$Bias\ in\ \bar{y}_{SRS} = \alpha \bar{Y} \{c_x^2 - c_{yx}\} \tag{16}$$

The mean square error (MSE) and bias of l_{13} up to 1st degree and 2nd degree approximation respectively, under RSS design are given as

$$MSE_{l_{13}} = \bar{Y}^2 [\left(\gamma c_y^2 - w_{y[i]}^2\right) + \left(\gamma c_x^2 - w_{x(i)}^2\right) - 2\left(\gamma c_{yx} - w_{yx(i)}\right)] \tag{17}$$

$$\text{Bias in } l_{13} = -\frac{n(N-1)}{N(n-1)} (\bar{y}_{[i]} - \bar{r}_{(i)} \bar{x}_{(i)}) \tag{18}$$

The MSE and Bias of l_{14} turned into RSS design using expectations in Section 3.1 respectively are

$$MSE_{l_{14}} = \bar{Y}^2 \{\left(\gamma c_y^2 - w_{y[i]}^2\right) + g^2 \left(\gamma c_x^2 - w_{x(i)}^2\right) - 2g\left(\gamma c_{yx} - \ w_{yx(i)}\right) \tag{19}$$

where, $g = \frac{\bar{X}}{\bar{X} + C_x}$

$$\text{Bias in } l_{14} = g\bar{Y}[g(\ \gamma c_x^2 - w_{x(i)}^2) - (\gamma c_{yx} - \ w_{yx(i)})] \tag{20}$$

The MSE and Bias of l_{15} turned into RSS design using expectations as well are

$$MSE_{l_{15}} = \bar{Y}^2\{\left(\gamma c_y^2 - w_{y[i]}^2\right) + (\tfrac{\bar{X}C_x}{\bar{X}C_x+\rho})^2\left(\gamma c_x^2 - w_{x(i)}^2\right) - 2(\tfrac{\bar{X}C_x}{\bar{X}C_x+\rho})\left(\gamma c_{yx} - w_{yx(i)}\right) \quad (21)$$

and

$$\text{Bias in } l_{15} = -\frac{n(N-1)}{N(n-1)}\left(\bar{y}_{(i)} - r_2^s x''_{(i)}\right). \quad (22)$$

The MSE and Bias of l_{16} turned into RSS design to make comparison with the proposed are given by

$$MSE_{l_{16}} = \bar{Y}^2\{\left(\gamma c_y^2 - w_{y[i]}^2\right) + (\tfrac{\bar{X}}{\bar{X}+C_x})^2\left(\gamma c_x^2 - w_{x(i)}^2\right) - 2(\tfrac{\bar{X}}{\bar{X}+C_x})\left(\gamma c_{yx} - w_{yx(i)}\right)\} \quad (23)$$

$$\text{Bias in } l_{16} = \frac{\bar{X}}{\bar{X}+\rho}\,\bar{Y}[\frac{\bar{X}}{\bar{X}+\rho}(\,\gamma c_x^2 - w_{x(i)}^2) - (\gamma c_{yx} - w_{yx(i)})] \quad (24)$$

The MSE and Bias of an exponential ratio type estimator of ${l_{17}}^e$ respectively are given as

$$MSE_{{l_{17}}^e} = \bar{Y}^2\left[\left(\gamma c_y^2 - w_{y[i]}^2\right) + \frac{1}{4}\left(\gamma c_x^2 - w_{x(i)}^2\right) - \left(\gamma c_{yx} - w_{yx(i)}\right)\right] \quad (25)$$

$$\text{Bias in } {l_{17}}^e = \bar{Y}[-\frac{\gamma C_{yx} - W_{yx(i)}}{2} + \frac{3\left(\gamma c_x^2 - w_{x(i)}^2\right)}{8}] \quad (26)$$

The MSE and Bias of an exponential ratio type estimator of ${l_{18}}^e$ respectively are given as

$$MSE_{{l_{18}}^e} = \bar{Y}^2[\left(\gamma c_y^2 - w_{y[i]}^2\right) + \delta\left(\gamma c_x^2 - w_{x(i)}^2\right)(\delta - 2H_{yx}] \quad (27)$$

$$\text{Bias in } {l_{18}}^e = \bar{Y}\{-\delta\left(\gamma c_{yx} - w_{yx(i)}\right) + (1+\delta^2)\left(\gamma c_x^2 - w_{x(i)}^2\right)\} \quad (28)$$

where, $H_{yx} = \rho_{yx}\frac{(\gamma c_y^2 - w_{y[i]}^2)^{1/2}}{(\gamma c_x^2 - w_{x(i)}^2)^{1/2}}$, y≠x, $\rho_{yx} = \frac{\gamma c_{yx} - w_{yx(i)}}{\left\{\left(\gamma c_y^2 - w_{y[i]}^2\right) * (\gamma c_x^2 - w_{x(i)}^2)\right\}^{1/2}}$,
and $\delta = \frac{a\bar{X}}{2(a\,\bar{X}+b)}$.

In the next section new family of estimators is obtained and shows some estimators discussed in section 3 are special case of it.

4. Proposed class of estimators

The proposed estimator is given by

$$\bar{y}_{RSS}^P = \bar{y}\exp\,[\theta\{\frac{\bar{X}^{\frac{1}{h}} - \bar{x}^{\frac{1}{h}}_{(i)}}{\bar{X}^{\frac{1}{h}} + (a-1)\bar{x}^{\frac{1}{h}}_{(i)}}\}] \quad (29)$$

where θ, a, h are some auxiliary constants. These constants are used to make a family of estimators by giving some numerical values. By converting above equation in to e_i's, it becomes

$$\bar{y}_{RSS}^P = \bar{Y}(1+e_0)\exp\,[\theta\left\{\frac{\bar{X}^{\frac{1}{h}}(1-(1+e_1)^{\frac{1}{h}})}{\bar{X}^{\frac{1}{h}}(1+(a-1)(1+e_1)^{\frac{1}{h}})}\right\}] \quad (30)$$

Expanding up to 1st order and after simplifications it becomes

$$\bar{y}_{RSS}^{P} = \bar{Y}(1+e_0)\exp[-\frac{t}{h}e_1 + \frac{t}{h^2}{e_1}^2 - \frac{t}{ah}{e_1}^2] \tag{31}$$

where, $t = \frac{\theta}{a}$.

$$\bar{y}_{RSS}^{P} = \bar{Y} + \bar{Y}e_0 - \bar{Y}\frac{t}{h}e_1 \tag{32}$$

Taking expectation on both sides of equation (31)

$$E(\bar{y}_{RSS}^{P} - \bar{Y}) = \bar{Y}E(e_0) - \bar{Y}\frac{t}{h}E(e_1)$$

$$E(\bar{y}_{RSS}^{P}) = \bar{Y}\ . \tag{33}$$

Hence proved from above result that $\bar{y}_{RSS}^{P}$ is an unbiased estimator up to 1st order of expansion. For MSE taking square and applying expectation on both sides of (31)

$$E(\bar{y}_{RSS}^{P} - \bar{Y})^2 = \bar{Y}^2 E(e_0 - \frac{t}{h}e_1)^2$$

$$MSE_{(\bar{y}_{RSS}^{P})} = \bar{Y}^2\left[\left(\gamma c_y^2 - w_{y[i]}^2\right) + \frac{t^2}{h^2}\left(\gamma c_x^2 - w_{x(i)}^2\right) - 2\frac{t}{h}\left(\gamma c_{yx} - w_{yx(i)}\right)\right] \tag{34}$$

To obtaining bias expand (29) up to 2nd order after simplification.

$$\bar{y}_{RSS}^{P} - \bar{Y} = \bar{Y}e_0 - \bar{Y}\theta(2-a)\frac{1}{h}e_1 - \bar{Y}(\theta)\frac{1}{h^2}{e_1}^2 - \bar{Y}\theta(2-a)\frac{1}{h}(e_0e_1) \tag{35}$$

After applying expectation on both sides of (34), we get

$$Bias\ in\ \bar{y}_{RSS}^{P} = -\bar{Y}(\theta)\frac{1}{h}[\frac{1}{h}\left(\gamma c_x^2 - w_{x(i)}^2\right) + (2-a)\left(\gamma c_{yx} - w_{yx(i)}\right) \tag{36}$$

In the next session optimal mean square error is obtained to meet the objective of minimum MSE.

4.1 Optimum mean square error

For obtaining minimum MSE it is required to partially differentiate (33) w.r.t the t, after that put equals to zero, we obtain

$$\frac{\partial(MSE_{(\bar{y}_{RSS}^{P})})}{\partial t} = \bar{Y}^2\frac{\partial}{\partial t}[\left(\gamma c_y^2 - w_{y[i]}^2\right) + \frac{t^2}{h^2}\left(\gamma c_x^2 - w_{x(i)}^2\right) - 2\frac{t}{h}\left(\gamma c_{yx} - w_{yx(i)}\right)] \tag{37}$$

$$t = h\frac{\left(\gamma c_{yx} - w_{yx(i)}\right)}{\left(\gamma c_x^2 - w_{x(i)}^2\right)}\ . \tag{38}$$

After substituting the (38) in to (34), it becomes

$$MSE_{(\bar{y}_{RSS}^{P})opt} = \bar{Y}^2\left(\gamma c_y^2 - w_{y[i]}^2\right)[1-\rho^2] \tag{39}$$

$$\text{where, } \rho = \sqrt{\frac{\left(\gamma c_{yx} - w_{yx(i)}\right)^2}{\left(\gamma c_x^2 - w_{x(i)}^2\right)\left(\gamma c_y^2 - w_{y[i]}^2\right)}} \tag{40}$$

4.2 Family of proposed estimator

A family can be determined by giving some numeric values to auxiliary constants.

Remark 1: when $\theta = 1$, h=1, a=2 then the proposed estimator turns in to existing estimator ${l_{17}}^{e}$ mentioned in section 3.

$$\bar{y}_{RSS}^{P} = \bar{y}\exp\left[\frac{\bar{X} - \bar{x}_{(i)}}{\bar{X} + \bar{x}_{(i)}}\right] \tag{41}$$

By substituting these values in equation (33),

$$MSE_{(\bar{y}_{RSS}^{P})} = \bar{Y}^{2}\left[\left(\gamma c_{y}^{2} - w_{y[i]}^{2}\right) + \frac{1}{4}\left(\gamma c_{x}^{2} - w_{x(i)}^{2}\right) - \left(\gamma c_{yx} - w_{yx(i)}\right)\right]. \tag{42}$$

Remark 2: when $\theta = 1$, h = 1, a = 1 then the proposed estimator turns in to simple ratio estimator mentioned in equation (3) as l_{13}. By substituting these values in equation (28)

$$\bar{y}_{RSS}^{P} = \bar{y}\exp\left[\frac{\bar{X} - \bar{x}_{(i)}}{\bar{X}}\right] \tag{43}$$

$$MSE_{(\bar{y}_{RSS}^{P})} = \bar{Y}^{2}\left[\left(\gamma c_{y}^{2} - w_{y[i]}^{2}\right) + \left(\gamma c_{x}^{2} - w_{x(i)}^{2}\right) - 2\left(\gamma c_{yx} - w_{yx(i)}\right)\right] \tag{44}$$

5. Efficiency comparison

The $\bar{y}_{RSS}^{P}$ is more efficient than existing $l_{13}, l_{14}, l_{15}, l_{16}, {l_{17}}^{e}$ and ${l_{18}}^{e}$ if the following conditions hold respectively.

i- $\left(\gamma c_{x}^{2} - w_{x(i)}^{2}\right)\left(1 - \frac{t^{2}}{h^{2}}\right) - 2\left(\gamma c_{yx} - w_{yx(i)}\right)\left(1 - \frac{t}{h}\right) \geq 0$

ii- $\left(g^{2} - \frac{t^{2}}{h^{2}}\right)\left(\gamma c_{x}^{2} - w_{x(i)}^{2}\right) - 2\left(g - \frac{t}{h}\right)\left(\gamma C_{yx} - W_{yx(i)}\right) \geq 0$

iii- $\left(\left(\frac{\bar{X}C_{x}}{\bar{X}C_{x}+\rho}\right)^{2} - \frac{t^{2}}{h^{2}}\right)\left(\gamma c_{x}^{2} - w_{x(i)}^{2}\right) - 2\left\{\left(\frac{\bar{X}C_{x}}{\bar{X}C_{x}+\rho}\right) - \frac{t}{h}\right\}\left(\gamma c_{yx} - w_{yx(i)}\right) \geq 0$

iv- $\left(l^{2} - \frac{t^{2}}{h^{2}}\right)\left(\gamma c_{x}^{2} - w_{x(i)}^{2}\right) - 2\left(l - \frac{t}{h}\right)\left(\gamma c_{yx} - w_{yx(i)}\right) \geq 0$, where $l = \frac{\bar{X}}{\bar{X}+\rho}$

v- $\left(\frac{1}{4} - \frac{t^{2}}{h^{2}}\right)\left(\gamma c_{x}^{2} - w_{x(i)}^{2}\right) - 2\left(1 - \frac{t}{h}\right)\left(\gamma c_{yx} - w_{yx(i)}\right) \geq 0$

vi- $\delta\left(\gamma c_{x}^{2} - w_{x(i)}^{2}\right)\left(\delta - 2H_{yx} - \frac{t^{2}}{h^{2}}\right) + 2\frac{t}{h}\left(\gamma c_{yx} - w_{yx(i)}\right) \geq 0$

The estimator $\bar{y}_{RSS}$ is more efficient than $\bar{y}_{SRS}$ if

$$\frac{1}{N}\left(c_{y}^{2} + c_{x}^{2} - 2c_{yx}\right) \leq \left(w_{y[i]}^{2} + w_{x(i)}^{2} - 2w_{yx(i)}\right).$$

6. Numerical illustration

To observe the performances of the estimators we use the following three real life data sets. The descriptions are given in the following tables.

Table 1: Population sources with characteristics.

Parameters	Population 1		Population 2		Population 3	
	Source: U.S. Environmental Protection Agency, 1991		Source: William.G,Cochran Sampling techniques,1909		Source: Applied Linear Statistical Models 2004, pg-1348, dataset 1	
	X Weight	Y MPG	X Weekly family income	Y Weekly expenditure	X Average no. of beds	Y Average no. of nurses
	N=83		N= 36		N=113	
$\bar{Y}$	33.8420		27.4909		173.2480	
$\bar{X}$	31.0494		72.5454		252.1680	
ρ	-0.9128		0.2521		0.9155	
c_y	0.2959		0.3629		0.8003	
c_x	0.2617		0.1436		0.7613	
c_{yx}	-0.0707		0.01314		0.5578	
V_y	100.2432		99.5226		19223.21	
V_x	66.02843		108.4904		36859.2	

Table 2: Different samples and MSE's results for population 1.

MSE \ n	9	12	15	30	12	16	20	40	15	20	25	50
$MSE_{l_{11}}$	5.96	0.92	3.59	1.76	3.64	2.72	2.17	1.09	2.50	1.87	1.48	0.75
$MSE_{l_{12}}$	11.14	2.51	6.68	3.34	8.35	6.26	5.01	2.50	6.68	5.01	4.01	2.00
$MSE_{l_{13}}$	1.82	8.49	1.08	0.54	1.35	1.00	0.81	0.40	1.06	0.80	0.64	0.31
$MSE_{l_{14}}$	1.82	4.53	1.08	0.54	1.34	1.00	0.80	0.40	1.06	0.79	0.64	0.31
$MSE_{l_{15}}$	2.20	1.57	1.29	0.64	1.36	1.01	0.80	0.41	0.98	0.72	0.57	0.28
$MSE_{l_{16}}$	1.83	1.37	1.09	0.55	1.36	1.02	0.82	0.41	1.08	0.81	0.65	0.32
$MSE_{l_{17}^e}$	2.78	0.15	1.66	0.82	1.82	1.36	1.09	0.55	1.33	0.99	0.79	0.39
$MSE_{l_{18}^e}$	3.97	2.90	2.38	1.17	2.49	1.86	1.48	0.74	1.76	1.31	1.04	0.53
$MSE_{(\bar{y}_{RSS}^P)opt}$	0.99	0.72	0.59	0.29	0.61	0.45	0.36	0.18	0.41	0.31	0.25	0.12

Table 3: Different samples and MSE's results for population 2.

MSE \ n	12	15	9	12	16	20	15	20	25
$MSE_{l_{11}}$	4.54	3.62	6.21	3.70	2.89	2.24	2.60	1.89	1.53
$MSE_{l_{12}}$	8.29	6.64	11.06	8.29	6.22	4.98	6.63	4.98	3.98
$MSE_{l_{13}}$	6.68	5.34	8.98	6.38	4.83	3.83	4.98	3.70	2.97
$MSE_{l_{14}}$	6.68	5.34	8.98	6.37	4.82	3.83	4.97	3.70	2.97
$MSE_{l_{15}}$	4.30	3.43	5.88	3.52	2.74	2.13	2.48	1.80	1.46
$MSE_{l_{16}}$	6.67	5.34	8.96	6.37	4.83	3.83	4.97	3.69	2.96
$MSE_{l_{17}^e}$	5.45	4.36	7.38	4.92	3.77	2.96	3.71	2.74	2.20
$MSE_{l_{18}^e}$	4.77	3.81	6.50	4.03	3.12	2.43	2.90	2.12	1.71
$MSE_{(\bar{y}_{RSS}^P)opt}$	4.25	3.39	5.82	3.47	2.70	2.10	2.44	1.77	1.44

Table 4: Different samples and MSE's results for population 3.

MSE \ n	9	12	15	30	12	16	20	40	15	20	25	50
$MSE_{l_{11}}$	1216	922.2	735.5	369.4	751.4	554.9	451.2	228.2	520.1	389.5	309.3	154.7
$MSE_{l_{12}}$	2136	1601	1281	640.8	1602	1202	961.2	480.6	1281	961.2	768.9	384.5
$MSE_{l_{13}}$	345.6	259.2	207.6	103.7	258.8	194.0	155.3	77.7	207.0	155.1	124.0	62.0
$MSE_{l_{14}}$	344.9	258.6	207.1	103.5	258.2	193.6	154.9	77.5	206.5	154.8	123.7	61.9
$MSE_{l_{15}}$	427.4	323.5	260.7	130.0	264.4	194.0	158.8	81.1	186.4	137.5	108.3	54.3
$MSE_{l_{16}}$	344.8	258.5	207.1	103.5	258.1	193.5	154.9	77.5	206.4	154.7	123.7	61.9
$MSE_{l_{17}{}^e}$	503.2	379.9	304.4	152.3	333.7	247.4	200.3	101.2	246.1	183.6	145.9	73.0
$MSE_{l_{18}{}^e}$	352.4	264.8	212.6	106.1	252.1	188.1	151.3	76.1	84.2	146.7	116.9	58.5
$MSE_{(\bar{y}_{RSS}^{P})opt}$	196.8	149.3	119.0	59.7	121.6	89.8	73.0	36.9	196.5	62.0	50.1	25.0

7. Conclusion

Using all three given populations, the proposed estimator is more efficient due to its minimum MSE. It is also concluded that the proposed exponentially ratio type estimator is most preferable over its competitive estimators under RSS. Thus, exponentially ratio type estimator gives better results than other ratio type under RSS design. In this study, ratio type existing estimators by Bhal and Tuteja (1991), Kadilar and Cingi (2009), Singh and Tailor (2003), Khan *et al* (2016) unbiased HR ratio type estimators and Sisodia and divedi were experienced with high positive, high negative and moderate correlations. It is illustrated by using these three correlation levels that MSE of proposed estimator is minimum than all other existing MSE's under RSS. Also, it is revealed by above numerical results that RSS design gives minimum MSE's than SRS design, so that RSS design is better than SRS design. At the end, it can be said that the proposed estimator can give a better practice for more studies.

Appendix A

In this section different sample sets with some useful information are given.

Table A.1: Different 12 sample sets of population 1 with characteristics.

n	9	12	15	30	12	16	20	40	15	20	25	50
m	3	3	3	3	4	4	4	4	5	5	5	5
r	3	4	5	10	3	4	5	10	3	4	5	10
$w_{x(i)}^2$	0.0037	0.0028	0.0022	0.0011	0.0034	0.0025	0.0025	0.0011	0.0030	0.0022	0.0018	0.0009
$w_{y[i]}^2$	0.0045	0.0035	0.0027	0.0014	0.0041	0.0031	0.0020	0.0012	0.0036	0.0027	0.0022	0.0011
w_{yx}	0.0041	0.0031	0.0024	0.0012	0.0038	0.0028	0.0022	0.0011	0.0033	0.0025	0.0020	0.0010

TABLE A.2. Different 12 sample sets of population 2 with characteristics.

n	**12**	**15**	**9**	**12**	**16**	**20**	**15**	**20**	**25**
m	3	3	3	4	4	4	5	5	5
r	4	5	3	3	4	5	3	4	5
$w^2_{x(i)}$	0.0009	0.0007	0.0012	0.0011	0.0008	0.0006	0.0009	0.0007	0.0006
$w^2_{y[i]}$	0.0050	0.0040	0.0064	0.0061	0.0044	0.0036	0.0053	0.0041	0.0033
w_{yx}	0.0021	0.0017	0.0028	0.0025	0.0019	0.0015	0.0022	0.0017	0.0013

TABLE A.3. Different 12 sample sets of population 2 with characteristics.

n	**9**	**12**	**15**	**30**	**12**	**16**	**20**	**40**	**15**	**20**	**25**	**50**
m	3	3	3	3	4	4	4	4	5	5	5	5
r	3	4	5	10	3	4	5	10	3	4	5	10
$w^2_{x(i)}$	0.0274	0.0202	0.0160	0.0081	0.0255	0.0193	0.0153	0.0076	0.0230	0.0172	0.0138	0.0069
$w^2_{y[i]}$	0.0307	0.0226	0.0000	0.0090	0.0283	0.0215	0.0170	0.0084	0.0254	0.0190	0.0153	0.0077
w_{yx}	0.0289	0.0214	0.0000	0.0086	0.0269	0.0204	0.0161	0.0080	0.0241	0.0181	0.0145	0.0073

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