Journal of Statistics | *ISSN-P: 1684-8403*
Volume 27, 2023, pp. 1-22 | *ISSN-E: 2957-8981*

Estimation of Population Mean using Supplementary Information in the Presence of Non-response

*Mustansar Aatizaz [*1], Javid Shabbir [2] and Ghazanfar Aatizaz [3]*

Abstract

Non-Response is the biggest problem of survey sampling, and it decreases the reliability of the estimators. To overcome this problem, we proposed a more efficient exponential-type estimator using two supplementary information's under the conditions of either nonresponse in both study and auxiliary variables or just in the in-study variable. In addition, some conditions are obtained under both situations which showed that our proposed estimator performs more efficiently. Finally, we use some real-life datasets to prove that the proposed estimator performs more accurately in both situations as compared to the existing estimators.

Keywords

Auxiliary variable, Bias, Mean square error, Exponential type estimator, Relative efficiency, Non-response.

1. Introduction

In survey sample, non-response is an inescapable fact, and it has a massive effect on results. It is observed from the literature that almost every human survey population is affected by the non-response. Non-response has been classified mainly as:(i) unit non-response: it means that respondent does not show any activity or refuse to give the response to all the questions. It has many causes like respondent not interested to taking part in the survey or respondents not being at home at the time of the survey. (ii) In item non-response, the respondent doesn't refuse the whole survey but does not give answer to some question. It can occur due to variety of reasons, some of which could be, the respondent feeling hesitant about certain private matters like sex, drug abuse, etc. Several researchers used the concept of Hansen and Hawatz and proposed different estimators to deal with this problem e.g., Azeem (2014), Singh and Kumar (2009), Singh and Kumar (2010), Singh and Kumar (2014), Khare and Rehman (2014), Yaqub *et al.* (2017), Azeem and Hanif (2017), Kuha *et al.* (2018) and Sharma *et al.* (2022).

Consider a finite population $P = P_1, P_2, P_3, \dots, P_N$ of size N. A sample of size n is drown using simple random sampling with out replacement (SRSWOR). Let y_i and (x_i, z_i) be the values of the study variable (y) and the auxiliary variables (x, z), respectively. From

[*] Corresponding author
[1] Department of Statistics, Allama Iqbal Open University, Islamabad-Pakistan.
Email: mustansar.aatizaz@aiou.edu.pk
[2] Department of Statistics, Quaid-i-Azam University, Islamabad-Pakistan.
Email: js@qau.edu.pk
[3] Core Statistics Department, State Bank of Pakistan, Karachi-Pakistan.
Email: gazanfar.aatizaz@sbp.org.pk

this sample n_1 units are responds and n_2= (n - n_1) units do not respond. A sub sample of size n_{2_r} is taken from the non-respondent sample n_2 of size $r = \frac{n_2}{k}$ where k > 1. Thus, the combine information (n_1+r) is used instead of n to estimate the population mean $\bar{Y}$ following the technique of Hansen and Hurwitz (1946).

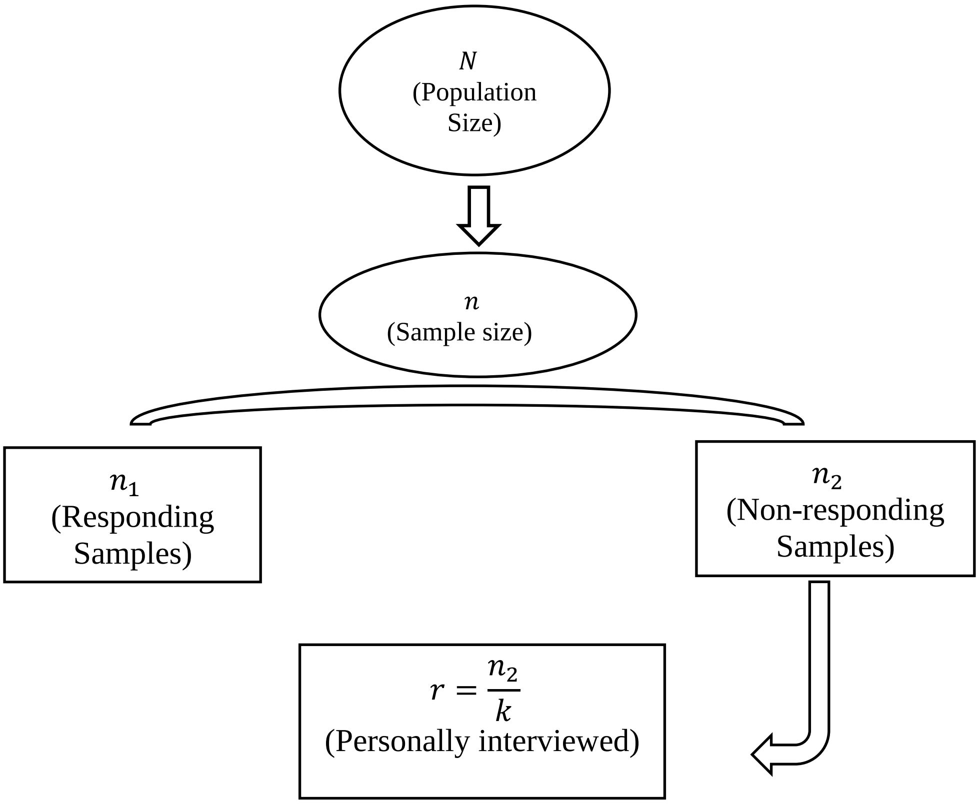

Figure 1: Hansen and Hurwitz (1946) Model.

In the estimation procedure proposed by Hansen and Hurwitz (1946), it is assumed that population is divided into two strata of respondent and non-respondent with population sizes N_1 and N_2. Let $\bar{Y} = \sum_{i=1}^{N}\frac{y_i}{N}$, $\bar{Y}_1 = \sum_{i=1}^{N_1}\frac{y_i}{N_1}$ and $\bar{Y}_2 = \sum_{i=1}^{N_2}\frac{y_i}{N_2}$ are the means of the total population, respondent and non-respondent groups respectively. Similarly, Let $S_y^2 = \sum_{i=1}^{N}\frac{(y_i-\bar{Y})^2}{N-1}$, $S_{y(1)}^2 = \sum_{i=1}^{N_1}\frac{(y_i-\bar{Y})^2}{N_1-1}$ and $S_{y(2)}^2 = \sum_{i=1}^{N_2}\frac{(y_i-\bar{Y})^2}{N_2-1}$ are the variances of total population, respondents and non-respondent groups respectively. Further, let $\bar{y} = \sum_{i=1}^{n}\frac{y_i}{n}$, $\bar{y}_1 = \sum_{i=1}^{n_1}\frac{y_i}{n_1}$ and $\bar{y}_2 = \sum_{i=1}^{n_2}\frac{y_i}{n_2}$ be the means of complete sample units n, respondents group n_1 and non-respondent group n_2 respectively. Let $\bar{y}_{r(2)} = \sum_{i=1}^{r}\frac{y_i}{r}$ is the mean of personally interviewed units r.

In Hansen and Hurwitz (1946) estimation procedure, it is assumed that population is divided into two strata of respondent and non-respondent with population sizes N_1 and N_2. Let $\bar{Y} = \sum_{i=1}^{N}\frac{y_i}{N}$, $\bar{Y}_1 = \sum_{i=1}^{N_1}\frac{y_i}{N_1}$ and $\bar{Y}_2 = \sum_{i=1}^{N_2}\frac{y_i}{N_2}$ are the means of the total population, respondent and non-respondent groups respectively. Similarly, Let $S_y^2 = \sum_{i=1}^{N}\frac{(y_i-\bar{Y})^2}{N-1}$, $S_{y(1)}^2 = \sum_{i=1}^{N_1}\frac{(y_i-\bar{Y})^2}{N_1-1}$ and $S_{y(2)}^2 = \sum_{i=1}^{N_2}\frac{(y_i-\bar{Y})^2}{N_2-1}$ are the variances of total population, respondents and non-respondent groups respectively. Further, let $\bar{y}=\sum_{i=1}^{n}\frac{y_i}{n}$, $\bar{y}_1=\sum_{i=1}^{n_1}\frac{y_i}{n_1}$ and $\bar{y}_2=\sum_{i=1}^{n_2}\frac{y_i}{n_2}$ be the means of complete sample units n, respondents group n_1 and non-respondent group n_2 respectively. Let $\bar{y}_{r(2)}=\sum_{i=1}^{r}\frac{y_i}{r}$ is the mean of personally interviewed units r.

Hansen and Hurwitz (1946) proposed estimator is given as:

$$\bar{Y}_H^*=h_1\bar{y}_1+h_2\bar{y}_{r(2)}, \tag{1}$$

where, $h_1=\frac{n_1}{n}$ and $h_2=\frac{n_2}{n}$ are the weights of respondents and non-respondents' groups respectively. $H_1=h_1+h_2$ which is the response rate after sub-sampling and $H_2=1-H_1$ is the reaming non-response rate. The variance of $\bar{Y}_H^*$ is given as:

$$\text{Var}\left(\hat{\bar{Y}}_H^*\right)=\bar{Y}^2\left(\theta_1C_y^2+\theta_2C_{y(2)}^2\right). \tag{2}$$

To obtain the bias and MSE of proposed and considered estimators, following error terms are used. Let

$$\xi_0^*=\frac{\bar{y}^*-\bar{Y}}{\bar{Y}},\quad \xi_1^*=\frac{\bar{x}^*-\bar{X}}{\bar{X}},\quad \xi_2^*=\frac{\bar{z}^*-\bar{Z}}{\bar{Z}},\qquad E(\xi_i^*)=0,\ i=(0,1,2),$$
$$\xi_1=\frac{\bar{x}-\bar{X}}{\bar{X}},\quad \xi_2=\frac{\bar{z}-\bar{Z}}{\bar{Z}},\quad E\left(\xi_j\right)=0,\ \ j=(1,2),$$
$$E(\xi_0^{*2})=\theta_1C_y^2+\theta_2C_{y(2)}^2=\vartheta_{200},$$
$$E(\xi_1^{*2})=\theta_1C_x^2+\theta_2C_{x(2)}^2=\vartheta_{020},$$
$$E(\xi_2^{*2})=\theta_1C_z^2+\theta_2C_{z(2)}^2=\vartheta_{002},$$
$$E(\xi_1^2)=\theta_1C_x^2=\upsilon_{020}E(\xi_2^2)=\theta_1C_z^2=\upsilon_{002},$$
$$E(\xi_0^*\xi_1)=\theta_1\rho_{yx}C_yC_x=\upsilon_{110},$$
$$E(\xi_0^*\xi_2)=\theta_1\rho_{yz}C_yC_z=\upsilon_{101},$$
$$E(\xi_1\xi_2)=\theta_1\rho_{xz}C_xC_z=\upsilon_{011},$$
$$E(\xi_0^*\xi_1^*)=\theta_1\rho_{yx}C_yC_x+\theta_2\rho_{yx(2)}C_{y(2)}C_{x(2)}=\vartheta_{110},$$
$$E(\xi_0^*\xi_2^*)=\theta_1\rho_{yz}C_yC_z+\theta_2\rho_{yz(2)}C_{y(2)}C_{z(2)}=\vartheta_{101},$$
$$E(\xi_1^*\xi_2^*)=\theta_1\rho_{xz}C_xC_z+\theta_2\rho_{xz(2)}C_{x(2)}C_{z(2)}=\vartheta_{011}$$

where $\theta_1=\left(\frac{1}{n}-\frac{1}{N}\right)$ and $\theta_2=\frac{h_2(k-1)}{n}$.

2. Adaptive Estimator for Literature

In this section, we discuss the bias and MSE of several adoptive estimators in the presence of non-response when the mean of auxiliary variables x and z known in advance. Two different situations of non-response are considered in the following subsections.

2.1 Situation-I

In Situation-I we assume that there is non-response on both the study variable y and the auxiliary variables (x,z).

(i) Ratio estimator

Ratio estimator in the presence of non-response, is given by

$$\hat{\bar{Y}}_R^* = \bar{y}^* \left(\frac{\bar{X}}{\bar{x}^*}\right). \tag{3}$$

The bias and MSE of $\hat{\bar{Y}}_R^*$ to first degree of approximation, are given as:

$$\text{Bias}\left(\hat{\bar{Y}}_R^*\right) \cong \bar{Y}(\vartheta_{020} - \vartheta_{110}),$$

and

$$\text{MSE}\left(\hat{\bar{Y}}_R^*\right) \cong \bar{Y}^2(\vartheta_{200} + \vartheta_{020} - 2\vartheta_{110}). \tag{4}$$

(ii) Product estimator

Product estimator in the presence of non-response, is given by:

$$\hat{\bar{Y}}_P^* = \bar{y}^* \left(\frac{\bar{x}^*}{\bar{X}}\right). \tag{5}$$

The bias and MSE of $\hat{\bar{Y}}_P^*$ to first degree of approximation, are given as:

$$\text{Bias}\left(\hat{\bar{Y}}_P^*\right) \cong \bar{Y}\vartheta_{110},$$

and

$$\text{MSE}\left(\hat{\bar{Y}}_P^*\right) \cong \bar{Y}^2(\vartheta_{200} + \vartheta_{020} + 2\vartheta_{110}). \tag{6}$$

(iii) Exponential type estimators

Proposed exponential type estimators under non-response are given by:

$$\hat{\bar{Y}}_{BT,R}^* = \bar{y}^* \exp\left(\frac{\bar{X} - \bar{x}^*}{\bar{X} + \bar{x}^*}\right) \tag{7}$$

$$\hat{\bar{Y}}_{BT,P}^* = \bar{y}^* \exp\left(\frac{\bar{x}^* - \bar{X}}{\bar{x}^* + \bar{X}}\right) \tag{8}$$

The bias and MSE of $\hat{\bar{Y}}_{BT,R}^*$ and $\hat{\bar{Y}}_{BT,P}^*$ are:

$$\text{Bias}\left(\hat{\bar{Y}}_{BR,R}^*\right) \cong \bar{Y}\left(\frac{3\vartheta_{020}}{8} - \frac{\vartheta_{110}}{2}\right)$$

$$\text{Bias}\left(\hat{\bar{Y}}_{BT,P}^*\right) \cong \bar{Y}\left(\frac{-\vartheta_{020}}{8} + \frac{\vartheta_{110}}{2}\right)$$

and

$$\text{MSE}\left(\hat{\bar{Y}}_{BT,R}^*\right) \cong \bar{Y}^2\left(\vartheta_{200} - \vartheta_{110} + \frac{\vartheta_{020}}{4}\right) \tag{9}$$

$$\text{MSE}\left(\hat{\bar{Y}}_{BT,P}^*\right) \cong \bar{Y}^2\left(\vartheta_{200} + \vartheta_{110} + \frac{\vartheta_{020}}{4}\right) \tag{10}$$

(iv) Traditional regression estimator

Traditional regression estimator for estimating the finite population mean in the presence of non-response, is given as:

$$\hat{\bar{Y}}_{Reg}^{*} = \bar{y}^{*} + b_{yx}^{*}(\bar{X} - \bar{x}^{*}) \quad (11)$$

where, b_{yx} is sample regression co-efficient whose population regression co-efficient is β_{yx}.

The bias and MSE of $\hat{\bar{Y}}_{Reg}^{*}$, are given by:

$$\text{Bias}\left(\hat{\bar{Y}}_{Reg}^{*}\right) \cong \beta_{yx}\left\{\theta_1\left(\frac{N}{N-2}\right)\left(\frac{\mu_{30}}{\mu_{20}} - \frac{\mu_{21}}{\mu_{11}}\right) + \theta_2\left(\frac{\mu_{30(2)}}{\mu_{20}} - \frac{\mu_{21(2)}}{\mu_{11}}\right)\right\},$$

and

$$\text{MSE}\left(\hat{\bar{Y}}_{Reg}^{*}\right) \cong \frac{\bar{Y}^2\left(\vartheta_{200}\vartheta_{020} - \vartheta_{110}^2\right)}{\vartheta_{020}}, \quad (12)$$

where,

$$\mu_{rs} = \frac{\sum_{i=1}^{N}(x_i - \bar{X})^r(y_i - \bar{Y})^s}{N} \text{ and } \mu_{rs(2)} = \frac{\sum_{i=1}^{N_2}(x_i - \bar{X})^r(y_i - \bar{Y})^s}{N_2}$$

(v) Singh et al (2009) exponential type estimator

Singh *et al* (2009), exponential type estimator in the presence of non-response, is given by:

$$\hat{\bar{Y}}_{SR}^{*} = \bar{y}^{*}\left[\gamma_1 \exp\left(\frac{\bar{X} - \bar{x}^{*}}{\bar{X} + \bar{x}^{*}}\right) + (1 - \gamma_1)\exp\left(\frac{\bar{x}^{*} - \bar{X}}{\bar{x}^{*} + \bar{X}}\right)\right] \quad (13)$$

where,

γ_1 is a constant. The bias and minimum MSE of $\hat{\bar{Y}}_{SR}^{*}$ using $\gamma_{1(opt)}=\frac{\vartheta_{020} + 2\vartheta_{110}}{2\vartheta_{020}}$, are given as:

$$\text{Bias}\left(\hat{\bar{Y}}_{SR}^{*}\right) \cong \frac{\bar{Y}}{8}\{4\vartheta_{110}(1 - 2\gamma_1) + \vartheta_{020}(4\gamma_1 - 1)\},$$

and

$$\text{MSE}_{min}\left(\hat{\bar{Y}}_{SR}^{*}\right) \cong \frac{\bar{Y}^2\left(\vartheta_{200}\vartheta_{020} - \vartheta_{110}^2\right)}{\vartheta_{020}} = \text{MSE}\left(\hat{\bar{Y}}_{Reg}^{*}\right). \quad (14)$$

(vi) *Riaz et al. (2014) estimator*

He suggested the difference estimator under non-response, is given as:

$$\hat{\bar{Y}}_{RI}^{*} = w_1\bar{y}^{*} + w_2(\bar{X} - \bar{x}^{*}) \quad (15)$$

where, w_1 and w_2 both are constants. The optimum values of w_1 and w_2, are given by

$$w_{1(opt)} \cong \frac{\vartheta_{020}}{\vartheta_{200}\vartheta_{020} - \vartheta_{110}^2 + \vartheta_{020}},$$

$$w_{2(opt)} \cong \frac{\bar{Y}\vartheta_{110}}{\bar{X}(\vartheta_{200}V_{020} - \vartheta_{110}^2 + \vartheta_{020})}.$$

The bias and minimum MSE of $\hat{\bar{Y}}_{RI}^{*}$, are given by

$$\text{Bias}\left(\hat{\bar{Y}}_{RI}^{*}\right) \cong \bar{Y}(w_1 - 1),$$

and

$$\text{MSE}_{min}\left(\hat{\bar{Y}}_{RI}^{*}\right) \cong \frac{\bar{Y}^2\left(\vartheta_{200}\vartheta_{020} - \vartheta_{110}^2\right)}{\vartheta_{200}\vartheta_{020} - \vartheta_{110}^2 + \vartheta_{020}}. \quad (16)$$

(vii) Difference estimator

Usual difference estimator using two auxiliary variables in the presence of non-response, is given as:

$$\hat{\hat{Y}}_D^* = \bar{y}^* - U_4(\bar{x}^* - \bar{X}) - U_5(\bar{z}^* - \bar{Z}), \tag{17}$$

where, U_4 and U_5 are constants. The optimum values of U_4 and U_5, are given as:

$$U_{4(opt)} \cong \frac{\bar{Y}(\vartheta_{002}\vartheta_{110} - \vartheta_{101}\vartheta_{011})}{\bar{X}(\vartheta_{200}\vartheta_{002} - \vartheta_{011}^2)},$$

and

$$U_{5(opt)} \cong \frac{\bar{Y}(\vartheta_{200}\vartheta_{101} - \vartheta_{110}\vartheta_{011})}{\bar{Z}(\vartheta_{200}\vartheta_{002} - \vartheta_{011}^2)}.$$

Minimum MSE of $\hat{\hat{Y}}_D^*$, using the optimum values of U_4 and U_5 is given as:

$$\text{MSE}_{min}\left(\hat{\hat{Y}}_D^*\right) \cong \frac{\bar{Y}^2\{-\vartheta_{200}\vartheta_{011}^2 + 2\vartheta_{110}\vartheta_{101}\vartheta_{011} + \vartheta_{020}(\vartheta_{200}\vartheta_{002} - \vartheta_{101}^2) - \vartheta_{002}\vartheta_{110}^2\}}{\vartheta_{020}\vartheta_{002} - \vartheta_{011}^2}. \tag{18}$$

(viii) Multivariate ratio estimator

Olkin (1958), proposed multivariate ratio estimator under non-response, is given by:

$$\hat{\hat{Y}}_{MR}^* = \bar{y}^*\left\{w_3\left(\frac{\bar{X}}{\bar{x}^*}\right) + w_4\left(\frac{\bar{Z}}{\bar{z}^*}\right)\right\}, \tag{19}$$

where, w_3 and w_4 both are weights with satisfying the condition w_3+ w_4=1. The bias and minimum MSE of $\hat{\hat{Y}}_{MR}^*$ using

$w_{3(opt)} \cong \frac{\vartheta_{002}+\vartheta_{110}-\vartheta_{101}-\vartheta_{011}}{\vartheta_{200}+\vartheta_{002}-2\vartheta_{011}}$ and $w_{4(opt)} = 1 - w_{3(opt)}$, are given by:

$$\text{Bias}\left(\hat{\hat{Y}}_{MR}^*\right) \cong \bar{Y}\{w_3(\vartheta_{020} - \vartheta_{002} - \vartheta_{110} + \vartheta_{101}) + \vartheta_{002} - \vartheta_{101}\},$$

and

$$\text{MSE}_{min}\left(\hat{\hat{Y}}_{MR}^*\right) \cong \frac{\bar{Y}\left\{\begin{array}{c}-\vartheta_{011}^2 + 2\vartheta_{011}(-\vartheta_{200} + \vartheta_{110} + \vartheta_{101}) + \vartheta_{020}(\vartheta_{200} + \\ \vartheta_{002} - 2\vartheta_{101}) + \vartheta_{002}(\vartheta_{200} - 2\vartheta_{110}) - (\vartheta_{110} - \vartheta_{101})^2\end{array}\right\}}{\vartheta_{020} + \vartheta_{002} - 2\vartheta_{011}}. \tag{20}$$

2.2 Situation-II

In Situation-II we assume that there is non-response on just the study variable y and the auxiliary variables (x, z) have complete information.

(i) Ratio estimator

Ratio estimator in the presence of non-response, is given by:

$$\hat{\hat{Y}}_R = \bar{y}^*\left(\frac{\bar{X}}{\bar{x}}\right). \tag{21}$$

The bias and MSE of $\hat{\hat{Y}}_R$, are given by:

$\text{Bias}\left(\hat{\hat{\bar{Y}}}_R\right) \cong barY(\upsilon_{020} - \upsilon_{110}),$
and

$$\text{MSE}\left(\hat{\hat{\bar{Y}}}_R\right) \cong \bar{Y}^2(\upsilon_{200} + \upsilon_{020} - 2\upsilon_{110}) \tag{22}$$

(ii) Product estimator

Product estimator in the presence of non-response, is given by:

$$\hat{\hat{\bar{Y}}}_P = \bar{y}^* \left(\frac{\bar{x}}{\bar{X}}\right). \tag{23}$$

The bias and MSE of $\hat{\hat{\bar{Y}}}_P$, are given by:

$\text{Bias}\left(\hat{\hat{\bar{Y}}}_P\right) \cong \bar{Y}\upsilon_{110},$
and

$$\text{MSE}\left(\hat{\hat{\bar{Y}}}_P\right) \cong \bar{Y}^2(\upsilon_{200} + \upsilon_{020} + 2\upsilon_{110}). \tag{24}$$

(iii) Bahl and Tuteja exponential type estimators

Bahl and Tuteja proposed the exponential type estimators which are given by:

$$\hat{\hat{\bar{Y}}}_{BT,R} = \bar{y}^* \exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right), \tag{25}$$

and

$$\hat{\hat{\bar{Y}}}_{BT,P} = \bar{y}^* \exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right). \tag{26}$$

Biases and MSEs of $\hat{\hat{\bar{Y}}}_{BT,R}$ and $\hat{\hat{\bar{Y}}}_{BT,P}$, are given by:

$$\text{Bias}\left(\hat{\hat{\bar{Y}}}_{BR,R}\right) \cong \bar{Y}\left(\frac{3\upsilon_{020}}{8} - \frac{\upsilon_{110}}{2}\right),$$

$$\text{Bias}\left(\hat{\hat{\bar{Y}}}_{BT,P}\right) \cong \bar{Y}\left(\frac{-\upsilon_{020}}{8} + \frac{\upsilon_{110}}{2}\right),$$

and

$$\text{MSE}\left(\hat{\hat{\bar{Y}}}_{BT,R}\right) \cong \bar{Y}^2\left(\upsilon_{200} - \upsilon_{110} + \frac{\upsilon_{020}}{4}\right), \tag{27}$$

$$\text{MSE}\left(\hat{\hat{\bar{Y}}}_{BT,P}\right) \cong \bar{Y}^2\left(\upsilon_{200} + \upsilon_{110} + \frac{\upsilon_{020}}{4}\right). \tag{28}$$

(iv) Traditional regression type estimator

Traditional regression type estimator for estimating the population mean in the presence of non-response is given below:

$$\hat{\hat{\bar{Y}}}_{Reg} = \bar{y}^* + b^*_{yx}(\bar{X} - \bar{x}), \tag{29}$$

where b_{yx} is sample regression co-efficient whose population regression co-efficient is β_{yx}.

The bias and MSE of $\hat{\bar{Y}}_{Reg}$, are given by:

$$\text{Bias}\left(\hat{\bar{Y}}_{Reg}\right) \cong \beta_{yx}\left\{\theta_1\left(\frac{N}{N-2}\right)\left(\frac{\mu_{30}}{\mu_{20}}-\frac{\mu_{21}}{\mu_{11}}\right)\right\},$$

and

$$\text{MSE}\left(\hat{\bar{Y}}_{Reg}\right) \cong \frac{\bar{Y}^2\left(\upsilon_{200}\upsilon_{020}-\upsilon_{110}^2\right)}{\upsilon_{020}}. \tag{30}$$

(v) Singh et al. exponential type estimator

Singh *et al*. (2009), exponential type estimator in the presence of non-response, is given by:

$$\hat{\bar{Y}}_{SR} = \bar{y}^*\left[\gamma_1 \exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)+(1-\gamma_1)\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)\right], \tag{31}$$

Where, γ_1 is the constant. The bias and minimum MSE of $\hat{\bar{Y}}_{SR}$ using of $\gamma_{1(opt)}=\frac{\upsilon_{020}+2\upsilon_{110}}{2\upsilon_{020}}$, is given as:

$$\text{Bias}\left(\hat{\bar{Y}}_{SR}\right) \cong \frac{\bar{Y}}{8}\{\upsilon_{200}(4\gamma_1-1)+4\upsilon_{110}(1-2\gamma_1)\},$$

and

$$\text{MSE}_{min}\left(\hat{\bar{Y}}_{SR}\right) \cong \frac{\bar{Y}^2\left(\upsilon_{200}\upsilon_{020}-\upsilon_{110}^2\right)}{\upsilon_{020}} = \text{MSE}\left(\hat{\bar{Y}}_{Reg}\right). \tag{32}$$

(vi) Riaz et al. estimator

Riaz *et al*. (2014) suggested the difference estimator using constants w_1 and w_2, is given by:

$$\hat{\bar{Y}}_{RI} = w_1\bar{y}^* + w_2(\bar{X}-\bar{x}), \tag{33}$$

where w_1 and w_2 are both constants. The optimum values of w_1 and w_2 are give by:

$$w_{1(opt)} = \frac{\upsilon_{020}}{\upsilon_{200}\upsilon_{020}-\upsilon_{110}^2+\upsilon_{020}},$$

and

$$w_{2(opt)} = \frac{\bar{Y}\upsilon_{110}}{\bar{X}(\upsilon_{200}\upsilon_{020}-\upsilon_{110}^2+\upsilon_{020})}.$$

The bias and minimum MSE of $\hat{\bar{Y}}_{RI}$, are given by:

$$\text{Bias}\left(\hat{\bar{Y}}_{RI}\right) \cong \bar{Y}(w_1-1),$$

$$\text{MSE}_{min}\left(\hat{\bar{Y}}_{RI}\right) \cong \frac{\bar{Y}^2\left(\upsilon_{200}\upsilon_{020}-\upsilon_{110}^2\right)}{\upsilon_{200}\upsilon_{020}-\upsilon_{110}^2+\upsilon_{020}} \tag{34}$$

(vii) Difference estimator

Usual difference estimator using two auxiliary variables in the presence of non-response is given by:

$$\hat{\bar{Y}}_{D} = \bar{y}^* - U_4(\bar{x}-\bar{X}) - U_5(\bar{z}-\bar{Z}), \tag{35}$$

where U_4 and U_5 are both constants. The optimum values of U_4 and U_5 are given by:

$$U_{4(opt)} = \frac{\bar{Y}(\upsilon_{002}\upsilon_{110} - \upsilon_{101}\upsilon_{011})}{\bar{X}(\upsilon_{200}\upsilon_{002} - \upsilon_{011}^2)},$$

and

$$U_{5(opt)} = \frac{\bar{Y}(\upsilon_{200}\upsilon_{101} - \upsilon_{110}\upsilon_{011})}{\bar{Z}(\upsilon_{200}\upsilon_{002} - \upsilon_{011}^2)},$$

Minimum MSE of $\hat{\hat{\bar{Y}}}_D$ using the optimum values of U_4 and U_5 is given as:

$$\text{MSE}_{min}\left(\hat{\bar{Y}}_D\right) \cong \frac{\bar{Y} - \upsilon_{200}\upsilon_{011}^2 + 2\upsilon_{110}\upsilon_{101}\upsilon_{011} + \upsilon_{020}(\upsilon_{200}\upsilon_{002} - \upsilon_{101}^2) - \upsilon_{002}\upsilon_{110}^2}{\upsilon_{020}\upsilon_{002} - \upsilon_{011}^2} \tag{36}$$

(viii) Multivariate ratio estimator

Multivariate ratio estimator under non-response is given by:

$$\hat{\hat{\bar{Y}}}_{MR} = \bar{y}^* \left\{w_3\left(\frac{\bar{X}}{\bar{x}}\right) + w_4\left(\frac{\bar{Z}}{\bar{z}}\right)\right\}, \tag{37}$$

where w_3 and w_4 are both weights with satisfying the condition $w_3 + w_4 = 1$.
The bias and minimum MSE of $\hat{\hat{\bar{Y}}}_{MR}$ using $w_{3(opt)} = \frac{\upsilon_{002}+\upsilon_{110}-\upsilon_{101}-\upsilon_{011}}{\upsilon_{200}+\upsilon_{002}-2\upsilon_{011}}$ and $w_{4(opt)} = 1 - w_{3(opt)}$ are given by:

$$\text{Bias}\left(\hat{\hat{\bar{Y}}}_{MR}\right) \cong \bar{Y}\{w_3(\upsilon_{020} - \upsilon_{002} - \upsilon_{110} + \upsilon_{101}) + \upsilon_{002} - \upsilon_{101}\},$$

and

$$\text{MSE}\left(\hat{\hat{\bar{Y}}}_{MR}\right) \cong \frac{\bar{Y}\left\{\begin{matrix} -\upsilon_{011}^2 + 2\upsilon_{011}(-\upsilon_{200} + \upsilon_{110} + \upsilon_{101}) + \upsilon_{020}(\upsilon_{200} + \\ \upsilon_{002} - 2\upsilon_{101}) + \upsilon_{002}(\upsilon_{200} - 2\upsilon_{110}) - (\upsilon_{110} - \upsilon_{101})^2 \end{matrix}\right\}}{\upsilon_{020} + \upsilon_{002} - 2\upsilon_{011}}. \tag{38}$$

2.3 Proposed estimator in Situation-I

We propose a new exponential type of estimator for estimating the finite population mean using two auxiliary variables under SRSWOR in the presence of non-response. We suggested two situations: (i) when non-response exists on both the study variable and the auxiliary variables, and (ii) when non-response exists just on the study variable. The proposed estimator under Situation-I is given by:

$$\hat{\hat{\bar{Y}}}_{Prop}^{**} = \Bigg[\frac{\bar{y}^*}{2}\left\{\exp\left(\frac{\bar{X} - \bar{x}^*}{\bar{X} + \bar{x}^*}\right)\exp\left(\frac{\bar{Z} - \bar{z}^*}{\bar{Z} + \bar{z}^*}\right) + \exp\left(\frac{\bar{x}^* - \bar{X}}{\bar{x}^* + \bar{X}}\right)\exp\left(\frac{\bar{z}^* - \bar{Z}}{\bar{z}^* + \bar{Z}}\right)\right\}$$
$$+ a_1^*(\bar{X} - \bar{x}^*) + a_2^*\bar{y}^* + a_3^*(\bar{Z} - \bar{z}^*)\Bigg]\left\{\exp\left(\frac{\bar{X} - \bar{x}^*}{\bar{X} + \bar{x}^*}\right)\exp\left(\frac{\bar{Z} - \bar{z}^*}{\bar{Z} + \bar{z}^*}\right)\right\},$$

where, $a_i^*(i = 1,2,3)$ are constants.

Expanding the above estimator in terms ξ_i's, upto the first order of approximation, we get the following expression:

$$\left(\hat{\hat{Y}}_{Prop}^{**} - \bar{Y}\right) \cong \frac{\bar{Y}}{8}[(3a_2^* + 4)(\xi_1^{*2} + \xi_2^{*2}) + (2a_2^* + 4)\xi_1^*\xi_2^* - 4(a_2^* + 1)(\xi_0^* + 1)(\xi_1^* + \xi_2^*) + 8\xi_0^* + 8(\xi_0^* + 1)a_2^*] + \frac{1}{2}(\xi_1^* + \xi_2^* - 2)(\bar{X}a_1^*\xi_1^* + \bar{Z}a_3^*\xi_2^*), \tag{39}$$

Using the equation (39), the expression of bias up to the first order of approximation, is given as:

$$\text{Bias}\left(\hat{\hat{Y}}_{Prop}^{**}\right) \cong \frac{\bar{Y}}{8}\begin{Bmatrix}(-4\vartheta_{101} + 2\vartheta_{011} + 3\vartheta_{020} + 3\vartheta_{003} - 4\vartheta_{110} + 8)a_2^* + \\ 4(\vartheta_{020} + \vartheta_{002} - \vartheta_{110} - \vartheta_{101} + \vartheta_{011})\end{Bmatrix} + \frac{1}{2}\bar{X}a_1^*(\vartheta_{020} + \vartheta_{011}) + \bar{Z}a_3^*(\vartheta_{002} + \vartheta_{011}).$$

By taking the square of the above equation (39) up to the first degree of approximation, the expression is given as:

$$\left(\hat{\hat{Y}}_{Prop}^{**} - \bar{Y}\right)^2 \cong \frac{\bar{Y}}{4}[\{4\xi_1^{*2} + 4\xi_1^*(\xi_2^* - 2\xi_0^*) + 4(\xi_0^{*2} - 2\xi_0^*\xi_2^* + \xi_2^{*2} + 1)\}a_2^{*2} + a_2^*\{6\xi_1^{*2} + 4\xi_1^*(2\xi_2^* - 3\xi_0^*) + 8\xi_0^{*2} - 12\xi_0^*\xi_2^* + 6\xi_2^{*2}\}] + \bar{Y}\left(\xi_0^* - \frac{1}{2}\xi_1^* - \frac{1}{2}\xi_2^*\right)\{4\bar{Y}\left(\xi_0^* - \frac{1}{2}\xi_1^* - \frac{1}{2}\xi_2^*\right) - 1\} + a_2^*\bar{Y}(\xi_0^* - \xi_1^* - \xi_2^*) - (\bar{Z}a_3^*\xi_2^* + \bar{X}a_1^*\xi_1^*)\{\bar{Z}a_3^*\xi_2^* + \bar{X}a_1^*\xi_1^* - 2\} \tag{40}$$

applying expectations on the equation (40), MSE of $\hat{\hat{Y}}_{Prop}^{**}$ upto the first order of approximation is given as:

$$\text{MSE}\left(\hat{\hat{Y}}_{Prop}^{**}\right) \cong \frac{\bar{Y}}{4}\Big[\{-8(\vartheta_{110} + \vartheta_{101}) + 4(\vartheta_{200} + \vartheta_{020} + \vartheta_{011} + 1)\}a_2^{*2} + \{-12(\vartheta_{110} + \vartheta_{101}) + 8(\vartheta_{200} + \vartheta_{011}) + 6(\vartheta_{020} + \vartheta_{002})\}a_2^* - 4(\vartheta_{110} + \vartheta_{101} - \vartheta_{200}) + 2\vartheta_{011} + \vartheta_{020} + \vartheta_{002} - 8a_1^*\bar{X}\{(\vartheta_{110} - \vartheta_{101} - \vartheta_{020})a_2^* + \vartheta_{101} - \frac{1}{2}(\vartheta_{020} + \vartheta_{011}) - 8\bar{Z}a_3^*\{(\vartheta_{101} - \vartheta_{011} - \vartheta_{002})a_2^* + \vartheta_{101} - \frac{1}{2}(\vartheta_{002} + \vartheta_{011})\}\Big] + 2\bar{X}\bar{Z}a_1^*a_3^*\vartheta_{011} + \overline{X^2}a_1^{2*}\vartheta_{020} + \overline{Z^2}a_3^{*2}\vartheta_{002} \tag{41}$$

$$\hat{\hat{Y}}_{Prop}^{**} = [\ \frac{\overline{y^*}}{2}\left\{\exp\left(\frac{\bar{X} - \overline{x^*}}{\bar{X} + \overline{x^*}}\right)\exp\left(\frac{\bar{Z} - \overline{z^*}}{\bar{Z} + \overline{z^*}}\right) + \exp\left(\frac{\overline{x^*} - \bar{X}}{\overline{x^*} + \bar{X}}\right)\exp\left(\frac{\overline{z^*} - \bar{Z}}{\overline{z^*} + \bar{Z}}\right)\right\} + a_1^*(\bar{X} - \overline{x^*}) + a_2^*\overline{y^*} + a_3^*(\bar{Z} - \overline{z^*})]\left\{\exp\left(\frac{\bar{X} - \overline{x^*}}{\bar{X} + \overline{x^*}}\right)\exp\left(\frac{\bar{Z} - \overline{z^*}}{\bar{Z} + \overline{z^*}}\right)\right\}.$$

Differentiate equation (41) with respect to a_1^*, a_2^* and a_3^*, we get the following expressions for optimum values of a_1^*, a_2^* and a_3^* are, respectively.

$$a^*_{1(opt)} \cong \frac{\begin{array}{c}\bar{Y}[\vartheta^2_{011}(A+2\vartheta_{101}-\vartheta_{011})+\left(\vartheta_{020}-2\vartheta_{110}+\frac{\vartheta_{101}}{2}\right)\vartheta_{002}\vartheta_{011}+\frac{\vartheta_{101}\vartheta_{011}}{2} \\ (\vartheta_{020}+4\vartheta_{110}-4)+\frac{\vartheta_{002}}{2}(\vartheta_{020}-\vartheta_{110})+\left(\vartheta_{200}-\frac{\vartheta_{110}}{2}-1\right)\vartheta_{020}\vartheta_{002} \\ +\left(-\vartheta^2_{110}+2\vartheta_{110}+\frac{\vartheta^2_{020}}{2}\right)\vartheta_{002}-\vartheta_{020}\vartheta^2_{101}]\end{array}}{\begin{array}{c}\bar{X}[\vartheta^2_{011}(\vartheta_{011}-\vartheta_{200}-1)+(2\vartheta_{110}\vartheta_{101}-\vartheta_{020}\vartheta_{002})\vartheta_{011} \\ +\vartheta_{020}(\vartheta_{200}+1)-\vartheta^2_{110}\vartheta_{002}-\vartheta_{020}\vartheta^2_{101}]\end{array}}$$

$$a^*_{2(opt)} \cong \frac{\begin{array}{c}\vartheta^2_{011}(4\vartheta_{200}+\vartheta_{020}+\vartheta_{002})-8\vartheta_{110}\vartheta_{101}\vartheta_{011}-\vartheta^2_{020}\vartheta_{002} \\ +\vartheta_{020}(-4\vartheta_{200}\vartheta_{002}-\vartheta^2_{002}+4\vartheta^2_{101})+4\vartheta_{002}\vartheta^2_{110}\end{array}}{\begin{array}{c}4\vartheta_{011}\{\vartheta_{011}(\vartheta_{011}-\vartheta_{200}-1)+(-\vartheta_{020}\vartheta_{002}+2\vartheta_{110}\vartheta_{101})\} \\ +4\vartheta_{020}\{\vartheta_{002}(\vartheta_{200}+1)-\vartheta^2_{101}\}-4\vartheta_{002}\vartheta^2_{110}\end{array}}$$

$$a^*_{3(opt)} \cong \frac{\begin{array}{c}\bar{Y}[\vartheta^2_{011}(A+2\vartheta_{110}-\vartheta_{011})+(\vartheta_{002}-2\vartheta_{101}+\backslash frac\vartheta_{110}2)\vartheta_{020}\vartheta_{011} \\ +\frac{\vartheta_{110}\vartheta_{011}}{2}(\vartheta_{002}+4\vartheta_{101}-4)+\frac{\vartheta_{020}}{2}(\vartheta_{002}-\vartheta_{101}) \\ +\left(\vartheta_{200}-\frac{\vartheta_{101}}{2}-1\right)\vartheta_{002}\vartheta_{020}+\left(-\vartheta^2_{101}+2\vartheta_{101}+\frac{\vartheta^2_{002}}{2}\right)\vartheta_{020}-\vartheta_{002}\vartheta^2_{110}]\end{array}}{\begin{array}{c}\bar{Z}[\vartheta^2_{011}(\vartheta_{011}-\vartheta_{200}-1)+(2\vartheta_{110}\vartheta_{101}-\vartheta_{020}\vartheta_{002})\vartheta_{011} \\ +\vartheta_{002}(\vartheta_{200}+1)-\vartheta^2_{101}\vartheta_{020}-\vartheta_{002}\vartheta^2_{110}]\end{array}}$$

where,
$A^* = -\vartheta_{200} - \frac{\vartheta_{020}}{2} - \frac{\vartheta_{002}}{2} + 1.$

Substituting the optimum values of a_1^*, a_2^* and a_3^* in Equation (41), we obtain the minimum MSE of $\hat{\hat{Y}}^{**}_{Prop}$, is given by:

$$\mathrm{MSE}_{\min}\left(\hat{\hat{Y}}^{**}_{Prop}\right) \cong \frac{\bar{Y}^2\alpha_1^*}{2\alpha_2^*}, \tag{42}$$

where,

$$\alpha_1^* = \frac{\vartheta^2_{020}\vartheta_{002}}{8} + C^*_{123}\vartheta^2_{002} + F^*_{123}\vartheta_{020} - \vartheta^2_{002}\left(\frac{\vartheta_{011}}{2}\right) + E^*_{123}\vartheta_{002} - 2\vartheta_{011}(\vartheta_{011}-1)(\vartheta_{200}\vartheta_{011}-2\vartheta_{110}\vartheta_{101}),$$

and

$$\alpha_2^* = \vartheta_{002}\vartheta^2_{110} - \vartheta_{011}\{\vartheta^2_{011} + 2\vartheta_{110}\vartheta_{101} - \vartheta_{011}(\vartheta_{200}+1)\} - \vartheta_{020}\{\vartheta_{002}(\vartheta_{200} - \vartheta_{011} + 1) - \vartheta^2_{101}\},$$

where,
$C^*_{123} = \vartheta_{200}\vartheta_{002} - \frac{\vartheta_{011}}{8} + \frac{\vartheta_{002}}{4} - \vartheta_{101},$
$D^*_{123} = -2\vartheta_{200} - 2\vartheta_{200}\vartheta_{011} - \vartheta^2_{101} - \vartheta^2_{110} - \frac{\vartheta^2_{011}}{4},$
$E^*_{123} = -\vartheta_{200}\vartheta_{011} - 2\vartheta_{110}\vartheta_{011}(\vartheta_{110} - \vartheta_{101}) + 2\vartheta^2_{102},$
$F^*_{123} = \frac{\vartheta^2_{002}}{8} + \vartheta_{200}\vartheta^2_{002} + D^*_{123}\vartheta_{002} - \vartheta_{200}\vartheta^2_{011} + 2\vartheta_{101}\vartheta_{011}(\vartheta_{110} - \vartheta_{101}) + 2\vartheta^2_{101}.$

2.4 Proposed estimator in Situation-II

We propose a new exponential type of estimator for estimating the finite population mean using two auxiliary variables under SRSWOR. The proposed estimator under Situation-II is given by:

$$\hat{\hat{Y}}^*_{Prop} = [\frac{\overline{y^*}}{2}\left\{\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)\exp\left(\frac{\bar{Z}-\bar{z}}{\bar{Z}+\bar{z}}\right)+\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)\exp\left(\frac{\bar{z}-\bar{Z}}{\bar{z}+\bar{Z}}\right)\right\}$$
$$+a_1^*(\bar{X}-\bar{x})+a_2^*\overline{y^*}]\ +a_3^*(\bar{Z}-\bar{z})\left\{\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)\exp\left(\frac{\bar{Z}-\bar{z}}{\bar{Z}+\bar{z}}\right)\right\}$$

where, $a_i(i=1,2,3)$ are constants.
Expanding the above estimator in terms ξ_i's, upto the first order of approximation, we get the following expression.

$$\left(\hat{\hat{Y}}^*_{Prop}-\bar{Y}\right)\cong\frac{\bar{Y}}{8}[(3a_2^*+4)(\xi_1^2+\xi_2^2)+(2a_2+4)\xi_1\xi_2-4(a_2+1)(\xi_0^*+1)(\xi_1+\xi_2)+8\xi_0^*+8(\xi_0^*+1)a_2^*]+\frac{1}{2}(\xi_1+\xi_2-2)(\bar{X}a_1^*\xi_1+\bar{Z}a_3^*\xi_2). \quad (43)$$

Bias of $\hat{\hat{Y}}^*_{Prop}$ upto the first order of approximation, is given by:

$$\text{Bias}\left(\hat{\hat{Y}}^*_{Prop}\right)\ \cong\frac{\bar{Y}}{8}\begin{bmatrix}-4(\upsilon_{110}+\upsilon_{110})+2(\upsilon_{020}+\upsilon_{011}+4)a_2^*+\\4(\upsilon_{020}\upsilon_{002}-\upsilon_{110}-\upsilon_{101}+\upsilon_{011})\end{bmatrix}$$
$$+\frac{1}{2}\bar{X}a_1^*(\upsilon_{020}+\upsilon_{011})+\bar{Z}a_3^*(\upsilon_{002}+\upsilon_{011}),$$

By taking the square of the equation (43) up to the first degree of approximation, the expression is given as:

$$\left(\hat{\hat{Y}}^*_{Prop}-\bar{Y}\right)^2\cong\frac{\bar{Y}}{4}\{4\xi_1^2+4\xi_1(\xi_2-2\xi_0^*)+4(\xi_0^{*2}-2\xi_0^*\xi_2+\xi_2^2+1)\}a_2^{*2}+a_2^*\{6\xi_1^2$$
$$+4\xi_1(2\xi_2-3\xi_0^*)+8\xi_0^2-12\xi_0^*\xi_2+6\xi_2^2\}+\bar{Y}\left(\xi_0^*-\frac{1}{2}\xi_1-\frac{1}{2}\xi_2\right)$$
$$\{4\bar{Y}\left(\xi_0^*-\frac{1}{2}\xi_1-\frac{1}{2}\xi_2\right)-1\}+a_2^*\bar{Y}(\xi_0^*-\xi_1-\xi_2)-(\bar{Z}a_3^*\xi_2+\bar{X}a_1^*\xi_1)\{\bar{Z}a_3^*\xi_2$$
$$+\bar{X}a_1\xi_1-2\}. \quad (44)$$

Applying expectations on equation (44), MSE up to the first order of approximation is given by:

$$\text{MSE}\left(\hat{\hat{Y}}^*_{Prop}\right)\cong\frac{\bar{Y}}{4}\Big[\{-8(\upsilon_{110}+\upsilon_{101})+4(\upsilon_{200}+\upsilon_{020}+\upsilon_{011}+1)\}a_2^{*2}$$
$$+\{-12(\upsilon_{110}+\upsilon_{101})+8(\upsilon_{200}+\upsilon_{011})+6(\upsilon_{020}+\upsilon_{002})\}a_2^*$$
$$-4(\upsilon_{110}+\upsilon_{101}-\upsilon_{200})+2\upsilon_{011}+\upsilon_{020}+\upsilon_{002}$$
$$-8a_1^*\bar{X}\{(\upsilon_{110}-\upsilon_{101}-\upsilon_{020})a_2^*+\upsilon_{101}-\frac{1}{2}(\upsilon_{020}+\upsilon_{011})\}$$
$$-8\bar{Z}a_3^*\{(\upsilon_{101}-\upsilon_{011}-\upsilon_{002})a_2^*+\upsilon_{101}-\frac{1}{2}(\upsilon_{002}+\upsilon_{011})\}\Big]$$
$$+2\bar{X}\bar{Z}a_1^*a_3^*\upsilon_{011}+\overline{X^2}a_1^{2*}\upsilon_{020}+\overline{Z^2}a_3^{*2}\upsilon_{002}. \quad (45)$$

Differentiate equation (45) with respect to the a_1^*, a_2^* and a_3^*, we get the following expressions:

$$a^*_{1(opt)} = \frac{\bar{Y}}{2B\bar{X}}(\upsilon^2_{011} - \upsilon_{020}\upsilon_{002} + 2\upsilon_{002}\upsilon_{110} - 2\upsilon_{101}\upsilon_{011})$$

$$a^*_{2(opt)} = \frac{1}{B}(\upsilon_{200}\upsilon^2_{011} + \upsilon_{020}\upsilon^2_{101} + \upsilon_{002}\upsilon^2_{110} - \upsilon_{200}\upsilon_{020}\upsilon_{002} - 2\upsilon_{110}\upsilon_{101}\upsilon_{011})$$

$$a^*_{3(opt)} = \frac{\bar{Y}}{2B\bar{Z}}(\upsilon^2_{011} - \upsilon_{020}\upsilon_{002} + 2\upsilon_{002}\upsilon_{110} - 2\upsilon_{101}\upsilon_{011})$$

Substituting of the optimum values of a^*_1, a^*_2 and a^*_3 in equation (45), we obtain the minimum MSE of $\hat{\hat{Y}}^*_{Prop}$, is given by:

$$\text{MSE}_{\min}\left(\hat{\hat{Y}}^*_{Prop}\right) \cong \frac{\bar{Y}^2\alpha_1}{2\alpha_2} \quad (46)$$

where,

$$\alpha_1 = \frac{\upsilon^2_{020}\upsilon_{002}}{8} + C_{123}\upsilon^2_{002} + F_{123}\upsilon_{020} - \upsilon^2_{002}\left(\frac{\upsilon_{011}}{2}\right) + E_{123}\upsilon_{002}2\upsilon_{011}(\upsilon_{011} - 1) - \upsilon_{200}\upsilon_{011} - 2\upsilon_{110}\upsilon_{101},$$

And

$$\alpha_2 = \upsilon_{002}\upsilon^2_{110} - \upsilon_{011}\{\upsilon^2_{011} + 2\upsilon_{110}\upsilon_{101} - \upsilon_{011}(\upsilon_{200} + 1)\} - \upsilon_{020}\{\upsilon_{002}(\upsilon_{200} - \upsilon_{011} + 1) - \upsilon^2_{101}\},$$

where,

$$C_{123} = \upsilon_{200}\upsilon_{002} - \frac{\upsilon_{011}}{8} + \frac{\upsilon_{002}}{4} - \upsilon_{101},$$

$$D_{123} = -2\upsilon_{200} - 2\upsilon_{200}\upsilon_{011} - \upsilon^2_{101} - \upsilon^2_{110} - \frac{\upsilon^2_{011}}{4},$$

$$E_{123} = -\upsilon_{200}\upsilon_{011} - 2\upsilon_{110}\upsilon_{011}(\upsilon_{110} - \upsilon_{101}) + 2\upsilon^2_{102},$$

$$F_{123} = \frac{\upsilon^2_{002}}{8} + \upsilon_{200}\upsilon^2_{002} + D^*_{123}\upsilon_{002} - \upsilon_{200}\upsilon^2_{011} + 2\upsilon_{101}\upsilon_{011}(\upsilon_{110} - \upsilon_{101}) + 2\upsilon^2_{101}.$$

3. Efficiency comparison

In this section some conditions are constructed under which proposed estimator is performing more efficiently than the existing estimators.

3.1 Efficiency in Situation-I

Condition (i)

From equation (2) and (42), $\frac{\text{MSE}_{\min}(\hat{\hat{Y}}^{**}_{Prop})}{\text{Var}(\hat{Y}^*_H)} < 1$ if $2\alpha_2\left(\theta_1 C^2_y + \theta_2 C^2_{y(2)}\right) - \alpha_1 > 0.$

Condition (ii)

From equation (4) and (42), $\frac{\text{MSE}_{\min}(\hat{\hat{Y}}^{**}_{Prop})}{\text{MSE}(\hat{Y}^*_R)} < 1$ if $2\alpha_2(\vartheta_{200} - \vartheta_{020} + 2\vartheta_{110}) - \alpha_1 > 0.$

Condition (iii)

From equation (6) and (42), $\frac{\text{MSE}_{\min}(\hat{\hat{Y}}^{**}_{Prop})}{\text{MSE}(\hat{Y}^*_P)} < 1$ if $2\alpha_2(\vartheta_{200} + \vartheta_{020} + 2\vartheta_{110}) - \alpha_1 > 0.$

Condition (iv)

From equation (9) and (42), $\frac{\text{MSE}_{\min}(\hat{\hat{Y}}^{**}_{Prop})}{\text{MSE}(\hat{Y}^*_{BT,R})} < 1$ if $2\alpha_2\left(\vartheta_{200} - \vartheta_{110} + \frac{\vartheta_{020}}{4}\right) - \alpha_1 > 0.$

Condition (v)

From equation (10) and (42), $\frac{\mathrm{MSE_{min}}(\hat{\bar{Y}}^{**}_{Prop})}{\mathrm{MSE}(\hat{\bar{Y}}^{*}_{BT,P})} < 1$ if $2\alpha_2\left(\vartheta_{200}+\vartheta_{110}+\frac{\vartheta_{020}}{4}\right)-\alpha_1 > 0.$

Condition (vi)

From equation (12) and (42), $\frac{\mathrm{MSE_{min}}(\hat{\bar{Y}}^{**}_{Prop})}{\mathrm{Var}(\hat{\bar{Y}}^{*}_{Reg})} < 1$ if $2\alpha_2(\vartheta_{200}\vartheta_{020}-\vartheta_{110}^2)-\alpha_1\vartheta_{020} > 0.$

Condition (vii)

From equation (14) and (42), $\frac{\mathrm{MSE_{min}}(\hat{\bar{Y}}^{**}_{Prop})}{\mathrm{Var}(\hat{\bar{Y}}^{*}_{SR})} < 1$ if $2\alpha_2(\vartheta_{200}\vartheta_{020}-\vartheta_{110}^2)-\alpha_1\vartheta_{020} > 0.$

Condition (viii)

From equation (16) and (42), $\frac{\mathrm{MSE_{min}}(\hat{\bar{Y}}^{**}_{Prop})}{\mathrm{MSE}_{min}(\hat{\bar{Y}}^{*}_{RI})} < 1$ if $(2\alpha_2-\alpha_1)(\vartheta_{200}\vartheta_{020}+\vartheta_{110}^2)-$ $\alpha_1\vartheta_{020} > 0.$

Condition (ix)

From equation (18) and (42), $\frac{\mathrm{MSE_{min}}(\hat{\bar{Y}}^{**}_{Prop})}{\mathrm{MSE}_{min}(\hat{\bar{Y}}^{*}_{D})} < 1$ if

$$2\alpha_2(-\vartheta_{200}\vartheta_{011}^2+2\vartheta_{110}\vartheta_{101}\vartheta_{011}+\vartheta_{020}(\vartheta_{200}\vartheta_{002}-\vartheta_{101}^2)-\vartheta_{002}\vartheta_{110}^2)$$
$$-\alpha_1(\vartheta_{020}\vartheta_{002}-\vartheta_{011}^2) > 0.$$

Condition (x)

From equation (20) and (42), $\frac{\mathrm{MSE_{min}}(\hat{\bar{Y}}^{**}_{Prop})}{\mathrm{MSE}_{min}(\hat{\bar{Y}}^{*}_{MR})} < 1$ if $2\alpha_2T^*-\alpha_1(\vartheta_{020}\vartheta_{002}-2\vartheta_{011}) > 0,$

where,

$$T^* = -\vartheta_{011}^2+2\vartheta_{011}(-\vartheta_{200}+\vartheta_{110}+\vartheta_{101})+\vartheta_{020}(\vartheta_{200}+\vartheta_{002}-2\vartheta_{101})$$
$$+\vartheta_{002}(\vartheta_{200}-2\vartheta_{110})-(\vartheta_{110}-\vartheta_{101})^2.$$

3.2 Efficiency in Situation-II

Condition (i)

From equation (22) and (46), $\frac{\mathrm{MSE_{min}}(\hat{\bar{Y}}^{*}_{Prop})}{\mathrm{MSE}(\hat{\bar{Y}}_{R})} < 1$ if $2\alpha_2(\upsilon_{200}-\upsilon_{020}+2\upsilon_{110})-\alpha_1 > 0.$

Condition (ii)

From equation (24) and (46), $\frac{\mathrm{MSE_{min}}(\hat{\bar{Y}}^{*}_{Prop})}{\mathrm{MSE}(\hat{\bar{Y}}_{P})} < 1$ if $2\alpha_2(\upsilon_{200}+\upsilon_{020}+2\upsilon_{110})-\alpha_1 > 0.$

Condition (iii)

From equation (27) and (46), $\frac{\mathrm{MSE_{min}}(\hat{\bar{Y}}^{*}_{Prop})}{\mathrm{MSE}(\hat{\bar{Y}}_{BT,R})} < 1$ if $2\alpha_2\left(\upsilon_{200}-\upsilon_{110}+\frac{\upsilon_{020}}{4}\right)-\alpha_1 > 0.$

Condition (iv)

From equation (28) and (46), $\frac{\mathrm{MSE_{min}}(\hat{\bar{Y}}^{*}_{Prop})}{\mathrm{MSE}(\hat{\bar{Y}}_{BT,P})} < 1$ if $2\alpha_2\left(\upsilon_{200}+\upsilon_{110}+\frac{\upsilon_{020}}{4}\right)-\alpha_1 > 0.$

Condition (v)

From equation (30) and (46), $\frac{\mathrm{MSE_{min}}(\hat{\bar{Y}}^{*}_{Prop})}{\mathrm{Var}(\hat{\bar{Y}}_{Reg})} < 1$ if $2\alpha_2(\upsilon_{200}\upsilon_{020}-\upsilon_{110}^2)-\alpha_1\upsilon_{020} > 0.$

Condition (vi)

From equation (32) and (46), $\frac{\mathrm{MSE_{min}}(\hat{\bar{Y}}^{*}_{Prop})}{\mathrm{Var}(\hat{\bar{Y}}_{SR})} < 1$ if $2\alpha_2(\upsilon_{200}\upsilon_{020}-\upsilon_{110}^2)-\alpha_1\upsilon_{020} > 0.$

Condition (vii)

From equation (34) and (46), $\frac{\mathrm{MSE_{min}}(\hat{\bar{Y}}^{*}_{Prop})}{\mathrm{MSE}_{min}(\hat{\bar{Y}}_{RI})} < 1$ if $(2\alpha_2-\alpha_1)(\upsilon_{200}\upsilon_{020}+\upsilon_{110}^2)-$ $\alpha_1\upsilon_{020} > 0.$

Condition (viii)

From equation (36) and (46), $\frac{\text{MSE}_{\min}(\hat{\bar{Y}}^*_{Prop})}{\text{MSE}_{min}(\hat{\bar{Y}}_D)} < 1$ if

$$2\alpha_2(-\upsilon_{200}\upsilon_{011}^2 + 2\upsilon_{110}\upsilon_{101}\upsilon_{011} + \upsilon_{020}(\upsilon_{200}\upsilon_{002} - \upsilon_{101}^2) - \upsilon_{002}\upsilon_{110}^2) \\ -\alpha_1(\upsilon_{020}\upsilon_{002} - \upsilon_{011}^2) > 0.$$

Condition (ix)

From equation (38) and (46), $\frac{\text{MSE}_{\min}(\hat{\bar{Y}}^*_{Prop})}{\text{MSE}_{min}(\hat{\bar{Y}}_{MR})} < 1$ if $2\alpha_2 T - \alpha_1(\upsilon_{020}\upsilon_{002} - 2\upsilon_{011}) > 0,$

where,

$$T = -\upsilon_{011}^2 + 2\upsilon_{011}(-\upsilon_{200} + \upsilon_{110} + \upsilon_{101}) + \upsilon_{020}(\upsilon_{200} + \upsilon_{002} - 2\upsilon_{101}) \\ +\upsilon_{002}(\upsilon_{200} - 2\upsilon_{110}) - (\upsilon_{110} - \upsilon_{101})^2.$$

4. Numerical analysis

Four real life datasets are used in this section to assess the performance of proposed estimator as compared to several existing estimators. Estimators are compared using mean square error (MSE) and percentage relative efficiency (PREs).

Dataset 1. [Source: Sukhatme (1977), Page 183]
y = Total cultivated area in 1931
x = Total Area under wheat in 1936
z = Total Area under wheat in 1937

Table 1: Summary statistics of dataset-I.

Parameters	Values	Parameters	Values	Parameters	Values
N	34	$\bar{Z}$	201.4118	C_y	0.6169
n	7	ρ_{yx}	0.8308	C_x	0.7678
$\bar{Y}$	765.3529	ρ_{yz}	0.8993	C_z	0.7555
$\bar{X}$	218.4118	ρ_{xz}	0.9299	θ_1	0.1134
		Non-response $W = 30\%$			
N_2	11	$\bar{Z}_2$	155.4161	$C_{y(2)}$	0.5313
n_2	4	$\rho_{yx(2)}$	0.5410	$C_{x(2)}$	0.8741
$\bar{Y}_2$	615.4303	$\rho_{yz(2)}$	0.7802	$C_{z(2)}$	0.5812
$\bar{X}_2$	198.3120	$\rho_{xz(2)}$	0.9131	θ_2	0.0428
		Non-response $W = 25\%$			
N_2	9	$\bar{Z}_2$	159.4444	$C_{y(2)}$	0.5401
n_2	3	$\rho_{yx(2)}$	0.6277	$C_{x(2)}$	0.8832
$\bar{Y}_2$	594.7778	$\rho_{yz(2)}$	0.8252	$C_{z(2)}$	0.6027
$\bar{X}_2$	209.4444	$\rho_{xz(2)}$	0.8921	θ_2	0.0313
		Non-response $W = 20\%$			
N_2	7	$\bar{Z}_2$	132.2857	$C_{y(2)}$	0.6432
n_2	2	$\rho_{yx(2)}$	0.9537	$C_{x(2)}$	0.5588
$\bar{Y}_2$	541.8571	$\rho_{yz(2)}$	0.8793	$C_{z(2)}$	0.6115
$\bar{X}_2$	147.1429	$\rho_{xz(2)}$	0.9529	θ_2	0.0286

Dataset 2. [Source: Sarndal *et al*. (2003)]
y = Municipal taxation revenues in 1985 (in millions of kronor)
x = Total number of seats in municipal council.
z = Real estate values according to 1984 assessment (in millions of kronor)

Table 2: Summary statistics of dataset-II.

Parameters	Values	Parameters	Values	Parameters	Values
N	284	$\bar{Z}$	3077.5250	C_y	2.4331
n	57	ρ_{yx}	0.5806	C_x	0.2325
$\bar{Y}$	245.0880	ρ_{yz}	0.9359	C_z	1.5420
$\bar{X}$	47.5352	ρ_{xz}	0.6771	θ_1	0.0140
		Non-response $W = 30\%$			
N_2	85	$\bar{Z}_2$	2539.7290	$C_{y(2)}$	1.1178
n_2	17	$\rho_{yx(2)}$	0.8262	$C_{x(2)}$	0.2306
$\bar{Y}_2$	167.8118	$\rho_{yz(2)}$	0.8731	$C_{z(2)}$	0.9503
$\bar{X}_2$	45.8235	$\rho_{xz(2)}$	0.7643	θ_2	0.0052
		Non-response $W = 25\%$			
N_2	71	$\bar{Z}_2$	2614.268	$C_{y(2)}$	1.0727
n_2	15	$\rho_{yx(2)}$	0.8460	$C_{x(2)}$	0.2388
$\bar{Y}_2$	166.0704	$\rho_{yz(2)}$	0.8487	$C_{z(2)}$	0.9131
$\bar{X}_2$	46.1549	$\rho_{xz(2)}$	0.7543	θ_2	0.0043
		Non-response $W = 20\%$			
N_2	57	$\bar{Z}_2$	2750.2460	$C_{y(2)}$	1.1024
n_2	11	$\rho_{yx(2)}$	0.8643	$C_{x(2)}$	0.2559
$\bar{Y}_2$	172.3333	$\rho_{yz(2)}$	0.7656	$C_{z(2)}$	0.9347
$\bar{X}_2$	45.7719	$\rho_{xz(2)}$	0.7656	θ_2	0.0035

Dataset 3. [Source: Fox (2015)]
y = Percentage of occupational holders in the 1950 US Census
x = Percentage of occupational incumbents in 1950 who were high school graduates
z = Percentage of respondents in a social survey who rated the occupation as good in prestige

Table 3: Summary statistics of dataset-III.

Parameters	Values	Parameters	Values	Parameters	Values
N	45	$\bar{Z}$	47.6889	C_y	0.5836
n	9	ρ_{yx}	0.7245	C_x	0.5663
$\bar{Y}$	41.8667	ρ_{yz}	0.8378	C_z	0.6607
$\bar{X}$	52.5556	ρ_{xz}	0.8519	θ_1	0.0889
		Non-response $W = 30\%$			
N_2	14	$\bar{Z}_2$	14.5000	$C_{y(2)}$	0.6211
n_2	4	$\rho_{yx(2)}$	0.4719	$C_{x(2)}$	0.3914
$\bar{Y}_2$	16.6428	$\rho_{yz(2)}$	0.6678	$C_{z(2)}$	0.6635
$\bar{X}_2$	23.8571	$\rho_{xz(2)}$	0.6678	θ_2	0.0333

Parameters	**Values**	**Parameters**	**Values**	**Parameters**	**Values**
		Non-response $W = 25\%$			
N_2	11	$\bar{Z}_2$	14.4545	$C_{y(2)}$	0.4823
n_2	3	$\rho_{yx(2)}$	0.5181	$C_{x(2)}$	0.3459
$\bar{Y}_2$	15.9090	$\rho_{yz(2)}$	0.8694	$C_{z(2)}$	0.7459
$\bar{X}_2$	25.6363	$\rho_{xz(2)}$	0.6063	θ_2	0.0277
		Non-response $W = 20\%$			
N_2	9	$\bar{Z}_2$	13.5555	$C_{y(2)}$	0.5484
n_2	2	$\rho_{yx(2)}$	0.8001	$C_{x(2)}$	0.3187
$\bar{Y}_2$	14.7777	$\rho_{yz(2)}$	0.8955	$C_{z(2)}$	0.8500
$\bar{X}_2$	27.4444	$\rho_{xz(2)}$	0.7846	θ_2	0.0222

Dataset 4. [Source: Murthy *et al*. (1967), Page 93]
y = Total Area (acres)
x = Cultivated Area (acres)
z = Consumption of fertilizers (lbs)

Table 4: Summary statistics of dataset-IV.

Parameters	**Values**	**Parameters**	**Values**	**Parameters**	**Values**
N	36	$\bar{Z}$	97.1111	C_y	1.4432
n	8	ρ_{yx}	0.3685	C_x	0.5393
$\bar{Y}$	17.5678	ρ_{yz}	0.2545	C_z	1.3130
$\bar{X}$	02.3650	ρ_{xz}	0.7066	θ_1	0.0972
		Non-response $W = 30\%$			
N_2	12	$\bar{Z}_2$	135.8182	$C_{y(2)}$	0.9479
n_2	3	$\rho_{yx(2)}$	0.7771	$C_{x(2)}$	0.5830
$\bar{Y}_2$	14.9927	$\rho_{yz(2)}$	0.5915	$C_{z(2)}$	0.5793
$\bar{X}_2$	2.4036	$\rho_{xz(2)}$	0.7316	θ_2	0.0375
		Non-response $W = 25\%$			
N_2	11	$\bar{Z}_2$	17.9211	$C_{y(2)}$	0.6734
n_2	3	$\rho_{yx(2)}$	0.5613	$C_{x(2)}$	0.5866
$\bar{Y}_2$	13.6211	$\rho_{yz(2)}$	0.8834	$C_{z(2)}$	0.5972
$\bar{X}_2$	29.1362	$\rho_{xz(2)}$	0.6333	θ_2	0.0312
		Non-response $W = 20\%$			
N_2	8	$\bar{Z}_2$	159.1429	$C_{y(2)}$	0.5089
n_2	2	$\rho_{yx(2)}$	0.6562	$C_{x(2)}$	0.5089
$\bar{Y}_2$	15.3114	$\rho_{yz(2)}$	0.2259	$C_{z(2)}$	0.4562
$\bar{X}_2$	2.5685	$\rho_{xz(2)}$	0.5790	θ_2	0.0250

We use the following expressions to obtain the percentage relative efficiency under both Situation-I and Situation-II.

$$\text{PRE}_I = \frac{\text{Var}\left(\hat{\hat{Y}}_H^*\right)}{\text{MSE}\left(\hat{\hat{Y}}_i^*\right)} \times 100,$$

$$\text{PRE}_{II} = \frac{\text{Var}\left(\hat{\hat{Y}}_H^*\right)}{\text{MSE}\left(\hat{\hat{Y}}_j\right)} \times 100.$$

$i = j = R, P, (BT, R), (BT, P), Reg, (S, R), RI, D, MR, Prop$

4.1 Analysis in Situation-I

In Table 5, all the conditioned values are positive in Situation-I, which shows the superiority of the proposed estimator.

Table 5*:* Efficiency conditions result for Situation-I.

Conditions	Population-I	Population-II	Population-III	Population-IV
(i)	0.00077	0.00008	0.00005	0.00151
(ii)	0.00054	0.00009	0.00008	0.00196
(iii)	0.00444	0.00012	0.00017	0.00274
(iv)	0.00040	0.00008	0.00003	0.00118
(v)	0.00211	0.00010	0.00010	0.00203
(vi)	0.00008	0.00002	0.00001	0.00004
(vii)	0.03846	0.07165	0.00980	0.05284
(viii)	0.01522	0.00499	0.00171	0.08066
(ix)	0.01205	0.00068	0.00145	0.05274
(x)	0.01942	0.00075	0.00335	0.05302

Table 6: PRE values of different estimators with respect to $\hat{\hat{Y}}_H^*$ under Situation-I for $H_2 = 0.30$.

Dataset	$\frac{1}{k}$	$\hat{\hat{Y}}_H^*$	$\hat{\hat{Y}}_R^*$	$\hat{\hat{Y}}_P^*$	$\hat{\hat{Y}}_{BT,R}^*$	$\hat{\hat{Y}}_{BT,P}^*$	$\hat{\hat{Y}}_{Reg}^*$	$\hat{\hat{Y}}_{SR}^*$	$\hat{\hat{Y}}_{RI}^*$	$\hat{\hat{Y}}_D^*$	$\hat{\hat{Y}}_{MR}^*$	$\hat{\hat{Y}}_{Prop}^{**}$
1	1/5	100.00	64.64	15.85	140.23	32.32	142.10	142.10	261.83	295.71	227.20	**302.93**
	1/4	100.00	73.00	16.59	153.06	33.53	153.81	153.81	285.41	316.61	236.54	**323.11**
	1/3	100.00	87.09	17.62	172.86	35.18	172.86	172.86	321.74	348.20	250.48	**353.97**
	1/2	100.00	115.91	19.15	207.42	37.57	210.09	210.09	384.99	402.12	274.68	**407.17**
2	1/5	100.00	102.50	73.70	94.68	80.20	145.96	145.96	501.68	501.68	498.72	**511.22**
	1/4	100.00	104.37	76.71	96.94	83.04	146.44	146.44	542.29	543.15	541.99	**552.38**
	1/3	100.00	106.43	80.21	99.48	86.29	147.18	147.18	597.46	601.62	601.56	**610.53**
	1/2	100.00	108.74	84.34	102.35	90.08	148.45	148.45	677.45	690.25	689.30	**698.86**
3	1/5	100.00	66.81	18.84	66.95	29.42	70.64	70.64	101.52	104.73	98.38	**108.57**
	1/4	100.00	77.12	20.41	77.30	32.31	81.98	81.98	118.59	122.02	113.61	**125.66**
	1/3	100.00	93.20	22.51	93.46	36.30	101.01	101.01	146.16	149.73	137.45	**153.17**
	1/2	100.00	121.79	25.45	122.24	42.15	132.53	132.53	198.33	201.55	180.20	**204.78**
4	1/5	100.00	129.64	50.01	113.55	66.90	130.85	130.85	101.80	130.93	129.68	**161.00**
	1/4	100.00	126.84	52.85	113.16	69.65	127.67	127.67	102.38	127.75	126.87	**155.37**
	1/3	100.00	123.65	56.68	112.70	73.26	124.10	124.10	1103.29	124.19	123.68	**149.34**
	1/2	100.00	119.98	62.17	112.14	78.17	120.11	120.11	104.71	120.17	120.00	**142.88**

Table 7: PRE values of different estimators with respect to $\hat{\bar{Y}}_H^*$ under Situation-I for $H_2 = 0.25$.

Dataset	$\frac{1}{k}$	$\hat{\bar{Y}}_H^*$	$\hat{\bar{Y}}_R^*$	$\hat{\bar{Y}}_P^*$	$\hat{\bar{Y}}_{BT,R}^*$	$\hat{\bar{Y}}_{BT,P}^*$	$\hat{\bar{Y}}_{Reg}^*$	$\hat{\bar{Y}}_{SR}^*$	$\hat{\bar{Y}}_{RI}^*$	$\hat{\bar{Y}}_D^*$	$\hat{\bar{Y}}_{MR}^*$	$\hat{\bar{Y}}_{Prop}^{**}$
1	1/5	100.00	76.24	15.50	161.06	31.34	161.16	161.16	314.11	340.59	257.62	**346.43**
	1/4	100.00	85.55	16.30	173.95	32.69	173.99	173.99	337.26	360.52	265.27	**365.81**
	1/3	100.00	100.72	17.40	192.98	34.54	194.06	194.06	370.94	389.40	276.38	**394.14**
	1/2	100.00	129.76	19.02	223.93	37.18	230.60	230.60	424.48	435.38	294.85	**439.57**
2	1/5	100.00	104.70	75.78	96.90	82.36	146.46	146.46	504.62	504.66	500.22	**513.89**
	1/4	100.00	106.16	78.52	98.78	84.88	146.78	146.78	547.35	547.80	545.74	**556.80**
	1/3	100.00	107.74	81.62	100.84	87.71	147.39	147.39	604.20	607.54	607.25	**616.30**
	1/2	100.00	109.46	85.18	103.11	90.90	148.55	148.55	684.09	695.90	695.31	**704.43**
3	1/5	100.00	94.08	21.30	95.30	34.89	102.13	102.13	211.39	214.05	169.78	**217.45**
	1/4	100.00	106.14	22.79	107.40	37.68	115.77	115.77	229.29	232.38	186.07	**235.69**
	1/3	100.00	122.78	24.63	124.09	41.17	134.81	134.81	252.61	255.97	207.27	**259.18**
	1/2	100.00	147.24	26.92	148.54	45.64	163.29	163.29	284.97	287.86	236.32	**290.99**
4	1/5	100.00	113.91	51.11	108.04	68.26	114.21	114.21	98.60	114.26	113.94	**137.52**
	1/4	100.00	114.31	54.50	108.79	71.47	114.53	114.53	100.29	114.56	114.32	**137.07**
	1/3	100.00	114.74	58.66	109.61	75.25	114.87	114.87	102.21	114.89	114.74	**136.64**
	1/2	100.00	115.21	63.89	110.49	79.77	115.26	115.26	104.40	115.26	115.21	**136.27**

Table 8: PRE values of different estimators with respect to $\hat{\bar{Y}}_H^*$ under Situation-I for $H_2 = 0.20$.

Dataset	$\frac{1}{k}$	$\hat{\bar{Y}}_H^*$	$\hat{\bar{Y}}_R^*$	$\hat{\bar{Y}}_P^*$	$\hat{\bar{Y}}_{BT,R}^*$	$\hat{\bar{Y}}_{BT,P}^*$	$\hat{\bar{Y}}_{Reg}^*$	$\hat{\bar{Y}}_{SR}^*$	$\hat{\bar{Y}}_{RI}^*$	$\hat{\bar{Y}}_D^*$	$\hat{\bar{Y}}_{MR}^*$	$\hat{\bar{Y}}_{Prop}^{**}$
1	1/5	100.00	265.12	18.48	205.77	33.30	304.84	304.84	332.97	358.49	315.09	**364.20**
	1/4	100.00	252.91	18.97	216.11	34.47	302.93	302.93	353.92	371.39	314.44	**376.58**
	1/3	100.00	239.45	19.60	230.23	36.01	303.40	303.40	384.46	393.95	316.27	**398.62**
	1/2	100.00	224.51	20.45	250.70	38.15	308.39	308.39	433.13	435.62	322.59	**439.78**
2	1/5	100.00	106.72	77.63	98.90	84.27	148.34	148.34	525.44	525.45	521.67	**534.60**
	1/4	100.00	107.75	80.05	100.39	86.45	148.23	148.23	567.49	568.43	566.82	**577.37**
	1/3	100.00	108.87	82.77	101.99	88.86	148.41	148.41	622.29	626.69	626.55	**635.41**
	1/2	100.00	110.06	85.82	103.73	91.54	149.10	149.10	697.02	710.22	709.43	7**718.73**
3	1/5	100.00	103.75	22.09	103.00	36.33	111.90	111.90	220.70	224.03	183.20	**263.23**
	1/4	100.00	115.75	23.52	115.12	39.02	125.62	125.62	238.69	242.16	198.80	**278.41**
	1/3	100.00	131.67	25.22	131.29	42.30	144.09	144.09	261.50	264.84	218.43	**298.14**
	1/2	100.00	153.82	27.30	153.91	46.38	170.32	170.32	291.83	294.36	244.15	**324.69**
4	1/5	100.00	114.48	58.77	112.88	76.93	115.89	115.89	103.83	115.91	114.48	**138.12**
	1/4	100.00	114.76	61.17	112.55	78.72	115.71	115.71	104.53	115.73	114.76	**137.46**
	1/3	100.00	115.06	63.90	112.20	80.69	115.59	115.59	105.28	115.61	115.06	**136.84**
	1/2	100.00	115.38	67.03	111.84	82.87	115.56	115.56	106.08	115.57	115.38	**136.31**

In Tables 6-8, we obtain the PRE values to assess the performances of all considered estimators under Situation-I on different values of K. Considering the above results, we see that, the proposed estimator $\hat{\bar{Y}}_{Prop}^{**}$ is more efficient than all other existing estimators in all 4 real life datasets.

4.2 Analysis in Situation-II

In Table 9, all the conditioned values are positive in Situation-II which shows the superiority of the proposed estimator. In Tables 10-12, we obtain PRE values to assess the performances of all considered estimators under Situation-II on different values of k. Considering the above Tables, we see that, the proposed estimator $\hat{\bar{Y}}^*_{Prop}$ is more efficient than all other existing estimators in all 4 real life datasets.

Table 9: Efficiency conditions result for Situation-II.

Conditions	Population-I	Population-II	Population-III	Population-IV
i	0.00005	0.00011	0.00003	0.00153
ii	0.00014	0.00006	0.00006	0.00188
iii	0.00002	0.00003	0.00001	0.00123
iv	0.00008	0.00004	0.00004	0.00158
v	0.00001	0.00001	0.00003	0.00003
vi	0.00029	0.00170	0.00048	0.08767
vii	0.00460	0.04643	0.00584	0.07328
viii	0.00028	0.00126	0.00047	0.07326
ix	0.00410	0.00136	0.00224	0.07326

Table 10: PRE values of different estimators with respect to $\hat{\bar{Y}}^*_H$ under Situation-II for $H_2 = 0.30$.

Dataset	$\frac{1}{k}$	$\hat{\bar{Y}}^*_H$	$\hat{\bar{Y}}^*_R$	$\hat{\bar{Y}}^*_P$	$\hat{\bar{Y}}^*_{BT,R}$	$\hat{\bar{Y}}^*_{BT,P}$	$\hat{\bar{Y}}^*_{Reg}$	$\hat{\bar{Y}}^*_{SR}$	$\hat{\bar{Y}}^*_{RI}$	$\hat{\bar{Y}}^*_D$	$\hat{\bar{Y}}^*_{MR}$	$\hat{\bar{Y}}^{**}_{Prop}$
1	1/5	100.00	107.70	29.74	117.20	48.30	120.82	120.82	131.95	131.97	122.01	**138.15**
	1/4	100.00	116.94	28.02	129.66	46.99	134.64	134.64	150.39	150.42	136.29	**155.97**
	1/3	100.00	131.11	26.13	149.70	45.45	157.28	157.28	182.50	182.55	159.84	**187.46**
	1/2	100.00	155.55	24.02	187.25	43.58	201.18	201.18	252.44	252.56	206.03	**256.83**
2	1/5	100.00	94.53	79.93	90.89	83.55	117.22	117.22	260.43	263.54	262.82	**273.08**
	1/4	100.00	97.85	81.86	93.83	85.80	123.41	123.41	307.19	311.68	310.64	**320.91**
	1/3	100.00	101.67	84.02	97.20	88.33	130.79	131.91	380.17	387.31	385.65	**396.23**
	1/2	100.00	106.11	86.48	101.08	91.22	139.74	139.74	510.02	523.43	520.29	**532.04**
3	1/5	100.00	70.97	25.87	71.10	37.95	73.73	73.73	83.22	83.24	79.85	**100.58**
	1/4	100.00	82.51	26.67	82.69	40.36	86.39	86.39	100.21	100.25	95.22	**108.40**
	1/3	100.00	99.83	27.59	100.12	43.29	105.78	105.78	128.15	128.22	119.85	**137.92**
	1/2	100.00	111.14	28.29	111.49	45.16	118.59	118.59	147.63	147.72	136.67	**150.95**
4	1/5	100.00	97.40	66.94	94.82	77.62	97.41	97.41	92.00	97.41	97.41	**127.13**
	1/4	100.00	100.92	67.72	98.05	79.19	100.93	100.93	94.92	100.95	100.96	**127.63**
	1/3	100.00	103.79	71.08	101.01	82.53	103.79	103.79	97.98	103.80	103.80	**128.96**
	1/2	100.00	108.79	70.89	105.42	83.73	108.79	108.79	101.77	108.80	108.80	**131.50**

Table 11: PRE values of different estimators with respect to $\hat{\bar{Y}}_H^*$ under Situation-II for $H_2 = 0.25$.

Dataset	$\frac{1}{k}$	$\hat{\bar{Y}}_H^*$	$\hat{\bar{Y}}_R^*$	$\hat{\bar{Y}}_P^*$	$\hat{\bar{Y}}_{BT,R}^*$	$\hat{\bar{Y}}_{BT,P}^*$	$\hat{\bar{Y}}_{Reg}^*$	$\hat{\bar{Y}}_{SR}^*$	$\hat{\bar{Y}}_{RI}^*$	$\hat{\bar{Y}}_D^*$	$\hat{\bar{Y}}_{MR}^*$	$\hat{\bar{Y}}_{Prop}^{**}$
1	1/5	100.00	108.21	28.57	118.41	46.89	122.33	122.33	134.48	134.51	123.63	**140.35**
	1/4	100.00	117.93	27.08	131.57	45.83	136.96	136.96	154.17	154.21	138.75	**159.50**
	1/3	100.00	132.61	25.45	152.45	44.58	160.61	160.61	188.11	188.17	163.38	**192.91**
	1/2	100.00	157.34	23.65	190.65	43.09	205.43	205.43	260.54	260.68	210.60	**264.86**
2	1/5	100.00	97.40	81.54	93.42	85.45	122.68	122.68	302.65	307.00	305.99	**316.23**
	1/4	100.00	100.29	83.21	95.97	87.39	128.20	128.20	353.57	359.68	358.26	**368.67**
	1/3	100.00	103.54	85.03	98.82	89.53	134.58	134.58	429.69	438.99	436.82	**447.75**
	1/2	100.00	107.19	87.05	102.02	91.90	142.03	142.03	555.85	571.96	568.18	**580.49**
3	1/5	100.00	80.85	26.74	81.03	40.23	84.53	84.53	97.51	97.55	92.84	**100.95**
	1/4	100.00	92.91	27.40	93.15	42.37	97.94	97.94	116.41	116.46	109.64	**119.77**
	1/3	100.00	110.31	28.13	110.67	44.89	117.70	117.70	146.43	146.52	135.59	**149.73**
	1/2	100.00	137.64	28.95	138.21	47.91	149.71	149.71	201.51	201.68	181.00	**204.80**
4	1/5	100.00	99.69	67.45	96.92	78.65	99.70	99.70	93.90	99.70	99.70	**122.97**
	1/4	100.00	102.90	68.15	99.85	80.05	102.90	102.90	96.55	102.91	102.91	**125.42**
	1/3	100.00	106.56	68.91	103.18	81.60	106.56	106.56	99.54	106.57	106.59	**128.33**
	1/2	100.00	110.78	69.75	107.01	83.33	110.79	110.79	102.97	110.79	110.79	**131.80**

Table 12: PRE values of different estimators with respect to $\hat{\bar{Y}}_H^*$ under Situation-II for $H_2 = 0.20$.

Dataset	$\frac{1}{k}$	$\hat{\bar{Y}}_H^*$	$\hat{\bar{Y}}_R^*$	$\hat{\bar{Y}}_P^*$	$\hat{\bar{Y}}_{BT,R}^*$	$\hat{\bar{Y}}_{BT,P}^*$	$\hat{\bar{Y}}_{Reg}^*$	$\hat{\bar{Y}}_{SR}^*$	$\hat{\bar{Y}}_{RI}^*$	$\hat{\bar{Y}}_D^*$	$\hat{\bar{Y}}_{MR}^*$	$\hat{\bar{Y}}_{Prop}^{**}$
1	1/5	100.00	97.00	27.29	105.32	44.12	108.48	108.48	118.16	118.18	109.52	**123.90**
	1/4	100.00	107.00	26.10	118.33	43.60	122.75	122.75	136.69	136.72	124.22	**141.91**
	1/3	100.00	122.40	24.78	139.37	42.98	146.26	146.26	169.08	169.13	148.59	**173.80**
	1/2	100.00	149.25	23.31	179.17	42.23	192.27	192.27	240.20	240.31	196.83	**244.47**
2	1/5	100.00	99.81	83.10	95.60	87.20	126.85	126.85	334.22	339.59	338.34	**348.73**
	1/4	100.00	102.27	84.46	97.75	88.80	131.72	131.72	386.80	394.17	392.46	**403.11**
	1/3	100.00	104.98	85.93	100.12	90.55	137.24	137.24	463.18	474.06	471.52	**482.79**
	1/2	100.00	107.99	87.53	102.73	92.45	143.55	143.55	584.28	602.15	597.96	**610.66**
3	1/5	100.00	87.45	26.97	87.67	41.27	91.88	91.88	107.88	107.93	102.06	**116.48**
	1/4	100.00	99.83	27.59	100.12	43.29	105.78	105.78	128.15	128.22	119.85	**137.92**
	1/3	100.00	117.17	28.28	117.58	45.62	125.66	125.66	159.50	159.60	146.57	**171.31**
	1/2	100.00	143.23	29.04	143.85	48.36	156.44	156.44	214.41	214.60	191.21	**230.42**
4	1/5	100.00	110.59	71.11	107.03	84.36	110.59	110.59	103.20	110.60	110.60	**132.82**
	1/4	100.00	111.74	71.00	108.03	84.57	111.74	111.74	104.04	111.75	111.75	**133.48**
	1/3	100.00	112.97	70.90	109.10	84.80	112.98	112.98	104.94	112.98	112.98	**134.22**
	1/2	100.00	114.29	70.78	110.24	85.03	114.30	114.30	105.90	114.31	114.31	**135.05**

5. Conclusion

This study is proposed a new exponential type of estimator to minimize the mean square error for the finite population mean under simple random sampling, for this purpose we have used two supplementary information, Under the situations when the non-response just on study variable or on both study and auxiliary variables. The statistical properties of the estimator such as the bias and minimum MSE have been derived in the first order of approximation. In the first step, the proposed estimator is theoretically compared with the existing estimators. After that some real-life datasets are used in the numerical comparison. The numerical study confirms that the proposed estimator has the minimum MSE and maximum PRE values among compared estimators. Therefore, using the proposed estimators in practice for the present study and such issues is recommended.

References

1. Azeem, M. (2014). *On Estimation of Population Mean in the Presence of Measurement Error and Non-Response*. PhD thesis, National College of Business Administration and Economics, Lahore.
2. Azeem, M., and Hanif, M. (2017). Joint influence of measurement error and nonresponse on estimation of population mean. *Communications in Statistics-Theory and Methods*, 46(4), 1679-1693.
3. Hansen, M. H., and Hurwitz, W. N. (1946). The problem of non-response in sample surveys. *Journal of the American Statistical Association,* 41(236):517-529.
4. Khare, B. (2013). Separate generalized estimators for population mean in the presence of non-response in stratified random sampling. *Advances in Statistics and Optimization*,13(2), 150-159.
5. Khare, B., and Rehman, H. U. (2014). An improved chain ratio-cum-regression type estimator for population mean in the presence of non-response. *International Journal of Agricultura land Statistical Sciences,* 10(2), 281-284.
6. Kuha, J., Butt, S., Katsikatsou, M., and Skinner, C. J. (2018). The effect of probing “don't know” responses on measurement quality and nonresponse in surveys. *Journal of the American Statistical Association*, 113(521), 26-40.
7. Riaz, S., Diana, G., and Shabbir, J. (2014). Improved classes of estimators for population mean in presence of non-response. *Pakistan Journal of Statistics,* 30(1), 83-100.
8. Sharma, P., Pal, S. K., and Singh, H. P. (2022). Improved Estimators of Population Mean under Nonresponse in Successive Sampling. *Mathematical Problems in Engineering*, 2022(1), 1-8.
9. Singh, H. P., and Kumar, S. (2009). A general procedure of estimating the population mean in the presence of non-response under double sampling using auxiliary information. *SORT-Statistics and Operations Research Transactions,* 33(1), 71-84.
10. Singh, H. P., and Kumar, S. (2010). Estimation of mean in presence of non-response using two phase sampling schemes. *Statistical Papers,* 51(3), 559-582.
11. Singh, R., Chauhan, P., Sawan, N., and Smarandache, F. (2009). Ratio estimators in simple random sampling using information on auxiliary attribute. *Pakistan Journal of Statistics and Operation Research,* 4(1), 47-53.
12. Sinha, R., and Kumar, V. (2014). Improved classes of estimators for population mean using information on auxiliary character under double sampling the non-respondents. *National Academy Science Letters,* 37(1), 71-79.
13. Yaqub, M., Shabbir, J., and Gupta, S. N. (2017). Estimation of population mean based on dual use of auxiliary information in nonresponse. *Communications in Statistics-Theory and Methods,* 46(24), 12130-12151.