Journal of Statistics, Volume 25, pp. 32-48 *ISSN 1684-8403*

Some New Facts of Transmuted Moment Exponential Distribution

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Abstract

Probability distributions are useful for modeling datasets in several applied areas. Recently a new distribution, named transmuted size-biased exponential or transmuted moment exponential distribution has been proposed and studied. In this article, we derive some more properties of this distribution that have not been discussed earlier. The limit behavior of this distribution is studied. Some characterizations of transmuted moment exponential distribution are presented which are based on truncated moments and failure function. Different estimation methods for the estimation of model parameters can give guidelines to a researcher in choosing the feasible and appropriate method. An extensive simulation study is also performed to classify the best estimation method. The application of TME distribution is observed and different estimation methods are compared through a real data set.

Keywords: Moment exponential distribution, Parameter estimation, Estimation methods, Simulation.

1. Introduction

In 2012, Dara and Ahmed assigned linear weights to the exponential model and developed a moment exponential (ME) distribution. They explained the behavior of distribution, its hazard curves, and its interesting properties with a real data application. After that, Iqbal et al. (2014), Hasnain et al. (2015) and Hashmi et al. (2019) generalized the ME distribution for more flexibility. The ME model attained great attention due to its flexibility so various authors studied and further generalized it for modeling more complex datasets. For example, generalized exponentiated moment exponential (Iqbal, Hasnain, Salman, Ahmad, & Hamedani, 2014), Marshall-Olkin length biased exponential (Ahsan-ul-Haq et al., 2019), and Weibull-Moment Exponential (Hashmi, Ahsan-ul-Haq, & Ozel, 2019).

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Hussain et al. (2018) proposed a new version of moment exponential distribution, called transmuted moment exponential (TME), and derived its basic characteristics including method of moments and maximum likelihood for estimation of model parameters.

Our objective in this article is to explore the estimation of TME parameters by five different estimation methods, i.e., maximum likelihood estimator (MLE), method of Anderson-Darling estimation (ADE). Cramér-von Mises Minimum Distance estimator (CVME), ordinary least square estimator (OLSE), weighted least square estimator (WLSE), and method of maximum product spacing estimator (MPSE). We compare the suggested estimators by means of Monto Carlo simulations to design a guidance plan for selecting an appropriate estimation procedure which gives the best estimates for the model parameters of the TME model. As far as we know, there is no study is reported on the comparison of these five estimation methods for the estimation of TME parameters. Further, some more mathematical properties are also studied. An important feature of the distribution is limiting behaviour which is also gone through in this study.

The problem of characterizing a probability distribution is now common in the statistical literature and many researchers are interested in it. This research work also deals with characterizations that are related to truncated moments and hazard function (Glänzel, & Hamedani, 2001). Some well-known estimation techniques are utilized to estimate model parameters. The computational study is also performed to assess the performances of various estimation methods.

The rest of the article is structured in the following sequence. Section 2 is based on the cdf, pdf, survival, and hazard functions of TME distribution. We also discussed the limiting behavior of its density and hazard functions. In section 3, a relation is developed in which behavior of proposed distribution can be studied by the behavior of baseline model and mode is derived. In section 4, characterizations in terms of truncated moments and hazard rate function are presented. Different estimation methods are derived in section 5. Monte Carlo simulation study is performed in section 6. A comparison of different estimation methods is given in section 7. In section 8, we illustrated its application to a real data set to show the efficiency of TME distribution. Finally, in Section 9, we offer some comments.

2. Transmuted Moment Exponential Distribution

Hussain et al. (2018) introduced TME distribution with CDF and PDF are given, respectively, by

$$F(x)=\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}\left[1+\theta-\theta\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}\right] \quad (1)$$

where, $x>0\ \beta>0, |\theta|\le 1$ with pdf

$$f(x)=\frac{x}{\beta^2}\mathrm{e}^{-x/\beta}\left[1-\theta+2\theta\left(1+\frac{x}{\beta}\right)\mathrm{e}^{-x/\beta}\right],\ \ \beta>0, |\theta|\le 1. \quad (2)$$

Hussain et al. (2018) also derived corresponding survival function and hazard rate function (HRF) $h(x)$ of X are specified as

$$S(x) = 1 - \left[1 - \left(1 + \frac{x}{\beta}\right) e^{-\frac{x}{\beta}}\right]\left[1 + \theta\left(1 + \frac{x}{\beta}\right) e^{-\frac{x}{\beta}}\right], \qquad x > 0 \tag{3}$$

and

$$h(x) = \frac{\frac{x}{\beta^2} \mathrm{e}^{-x/\beta}\left[1 + \theta - 2\theta\left\{1 - \left(1 + \frac{x}{\beta}\right)\mathrm{e}^{-x/\beta}\right\}\right]}{1 - \left[1 - \left(1 + \frac{x}{\beta}\right)\mathrm{e}^{-x/\beta}\right]\left[1 + \theta\left(1 + \frac{x}{\beta}\right)\mathrm{e}^{-x/\beta}\right]}, \qquad x > 0 \tag{4}$$

$where\ , \beta > 0, |\theta| \leq 1, and$ β control the scale of the distribution while θ controls the skewness of the distribution. The moment exponential is a special case for θ=0. The TME distribution is a flexible model, and it generalizes some of the well-known distributions. Note that the TME model is used to analyze more complex data sets.

2.1. The shape of the density function

Figures 1(a) and (b) show graphs of the PDF & hazard rate function of TME distribution for some specific values of the parameter β and θ, respectively.

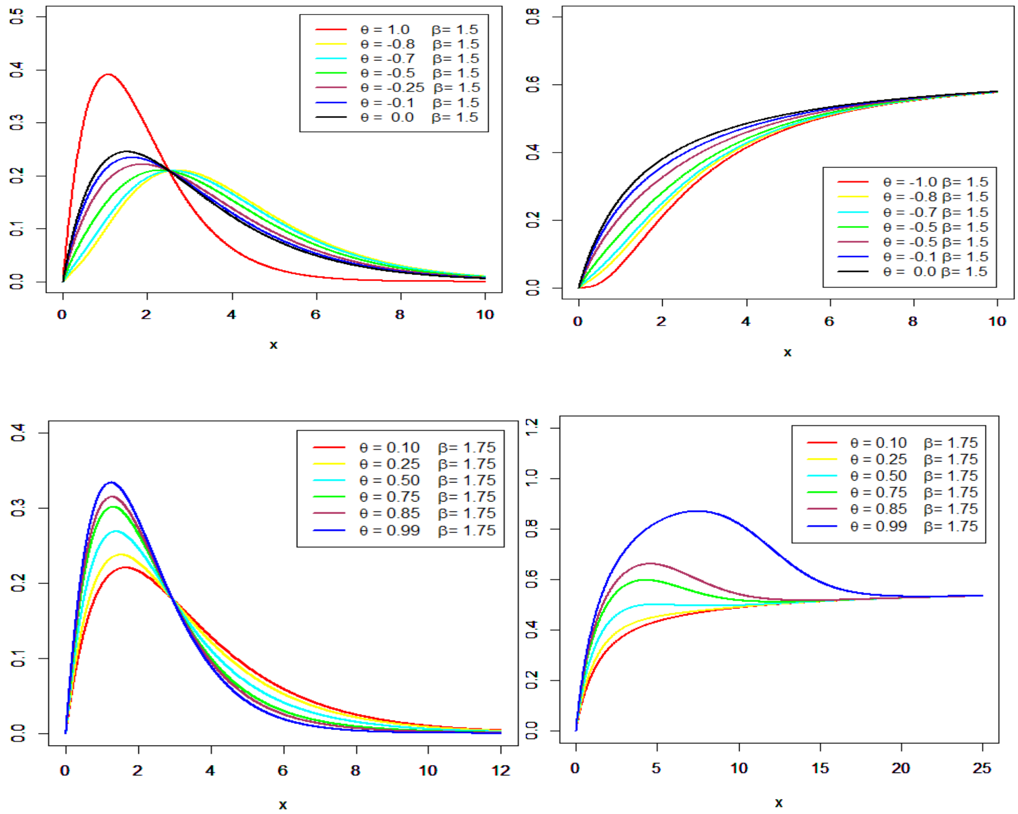

Figure 1 (a). Plots of density of TME (left), (b). h(x) of TME (right) for specified values of parameters.

2.2. The limiting behaviour of density and hazard functions of TME distribution

Following Lemmas are important for the proofs of theorems 1 and 2:

Lemma 1: If 'X' be a r.v., then for $\beta > 0$

$$\lim_{x\to\infty}\left(1+\frac{x}{\beta}\right)e^{-x/\beta} = 0$$

Proof: Applying L' Hospital rule and result follows.

Lemma 2: If 'X' be a r.v., then for $\beta > 0$

$$\lim_{x\to\infty} xe^{-x/\beta} = 0$$

Proof: Applying L' Hospital rule and result follows.

Theorem 1: The limit of PDF of TME distribution as $x \to \infty$ and at $x \to 0$ are zero.

Proof: Using Lemma 1 and 2, (2) becomes

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\frac{x}{\beta^2}\mathrm{e}^{-x/\beta}\left\{1-\theta+2\theta\left(1+\frac{x}{\beta}\right)\mathrm{e}^{-x/\beta}\right\}\right] = 0 \quad (5)$$

Also

$$\lim_{x\to\infty} f(x) = \frac{1}{\beta^2}\lim_{x\to\infty}\left[xe^{-x/\beta}\left\{1-\theta+2\theta\left(1+\frac{x}{\beta}\right)e^{-x/\beta}\right\}\right] = 0 \quad (6)$$

Theorem 2: The limit of HRF of TME model at $x \to 0$ is zero & as $x \to \infty$ is $1/\beta$.

Proof: We have

$$\lim_{x\to 0} h(x) = \lim_{x\to 0}\left[\frac{\frac{x}{\beta^2}\mathrm{e}^{-x/\beta}\left[1+\theta-2\theta\left\{1-\left(1+\frac{x}{\beta}\right)\mathrm{e}^{-x/\beta}\right\}\right]}{1-\left\{1-\left(1+\frac{x}{\beta}\right)\mathrm{e}^{-x/\beta}\right\}\left[1+\theta\left(1+\frac{x}{\beta}\right)\mathrm{e}^{-x/\beta}\right]}\right] = 0 \quad (7)$$

It is forthright to verify the result from above equation Eq. (7) using Lemma 1 and Lemma 2.

Now the limit of TME hazard rate function at infinity after simplification is

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty}\left[\frac{\frac{e^{-2x/\beta}x(-e^{x/\beta}\beta(\theta-1)+2(x+\beta)\theta)}{\beta^3}}{\frac{e^{-2x/\beta}(x+\beta)(-e^{x/\beta}\beta(\theta-1)+(x+\beta)\theta)}{\beta^2}}\right] = \frac{1}{\beta} \quad (8)$$

By applying L' Hospital rule, the result follows.

Remarks 1: The PDF of TME distribution has the following characteristics:

i. The curve is modal for all combinations of parameters β and θ.
ii. The curve begins from the origin, goes up, decreases after getting the highest point, and touches zero as x tends to infinity for all combinations of parameters β and θ.

Remarks 2: The properties related to h(x) of TME distribution are as follows:

i. The HRF's curve initiates from the origin and drives to the point $\frac{1}{\beta}$ as x approaches ∞ for all values of β and θ.
ii. The h(x) curve has an increasing trend in the beginning(origin) and touches the point $1/\beta$ as x →∞.
iii. For the same β, hazard curves coincide for different values of the transmuted variable.

3. General Properties

In this section, we study the characteristics of the transmuted variable with relation to the baseline random variable. The behaviour of transmuted distribution is assessed by the behavior of baseline distribution. The mean of the transmuted-G distribution is determined by the relation discussed in the following theorem provided the mean of baseline distribution also exists.

Theorem 3: Let $\pi(x)$ be a function of r. v. X with pdf ($g(x)$) and cdf ($G(x)$) of baseline distribution (ME) and pdf ($f(x)$) and cdf ($F(x)$) of transmuted distribution (TME). If $E_F(\pi(x))$ indicates $\int \pi(x)f(x)\ dx$, then

$$E_F(\pi(X)) = (1+\theta)E_G(\pi(X)) - 2\theta E_G[\pi(X)G(X)] \tag{9}$$

Proof: From (2)

$$\begin{aligned} E_F\big(\pi(X)\big) &= \int \pi(x)\,[(1+\theta)g(x) - 2\theta\ g(x)G(x)]\,dx \\ &= (1+\theta)\int \pi(x)g(x)\ dx - 2\theta \int \pi(x)\ G(x)\ \ g(x)\ \ dx \\ &= (1+\theta)E_G(\pi(X)) - 2\theta E_G(\pi(X)G(x)) \end{aligned}$$

3.1. Mode

The mode of transmuted moment exponential model is obtained as a result of following equation $f'(x) = 0$ if $f''(x) < 0$.

The density function of TME is differentiated with respect to x i.e., $f'(x)$ and equate it to zero.

$$f(x) = \frac{x}{\beta^2}\mathrm{e}^{-\frac{x}{\beta}}\left[1 - \theta + 2\theta\left(1 + \frac{x}{\beta}\right)\mathrm{e}^{-\frac{x}{\beta}}\right]$$

$$\begin{aligned} f'(x) = &\left[\frac{1}{\beta^2}\mathrm{e}^{-\frac{x}{\beta}} - \frac{x}{\beta^3}\mathrm{e}^{-\frac{x}{\beta}}\right]\left[1 - \theta + 2\theta\left(1 + \frac{x}{\beta}\right)\mathrm{e}^{-\frac{x}{\beta}}\right] \\ &+ \frac{2\theta x}{\beta^2}\mathrm{e}^{-\frac{x}{\beta}}\left(\mathrm{e}^{-\frac{x}{\beta}}\frac{1}{\beta} - \frac{1}{\beta}(1 + \frac{x}{\beta})\mathrm{e}^{-\frac{x}{\beta}}\right) \end{aligned}$$

$$\frac{e^{-\frac{2x}{\beta}}\left(e^{x/\beta}(x-\beta)\beta(-1+\theta) + 2(-2x^2 + \beta^2)\theta\right)}{\beta^4} = 0 \tag{10}$$

$$f''(x) = \frac{e^{-\frac{2x}{\beta}}}{\beta^4}\left[e^{x/\beta}(x-\beta)(\theta-1)+e^{x/\beta}\beta(\theta-1)-8x\theta\right] - \frac{2e^{-\frac{2x}{\beta}}}{\beta^5}\left(e^{x/\beta}(x-\beta)\beta(\theta-1)+2(\beta^2-2x^2)\theta\right) < 0\,,$$

$for\ \ \beta > 0, |\theta| \le 1$. Using (10), the mode of the distribution can be obtained.

4. Characterizations

Characterizing a distribution is a key feature in distribution theory and it helps the researcher in identifying the model.

4.1. Characterization based on two truncated moments

The characterization of TME distribution is carried out as follows: using the theorem proposed by Glänzel (1987) related to two truncated moments.

Theorem 4.1: Let X has the pdf given in (2) and

$$q_1(x) = \frac{\beta}{x}\left[1-\theta+2\theta\left(1+\frac{x}{\beta}\right)\mathrm{e}^{-x/\beta}\right]^{-1}, \tag{11}$$

$$q_2(x) = q_1(x)\mathrm{e}^{-\frac{x}{\beta}},\ \ x > 0. \tag{12}$$

The r. v. 'X' follows TME distribution iff the function η has the following expression:

$$\eta(x) = \frac{1}{2}\mathrm{e}^{-\frac{x}{\beta}}. \tag{13}$$

Proof: For x>0, we have

$$(1-F(x))E[q_1(X)|X \ge x] = \mathrm{e}^{-\frac{x}{\beta}},$$

$$(1-F(x))E[q_2(X)|X \ge x] = \frac{1}{2}\mathrm{e}^{-\frac{2x}{\beta}},$$

then

$$\eta(x) = \frac{1}{2}\mathrm{e}^{-\frac{x}{\beta}}.$$

It is obvious that

$\eta(x)q_1(x) - q_2(x) = q_1(x)\left\{\frac{1}{2}\mathrm{e}^{-\frac{x}{\beta}} - \mathrm{e}^{-\frac{x}{\beta}}\right\} \ne 0$ for all values of x, the proof completes.

Conversely, for $q_1(x)$, $q_2(x)$ and $\eta(x)$ we have r.v. X has TME distribution. now

$$\acute{s}(x) = \frac{\acute{\eta}(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)} = \frac{\acute{\eta}(x)}{\eta(x)-e^{-\frac{x}{\beta}}}$$

$$\acute{s}(x) = \frac{1}{\beta},\quad x > 0$$

and so $s(x) = \frac{x}{\beta}, \qquad x > 0 \qquad (14)$

Now using theorem of Glänzel (1987), the F(x) is

$$= c\int_0^x \frac{u}{\beta^2}\left[1-\theta+2\theta\left(1+\frac{u}{\beta}\right)\mathrm{e}^{-\frac{u}{\beta}}\right]exp\left[-\frac{u}{\beta}\right]du$$

which can be simplified to

$$\int_0^x f_{TMED}(u)du = F_{TMED}(x)$$

4.2. Characterization based hazard function

Hamedani et al. (2016) obtained characterization related to failure function. Using their concept, the characterization of TME model is offered as follows:

Theorem 4.2: The density function of TME model is (2) if its failure function h(x) satisfies the following equation

$$\acute{\mathrm{h}}(\mathrm{x}) - x^{-1}\mathrm{h}(\mathrm{x}) = \frac{\left\{-e^{\frac{2x}{\beta}}\beta^2(\theta-1)^2+e^{x/\beta}\left(x^2+4x\beta+3\beta^2\right)(\theta-1)\theta-2(x+\beta)^2\theta^2\right\}}{\beta(x+\beta)^2\left\{e^{x/\beta}\beta(\theta-1)-(x+\beta)\theta\right\}^2}. \quad (15)$$

Proof: If X has PDF (2), then

$$\acute{\mathrm{h}}(\mathrm{x}) - x^{-1}\mathrm{h}(\mathrm{x}) = \frac{\left[e^{\frac{2x}{\beta}}\beta^3(\theta-1)^2+e^{x/\beta}\left(x^3+x^2\beta-3x\beta^2-3\beta^3\right)(\theta-1)\theta+2\beta(x+\beta)^2\theta^2\right]}{\left[\beta(x+\beta)^2\left(e^{x/\beta}\beta(\theta-1)-(x+\beta)\theta\right)^2\right]} - x^{-1}\left(\frac{e^{-\frac{x}{\beta}}x\left[1+\theta-2\left\{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right\}\theta\right]}{\beta^2\left[1-\left(1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)\left\{1+\theta-\left(1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)\theta\right\}\right]}\right). \quad (16)$$

simplification follows (16). Now if (16) exists then

$$\frac{d}{dx}\left[x^{-1}\mathrm{h}(\mathrm{x})\right] = \frac{d}{dx}\left(\frac{\mathrm{e}^{-\frac{x}{\beta}}\left[1+\theta-2\theta\left\{1-(1+x/\beta)\mathrm{e}^{-\frac{x}{\beta}}\right\}\right]}{\beta^2\left[1-\left\{1-(1+x/\beta)\mathrm{e}^{-\frac{x}{\beta}}\right\}\left\{1+\theta(1+x/\beta)\mathrm{e}^{-\frac{x}{\beta}}\right\}\right]}\right)$$

and simplification results in Eq. (4).

5. Parameter estimation

Here in this part of research, we estimate the parameter using maximum likelihood estimation (MLE), Anderson Darling estimation (ADE), Cramer von misses (CVME), ordinary and weighted least squares estimation (OLSE & WLSE), and maximum product spacing estimation (MPSE) methods.

5.1. Maximum likelihood estimation

The ML method is the most advantageous parametric estimation technique. The reason is described by theoretical acceptance of the limiting properties of the estimators, such as consistency, efficiency, and asymptotic normality. Let $x_1, x_2, x_3, \dots\dots.. x_n$ be a r. sample from transmuted moment exponential model of size n with PDF Eq. (2), the 'log-likelihood' function (L) of TME is given by Hussain et al. (2018), given by

$$L = \sum \log x_i - 2n \log \beta - \sum \frac{x_i}{\beta} + \sum \log\left[1 - \theta + 2\theta\left(1 + \frac{x_i}{\beta}\right) e^{-\frac{x_i}{\beta}}\right]$$

The MLE of θ and β, as given in Hussain et al. (2018) is the solution of the following equations

$$\frac{\partial L}{\partial \theta} = \sum \frac{-1+2\left(1+\frac{x}{\beta}\right) e^{-\frac{x}{\beta}}}{1-\theta+2\theta\left(1+\frac{x}{\beta}\right) e^{-\frac{x}{\beta}}} = 0 \tag{17}$$

$$\frac{\partial L}{\partial \beta} = \frac{-2n}{\beta} + \sum \frac{x}{\beta^2} - \sum \frac{2\theta \frac{x}{\beta^2} e^{-\frac{x}{\beta}}}{1-\theta+2\theta\left(1+\frac{x}{\beta}\right) e^{-\frac{x}{\beta}}} = 0. \tag{18}$$

The exact solution of Eq. 17 and 18 is tedious to obtain. So it is recommended to utilize non-linear optimization algorithms for example the Newton-Raphson algorithm to maximize 'L' numerically. It can be solved using statistical software R.

5.2. Method of Anderson-Darling estimation

The Anderson-Darling estimators (ADE) of (θ, β) can be acquired by minimizing the following function, with respect to θ and β, as follows

$$A(\theta, \beta) = -n - \frac{1}{n} \sum_{i=1}^{n} (2i - 1)\left[\log F\left(x_{(i)}; \theta, \beta\right) + \log\left[1 - F\left(x_{(n+1-i)}; \theta, \beta\right)\right]\right]$$

Thus, the ADE of θ can be obtained by differentiating the above equation

$$\frac{\partial A(\theta,\beta)}{\partial \theta} = -\frac{1}{n}\left(\sum_{i=1}^{n} (-1+2i)\left(\frac{e^{-\frac{2x_i}{\beta}}\left(-\beta\left(1+e^{\frac{x_i}{\beta}}\right)-x_i\right)(\beta+x_i)}{\beta^2\left(1+\frac{e^{-\frac{2x_i}{\beta}}(\beta+x_i)\left(e^{\frac{x_i}{\beta}}\beta(-1+\theta)-\theta(\beta+x_i)\right)}{\beta^2}\right)} + \frac{-\beta+e^{\frac{x_{1-i+n}}{\beta}}\beta-x_{1-i+n}}{e^{\frac{x_{1-i+n}}{\beta}}\beta(-1+\theta)-\theta(\beta+x_{1-i+n})}\right)\right) \tag{19}$$

Similarly, we can obtain $\partial A(\theta, \beta)/\partial \beta$ and get estimates of θ and β by equating them to zero.

5.3. Cramer von Misses Minimum Distance Estimation

Another method for gaining estimates is CVME and estimates are obtained by minimizing the function with respect to θ and β. It is defined by

$$C(\theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{(i)};\theta\right)-\frac{2i-1}{2n}\right]^2$$

$$=\frac{1}{12n}+\sum_{i=1}^{n}\left[1-\frac{e^{-\frac{2x_{(i)}}{\beta}}(\beta+x_{(i)})\left(e^{\frac{x_{(i)}}{\beta}}\beta(\theta-1)-\theta(\beta+x_{(i)})\right)}{\beta^2}-\frac{2i-1}{2n}\right]^2. \qquad (20)$$

The CVME indicates that the bias of this estimator (θ, β) is least when compared with the bias of other minimum distance estimators.

5.4. Methods of least squares and weighted least squares estimation

We now consider the methods of least square estimation (OLSE) and weighted least square estimation (WLSE). OLSE method was firstly presented by Swain (1988). It is a non-linear method of estimation, especially when the MLEs cannot be obtained in an explicit form. The OLSE of (θ, β) can be intended by minimizing the least square function SSE, *L(θ, β)*.

$$L(\theta,\beta)=\sum_{i=1}^{n}\left(F\left(x_{(i)};\theta,\beta\right)-\frac{\mathrm{i}}{n+1}\right)^2$$

w. r. t θ, where $x_{(i)}$, (i=1, 2, 3, …..,n) is the *i*th element of the ordered observations $x_1, x_2, x_3, \ldots, x_n$ and $\hat{F}(.)$ is empirical CDF of *i*th observation. i.e., $\hat{F}(.)=\frac{\mathrm{i}}{n+1}$.

Using Eq. (1) and $\hat{F}(.)$, we have

$$=\sum_{i=1}^{n}\left(\left\{1-\left(1+\frac{x_{(i)}}{\beta}\right)e^{-x_{(i)}/\beta}\right\}\left[1+\theta-\theta\left\{1-\left(1+\frac{x_{(i)}}{\beta}\right)e^{-x_{(i)}/\beta}\right\}\right]-\frac{\mathrm{i}}{n+1}\right)^2.$$

Thus, the OLSE can be obtained by equating the equations to zero, i.e., $\partial L(\theta)/\partial\theta=0$

$$\frac{\partial L(\theta,\beta)}{\partial\theta}=\frac{1}{\beta^4}e^{-\frac{4x_{(i)}}{\beta}}n(x_{(i)}+\beta)\left(x_{(i)}+\beta-\beta e^{x_{(i)}/\beta}\right)\left(-e^{\frac{2x_{(i)}}{\beta}}\beta^2-2e^{x_{(i)}/\beta}\beta(x_{(i)}+\beta\right)(\theta-1)+2(x_{(i)}+\beta)^2\theta). \qquad (21)$$

$$\frac{\partial L(\theta,\beta)}{\partial\theta}=\frac{1}{\beta^6}e^{-\frac{4x_{(i)}}{\beta}}nx_{(i)}^2(e^{\frac{3x_{(i)}}{\beta}}\beta^3(\theta-1)-6e^{x_{(i)}/\beta}\beta\left(x_{(i)}+\beta\right)^2(\theta-1)\theta+4(x_{(i)}+\beta)^3\theta^2+2e^{\frac{2x}{\beta}}\beta^2(x_{(i)}+\beta)(1-3\theta+\theta^2)) \qquad (22)$$

The weighted least square estimate (WLSE) of θ and β, are obtained by minimizing the weighted least square function (Also called SSE) with respect to θ and β, defined by

$$WLS(\theta,\beta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(x_{(i)};\theta,\beta\right)-\frac{i}{n+1}\right]^2$$

$$=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[1+\frac{\mathrm{e}^{-\frac{2x_{(i)}}{\beta}}(\beta+x_{(i)})(\mathrm{e}^{\frac{x_{(i)}}{\beta}}\beta(\theta-1)-\theta(\beta+x_{(i)}))}{\beta^2}-\frac{i}{n+1}\right]^2$$

$\partial WLS(\theta,\beta)/_{\partial\theta}=0$ gives the WLSQE of θ.

$$\frac{\partial WLS(\theta,\beta)}{\partial\theta}=\frac{2}{i(1-i+n)\beta^2}\sum_{i=1}^{n}\mathrm{e}^{-\frac{2x_{(i)}}{\beta}}(1+n)^2(2+n)(-\beta+\mathrm{e}^{\frac{x_{(i)}}{\beta}}\beta-x_{(i)})(\beta+x_{(i)})\left(1-\frac{i}{1+n}-\frac{\mathrm{e}^{-\frac{2x_{(i)}}{\beta}}(\beta+x_{(i)})(\mathrm{e}^{\frac{x_{(i)}}{\beta}}\beta(\theta-1)-\theta(\beta+x_{(i)}))}{\beta^2}\right) \quad (23)$$

Similarly, we can get the expression of estimate β by minimizing the function SSE (WLS).

5.5. Maximum product spacing Estimation

The MPSE method is greatly used for continuous distributions and can be considered a powerful substitute for the MLE technique for estimating the parameters (see Cheng and Amin [25]). Let

$$D_i(\theta,\beta)=F\left(x_{(i)};\theta,\beta\right)-F\left(x_{(i-1)};\theta,\beta\right),\ \ i=1,2,3,\ldots\ldots,n+1$$

be the uniform spacing of a r. sample from the TME model, where

$$F\left(x_{(0)};\theta,\beta\right)=0,\qquad F\left(x_{(n+1)};\theta,\beta\right)=1,\text{and}\ \sum_{i=1}^{n+1}D_i(\theta,\beta)=1$$

The MPS estimator is found from the following function (GM), where GM is basically the geometric mean of the spacings. MPS are calculated by maximizing

$$GM(\theta,\beta)=\left\{\prod_{i=1}^{n+1}D_i(\theta,\beta)\right\}^{\frac{1}{n+1}},\ (24)$$

w. r. t. θ and β. The MPS estimator of θ and β can alternatively be solved by maximizing the logarithm of the GM (sample spacing's) stated above. There is no closed solution exist in Eq. (24), so the numerical method is used to find estimates.

Note that all estimation methods can be obtained by using numerical methods.

6. Simulation

A Monte Carlo simulation study is used to review the performance of different estimation methods for estimating TME parameters. Numerous sample sizes (n) are considered for specific values of parameters. The simulations are achieved as:

- Samples are generated from $F(x) = q$, where $q \sim U(0,1)$.
- Consider following values of n=20, 50, 100, 200 and 500.
- In the experiment N is considered as 10,000 for each 'n'.

The performance of these estimators is assessed by the average of ML estimates, mean square errors (MSEs) and biases. Therefore these measures are calculated for all sample sizes and all parametric values and reported in Tables 1-4.
The results of simulations specify that both selected criterions of comparison; the MSE and bias decreases and approaches towards zero as 'n' increases under the first-order asymptotic behaviour. The average of estimates of θ and β tends to be nearer to the true parametric vales as 'n' increases.

Table 1: Simulation results for $\theta = 0.5$ & $\beta = 0.5$

n	$Est.$	$Est.pra.$	MLE	ADE	CVME	OLSE	WLSE	MPSE
20	|Bias|	$\hat{\theta}$	0.01958	0.10931	0.14203	0.06241	0.07317	0.01031
		$\hat{\beta}$	0.01081	0.03977	0.04530	0.02741	0.03118	0.03017
	MSE	$\hat{\theta}$	0.18823	0.11637	0.13692	0.10877	0.10579	0.09392
		$\hat{\beta}$	0.01630	0.01816	0.02028	0.01742	0.01713	0.01396
50	|Bias|	$\hat{\theta}$	0.00392	0.10797	0.11794	0.06623	0.08352	0.04085
		$\hat{\beta}$	0.01038	0.03244	0.03367	0.02202	0.02682	0.02472
	MSE	$\hat{\theta}$	0.15397	0.10501	0.12271	0.10309	0.09994	0.08053
		$\hat{\beta}$	0.01038	0.01068	0.01200	0.01076	0.01042	0.00777
100	|Bias|	$\hat{\theta}$	0.01048	0.08810	0.09956	0.06864	0.08067	0.04413
		$\hat{\beta}$	0.00810	0.02713	0.02881	0.02164	0.02548	0.02156
	MSE	$\hat{\theta}$	0.12873	0.09301	0.10707	0.09251	0.08942	0.07064
		$\hat{\beta}$	0.00799	0.00789	0.00881	0.00791	0.00770	0.00564
200	|Bias|	$\hat{\theta}$	0.02152	0.07046	0.08282	0.05737	0.06337	0.04574
		$\hat{\beta}$	0.00248	0.02005	0.02193	0.01604	0.01840	0.01710
	MSE	$\hat{\theta}$	0.10080	0.07371	0.08562	0.07785	0.07352	0.05087
		$\hat{\beta}$	0.00559	0.00531	0.00601	0.00551	0.00525	0.00354
300	|Bias|	$\hat{\theta}$	0.04001	0.04314	0.04655	0.02531	0.03863	0.03245
		$\hat{\beta}$	0.00123	0.01494	0.01493	0.01003	0.01399	0.01409
	MSE	$\hat{\theta}$	0.08702	0.06278	0.07111	0.06680	0.06323	0.04202
		$\hat{\beta}$	0.00469	0.00452	0.00500	0.00468	0.00457	0.00287

Table 2: Simulation results for $\theta = 0.5$ & $\beta = 2.0$

n	$Est.$	$Est.pra.$	MLE	ADE	CVME	OLSE	WLSE	MPSE
20	|Bias|	$\hat{\theta}$	0.10164	0.00355	0.02598	0.07296	0.01618	0.07004
		$\hat{\beta}$	0.02507	0.06205	0.03274	0.00752	0.05125	0.05042
	MSE	$\hat{\theta}$	0.28954	0.19150	0.24284	0.17624	0.15500	0.13772
		$\hat{\beta}$	0.16995	0.20727	0.19263	0.17121	0.20306	0.15818
50	|Bias|	$\hat{\theta}$	0.05111	0.05295	0.05874	0.01733	0.03386	0.00893
		$\hat{\beta}$	0.00187	0.08142	0.08138	0.04384	0.06495	0.05554
	MSE	$\hat{\theta}$	0.18402	0.12620	0.14557	0.11875	0.12195	0.09322
		$\hat{\beta}$	0.13130	0.13802	0.15491	0.13701	0.13434	0.09490
100	|Bias|	$\hat{\theta}$	0.05075	0.07290	0.08201	0.04727	0.04989	0.01920
		$\hat{\beta}$	0.00179	0.09695	0.10106	0.06888	0.07568	0.06293
	MSE	$\hat{\theta}$	0.14326	0.09491	0.11227	0.09989	0.09828	0.06627
		$\hat{\beta}$	0.11075	0.11934	0.13242	0.11859	0.11248	0.07069
200	|Bias|	$\hat{\theta}$	0.04974	0.03745	0.05440	0.02829	0.03481	0.02578
		$\hat{\beta}$	0.00651	0.05964	0.07147	0.04716	0.05710	0.05972
	MSE	$\hat{\theta}$	0.09608	0.07031	0.08212	0.07569	0.07054	0.04364
		$\hat{\beta}$	0.07773	0.08071	0.09481	0.08622	0.07955	0.04577
300	|Bias|	$\hat{\theta}$	0.02636	0.03871	0.04543	0.02372	0.03242	0.03177
		$\hat{\beta}$	0.00931	0.05801	0.06152	0.04153	0.05230	0.05787
	MSE	$\hat{\theta}$	0.08306	0.06362	0.07252	0.06855	0.06383	0.04038
		$\hat{\beta}$	0.07260	0.07344	0.08246	0.07750	0.07258	0.04238

Table 3: Simulation results for $\theta = 0.8$ & $\beta = 0.5$

n	$Est.$	$Est.pra.$	MLE	ADE	CVME	OLSE	WLSE	MPSE
20	|Bias|	$\hat{\theta}$	0.01063	0.09893	0.07588	0.16820	0.13676	0.14800
		$\hat{\beta}$	0.00229	0.01583	0.01138	0.02909	0.02236	0.01818
	MSE	$\hat{\theta}$	0.05041	0.09612	0.07971	0.13776	0.11551	0.12310
		$\hat{\beta}$	0.00888	0.01813	0.01528	0.02666	0.02152	0.01955
50	|Bias|	$\hat{\theta}$	0.05041	0.09612	0.07971	0.13776	0.11551	0.12310
		$\hat{\beta}$	0.00888	0.01813	0.01528	0.02666	0.02152	0.01955
	MSE	$\hat{\theta}$	0.08716	0.09631	0.10440	0.11856	0.10310	0.07791
		$\hat{\beta}$	0.00680	0.00791	0.00866	0.00912	0.00829	0.00586
100	|Bias|	$\hat{\theta}$	0.05601	0.10328	0.09605	0.13922	0.10907	0.09811
		$\hat{\beta}$	0.00557	0.01579	0.01444	0.02317	0.01676	0.01245
	MSE	$\hat{\theta}$	0.07884	0.09661	0.10721	0.11898	0.09833	0.06573
		$\hat{\beta}$	0.00500	0.00603	0.00656	0.00691	0.00608	0.00415
200	|Bias|	$\hat{\theta}$	0.05983	0.08193	0.07181	0.10077	0.07686	0.07280
		$\hat{\beta}$	0.00879	0.01400	0.01199	0.01785	0.01276	0.01099
	MSE	$\hat{\theta}$	0.06847	0.07881	0.08574	0.09339	0.07484	0.04969
		$\hat{\beta}$	0.00394	0.00449	0.00486	0.00514	0.00442	0.00297
300	|Bias|	$\hat{\theta}$	0.05503	0.07510	0.07124	0.09642	0.07462	0.05804
		$\hat{\beta}$	0.00932	0.01422	0.01348	0.01856	0.01396	0.00995
	MSE	$\hat{\theta}$	0.05977	0.06697	0.07472	0.08334	0.06672	0.03909
		$\hat{\beta}$	0.00320	0.00360	0.00395	0.00426	0.00360	0.00225

Table 4: Simulation results for $\theta = -0.5$ & $\beta = 2.0$

n	$Est.$	$Est.pra.$	MLE	ADE	CVME	OLSE	WLSE	MPSE
20	|Bias|	$\hat{\theta}$	0.00078	0.10231	0.02291	0.19170	0.16868	0.31508
		$\hat{\beta}$	0.07204	0.17073	0.11317	0.27096	0.24319	0.40120
	MSE	$\hat{\theta}$	0.22614	0.25256	0.24100	0.30039	0.28727	0.37227
		$\hat{\beta}$	0.21496	0.25238	0.24082	0.34340	0.32619	0.50264
50	|Bias|	$\hat{\theta}$	0.02359	0.06480	0.02335	0.11157	0.08523	0.24392
		$\hat{\beta}$	0.05926	0.09534	0.06534	0.14437	0.11588	0.28153
	MSE	$\hat{\theta}$	0.16152	0.15654	0.15691	0.18551	0.16946	0.29145
		$\hat{\beta}$	0.14217	0.14159	0.13335	0.17025	0.15805	0.32910
100	|Bias|	$\hat{\theta}$	0.01355	0.02656	0.00258	0.05200	0.03022	0.14544
		$\hat{\beta}$	0.02395	0.03437	0.02024	0.06350	0.03768	0.15256
	MSE	$\hat{\theta}$	0.08701	0.07383	0.07959	0.09434	0.07457	0.16093
		$\hat{\beta}$	0.06809	0.05340	0.05709	0.07376	0.05391	0.15841
200	|Bias|	$\hat{\theta}$	0.00121	0.00999	0.00038	0.02320	0.01057	0.06045
		$\hat{\beta}$	0.01091	0.01905	0.01346	0.03233	0.01917	0.06509
	MSE	$\hat{\theta}$	0.03952	0.03660	0.03847	0.04113	0.03533	0.05583
		$\hat{\beta}$	0.03116	0.02769	0.02775	0.03133	0.02559	0.05065
300	|Bias|	$\hat{\theta}$	0.00272	0.00179	0.00495	0.00980	0.00154	0.03463
		$\hat{\beta}$	0.00411	0.00901	0.00628	0.01818	0.00876	0.03716
	MSE	$\hat{\theta}$	0.02479	0.02343	0.02761	0.02849	0.02367	0.02933
		$\hat{\beta}$	0.01625	0.01477	0.01757	0.01870	0.01490	0.02098

7. Data Analysis

Here we show the flexibility of the TME distribution. For this purpose, we used one real data set. The data set was analyzed by Lee (1992) and also by Hamedani, (2013). It consists of 121 observations of survival duration of patients. The following are the observations: 0.3, 0.3, 4, 5, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21, 21, 21.1, 23, 23.4, 23.6, 24, 24, 27.9, 28.2, 29.1, 30, 31, 31, 32, 35, 35, 37, 37, 37, 38, 38, 38, 39, 39, 40, 40, 40, 41, 41, 41, 42, 43, 43, 43, 44, 45, 45, 46, 46, 47, 48, 49, 51, 51, 51, 52, 54, 55, 56, 57, 58, 59, 60, 60, 60, 61, 62, 65, 65, 67, 67, 68, 69, 78, 80, 83, 88, 89, 90, 93, 96, 103, 105, 109, 109, 111, 115, 117, 125, 126, 127, 129, 129, 139, 154.

The exponential distribution is proposed by (Epstein, 1958)

$$g(x;\beta) = \frac{1}{\beta}\mathrm{e}^{-\frac{x}{\beta}}.$$

The Moment exponential (ME) distribution is given by (Iqbal, 2012)

$$g(x;\beta) = \frac{x}{\beta^2}\mathrm{e}^{-\frac{x}{\beta}}.$$

The exponentiated Moment exponential (EME) distribution is proposed by (Hasnain, 2015)

$$f(x) = \frac{\alpha}{\beta^2} x e^{\frac{-x}{\beta}} \left\{1 - \left(\frac{x+\beta}{\beta}\right) e^{\frac{-x}{\beta}}\right\}^{\alpha-1}.$$

The Kumaraswamy Moment exponential (KwME) distribution is given by Hashmi et al. (2017)

$$f(x) = \frac{abx}{\beta^2} e^{\frac{-x}{\beta}} \left\{1 - \left(\frac{x+\beta}{\beta}\right) e^{\frac{-x}{\beta}}\right\}^{a-1} \left[1 - \left\{1 - \left(\frac{x+\beta}{\beta}\right) e^{\frac{-x}{\beta}}\right\}^{a}\right]^{b-1}, \qquad x > 0$$

The Exponentiated Inverse Weibull (EIW) distribution is derived by (Flaih, 2012)

$$f(x) = \gamma\lambda\, x^{-\gamma-1} \left[e^{-x^{-\gamma}}\right]^{\lambda}, x > 0$$

The transmuted exponentiated Weibull (TEW) distribution is given by (Saboor, 2015)

$$f(x; \lambda, \beta, \gamma, a) = -\frac{1}{x} e^{-2(\beta x^{\gamma} + x\lambda)} (\gamma\beta x^{\gamma} + x\lambda)\left((a-1)e^{\beta x^{\gamma} + x\lambda} - 2\alpha\right)$$

Table 5: Descriptive statistics

Min.	Q1	Median	Q3	Mean	Max.
0.3	17.5	40	60	46.3289	154.0

Table 6: MLEs for real data sets

Model	Parameter Estimates				$-\hat{l}$	A*	W*
$TME\ (\hat{\beta}, \hat{\theta})$	25.1676	0.22479	-	-	582.81	1.0924	0.1180
$ME\ (\hat{\beta})$	23.1645	-	-	-	583.15	1.3380	0.1450
$E\ (\hat{\beta})$	0.02158	-	-	-	585.13	2.7083	0.4606
$KwME(\hat{a}, \hat{b}, \hat{\beta})$	0.86960	4999.99	4999.99	-	588.35	17.795	3.1798
$EME(\hat{a}, \hat{\beta})$	0.95981	20.0001	-	-	585.79	3.4181	0.4766
$EIW(\hat{\gamma}, \hat{\lambda})$	0.66602	6.81582	-	-	636.61	9.8468	1.6015
$TEW(\hat{a}, \hat{\gamma}, \hat{\beta}, \hat{\lambda})$	8.45*10-9	0.13193	$2.543*10^{-11}$	0.02158	585.13	2.7083	0.4606

In Table 6, different measures of goodness of fit are presented, and based on these measures, we have compared the TME distribution with some other distributions. These accuracy measures include AIC, BIC, Anderson-Darling statistic (A*) and Cramer-von Mises (W*), and log-likelihood. These measures show that TME distribution gives a better fit to this data set when compared with other distributions. The lower the values, the better the model is.

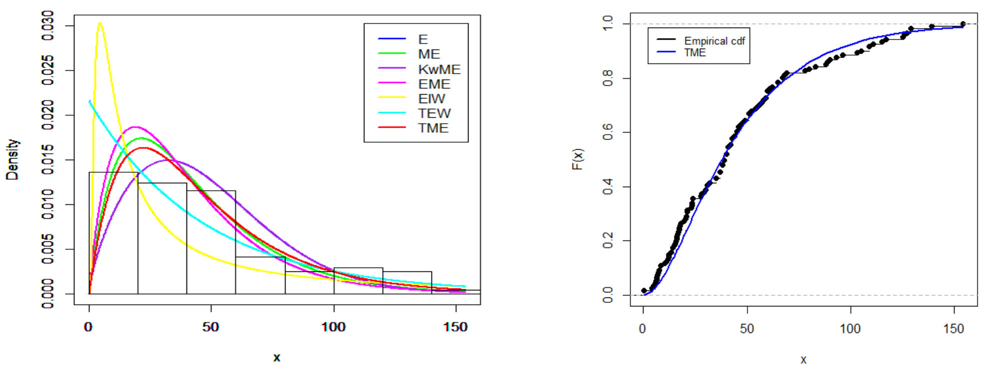

Figure 2: The fitted TME density and other densities for the first data set (left) and cdf (right).

Because one of the main goals of this study is to find the best estimators for the data set, a variety of estimating approaches have been used. Table 7 shows the various estimators for the data set that are based on various estimating methods.

Table 7. Estimation and Goodness for data set.

Method ↓ Statistics →	θ	β	KS	P-value
MLE	0.2248	25.168	0.075	0.50
ADE	0.3299	26.135	0.072	0.60
CVME	0.2955	25.799	0.072	0.60
OLSE	0.3005	25.881	0.073	0.50
WLSE	0.3258	26.158	0.072	0.50
MPSE	0.2725	25.891	0.076	0.50

It is noted that all the estimation techniques are well for assessing the data set, however, the ADE and CVME are the most effective.

8. Conclusion

The two-parameter TME distribution is a generalized distribution of ME distribution, and the shape PDF and hazard curves of the proposed model are studied at two important points; that is, origin and infinity. The density of TME distribution is modal and its hazard curve assumes increasing or upside-down bathtub behaviour. We obtain its characterizations based on truncated moments and hazard functions. A simulation study is also done, and it shows that mean square error decreases as sample size increases and ML estimators are efficient estimators. The application of the TME model to a data set shows that it gives a better fit. Since the TME model is a parsimonious model among other competitor models and hopefully is a simple model, it would provide wider applications in different fields of science and reliability.

Appendix A. R code

```
# Probaility density curves
rm(list=ls())
x=seq(0,10,length=1000)
Haq=function(par,x){
beta=par[1]
theta=par[2]
((x/beta^2)*(exp(-x/beta)))*(1- theta +2* theta *((1+(x/beta))*exp(-x/beta)))
}
y=3
plot(x,Haq(c(1.5,1.0),x),type="l",ylab="f(x)",ylim=c(0,y))
lines(x,Haq(c(1.5,-0.8),x),lwd=2,lty=1,col="red")
lines(x,Haq(c(1.5,-0.7),x),lwd=2,lty=1,col="purple")
lines(x,Haq(c(1.5,-0.5),x),lwd=2,lty=1,col="cyan")
```

```
lines(x,Haq(c(1.5,-0.25),x),lwd=2,lty=1,col="green")
lines(x,Haq(c(1.5,-0.1),x),lwd=2,lty=1,col="black")
lines(x,Haq(c(1.5,0.0),x),lwd=2,lty=1,col="blue")
colors <- c("red","purple","cyan", "green","black","blue")
labels <- c(expression(paste(beta," = 1.5 ",theta," = 1.0 ")),
expression(paste(beta," = 1.5 ", theta," = -0.8 ")),
expression(paste(beta," = 1.5 ", theta," = -0.7 ")),
expression(paste(beta," = 1.5 ", theta," = -0.5 ")),
expression(paste(beta," = 1.5 ", theta," = -0.1 ")),
expression(paste(beta," = 1.5 ", theta," = 0.0")))
legend("top", inset=.03,labels, lwd=2, lty=c(1), col=colors)
```

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