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On Exploring Order Effect and Ties in Paired Preferences with Bayesian Analysis

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Abstract

The paired comparison technique is used to rank stimuli on the basis of preference data obtained through presenting stimuli to some respondents in the form of pairs and asking them to record their preferences of the stimuli based on sensory evaluations. In this study, the Kuk (1995) model for paired comparisons, which accommodates home-ground advantage (equivalent to the order effect) and draws/ties in the paired comparison experiments, is considered for analysis in the Bayesian framework. Worth as well as tie parameters are estimated for both at home-ground and away-from-home matches. The entire estimation procedure is illustrated using a real dataset.

Keywords: Bayesian Analysis, Home-ground and away-from-home effects, Kuk's Model, Paired comparisons, Performance ranking, Ties/draws; Uninformative priors

Mathematical Subject Classification: 62J15, 62F07; 62F10; 62E15.

1. Introduction

In the method of paired comparisons, respondents are presented with stimuli in pairs and are asked to prefer one on the basis of sensory evaluations. If allowed, they may declare ties rendering the two stimuli equal in worth. By repeating this experiment, a fixed number of times under balanced or un-balanced patterns, preference datasets are generated and expressed in preference matrix. The preference matrix is then analyzed using the paired comparison models, which quantify these qualitative observations into ranks which are used to order the stimuli under study on the basis of their worth. The applications of the paired comparison method can be witnessed in different spheres of life ranging from consumers' behaviour in Psychology to ranking of universities and sport-teams.

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Bayesian inference is a technique widely used in the analysis of different types of datasets. The major difference between the classical and the Bayesian methods is that the former considers parameters as constant quantities and base their entire inference only on the current information (data) and do not utilize prior information about the unknown parameters; whereas the later incorporates prior information about the population parameters gathered through some reliable methods. The Bayesians treat population parameters as random quantities which thus has probability distributions. The data in the form of the likelihood function and the prior distribution are then merged together to yield posterior distribution where the current information is updated by the prior information. The posterior distribution is the workbench of the Bayesian statisticians, and they base all types of inferences on the posterior distribution.

The recent developments made in this field comprise construction of different paired comparison models and their extensions to include different factors affecting the preferences declared by the respondents about the competing stimuli as well as different estimation techniques like EM algorithm, method of moments, least squares, maximum likelihood, etc. David (1988) provides a nice review of the literature pertaining to the topic. Bradley and Terry (1952) and Thurstone (1927) assume the responses of the respondents to follow respectively the logistic and normal distributions. Bradley (1976) gives in detail the review of the work done on the method of paired comparisons considering different approaches to construct the paired comparisons models, their extensions for accommodating ties and inclusion of time factor to construct more dynamic models. He also discusses multiple comparisons along with certain relevant models along with certain other methods, like circular triads and the coefficient of concordance, quantification or scaling similar to discriminant analysis, ANOVA and iterative scoring system in paired comparisons. Glenn and David (1960), Joe (1990), Henery (1992) and Kuk (1995) consider the Thurstonian model to extend it for ties and home ground (order) effects. Stern (1990) considers the gamma distribution in devising the paired comparison models. Rao-Kupper (1967) and Davidson and Beaver (1977) attempt to accommodate ties and order effects in the Bradley-Terry model. Abbas and Aslam (2011) accommodate quantitative weighs in qualitative paired comparisons via the Bradley-Terry model. Different models for the paired comparisons have been studied in Bayesian paradigm by a large number of authors (Leonard, 1977; Chen and Smith, 1984; Aslam, 2002, 2003 & 2005). The recent developments regarding the study of the paired comparison models in Bayesian framework maybe seen in Abbas and Aslam (2009, 2010, 2012).

The Kuk (1995)'s model accommodates the home-ground and away-from-home effects on the strength and tie parameters of the teams under study. Whereas, we have also considered the effect of toss-results (winning or losing a toss) on the strength and tie parameters of the teams under study. Moreover, an attempt has also been made to incorporate prior information and the analysis is carried out in the Bayesian framework. Section 2 elaborates the Kuk (1995) model. The Bayesian analysis of the Kuk (1995) model under study has been carried out in Section 3.

Section 4 illustrates the entire estimation procedure using a real dataset. Section 5 concludes and discusses the entire study.

2. The Kuk's model for paired comparisons

Kuk (1995) considers an extension of Glenn and David (1960) model for paired comparison which itself is an extension of the famous Thurstone-Mosteller model, which assumes that the judges' responses follow the normal distribution. Glenn and David consider tie or draws by accommodating an additional parameter for this having a non-negative value. Kuk (1995) considers the effect of home-ground on the strength and tie parameter the teams of soccer. He splits the tie parameter into two parts, and each part is attributed to the home-ground and away-from-home effects. Kuk (1995) model is defined as follows:

$$\left.\begin{aligned} p_{i.ij} &= \Phi\left(\theta_i^H - \theta_j^A - \tau_{ij}\right) \\ p_{o.ij} &= \Phi\left(\theta_i^H - \theta_j^A + \tau_{ij}\right) - \Phi\left(\theta_i^H - \theta_j^A - \tau_{ij}\right) \\ p_{j.ij} &= 1 - p_{i.ij} - p_{o.ij} \end{aligned}\right\}$$

where $\Phi(.)$ denotes the cumulative distribution function of the Gaussian distribution, $p_{i.ij}$ denotes the winning probability of team i (playing at home-ground H) against team j (playing away-from-home A), τ_{ij} stands for the tie parameter and is split into two parts $\tau_i^H$and τ_j^A representing the tie values for the teams i (playing at home ground H) and j (playing away-from-home A) respectively, $\theta_1^H, \theta_2^H, \dots, \theta_t^H$ represent the worth/strength of t teams when playing at home grounds and $\theta_1^A, \theta_2^A, \dots, \theta_t^A$, the worth of the t teams when playing away-from-home. Also $\tau_1^H, \tau_2^H, \dots, \tau_t^H$ and $\tau_1^A, \tau_2^A, \dots, \tau_t^A$ denote the tie parameters for the t teams playing respectively at home and away-from-home. The first subscript denotes the team playing at his home-ground and $\theta_i^H - \theta_i^A$ denotes the home-ground effect.

3. Bayesian analysis of the Kuk's model

Now for the estimation of the model parameters in the Bayesian framework, we use the likelihood function which needs a regular distribution being followed by the observed data. And as far as the present situation is concerned, there are total three categories of the outcomes for each trial, i.e., winning, losing and drawing the match and hence follow the multinomial distribution with three categories of outcomes.

Before proceeding further, we first explain the notations used in this study. Of the total a_{ij} and a_{ji} matches played respectively at team i's and j's home-ground, $a_{i.ij}$and $a_{j.ij}$ be the respective number of matches won and lost by team i played at his home grounds, whereas $a_{i.ji}$ and $a_{j.ji}$ respectively denote the number of matches won and lost by team i played at team j's home grounds, $a_{o.ij}$ and $a_{o.ji}$ be the respective number of tied matches played respectively at i's and j's home grounds and obviously $a_{ij} = a_{i.ij} + a_{j.ij} + a_{o.ij}$ and $a_{ji} = a_{i.ji} + a_{j.ji} + a_{o.ji}$. Similarly,

$p_{i.ij}$, $p_{j.ij}$and $p_{o.ij}$ denote team i's corresponding probabilities for winning, losing and drawing a match played at the team i's home grounds and $p_{j.ij}$, $p_{i.ji}$ and $p_{o.ji}$ represent respectively the team j's probabilities of winning, losing and drawing a match played at j's home grounds. As a rule of thumb, the first subscript of 'p' before the dot (.) indicates the winner team at home ground. Also $n_{ij} = a_{ij} + a_{ji}$ be the total matches played between the teams i and j irrespective of the grounds. Then of the total n_{ij} matches played between the teams i and j at i's home grounds, the probability that $a_{i.ij}$ are won by team i , $a_{j.ij}$ are won by team j and $a_{o.ij}$ matches are tied between the teams$i$ and j, is given by

$$P_{ij} = c_{ij}(p_{i.ij})^{a_{i.ij}}(p_{j.ij})^{a_{j.ij}}(p_{o.ij})^{a_{o.ij}}, \qquad \text{for all } i \neq j,$$

where $c_{ij} = a_{ij}!/(a_{i.ij}!\,a_{j.ij}!\,a_{o.ij}!)$ is the normalizing constant. The similar probability between the teams i and j while playing at team$j$'s home is

$$P_{ji} = c_{ji}(p_{i.ji})^{a_{i.ji}}(p_{j.ji})^{a_{j.ji}}(p_{o.ji})^{a_{o.ji}}, \qquad \text{for all} i \neq j,$$

where the normalizing constant $c_{ji} = a_{ji}!/(a_{i.ji}!\,a_{j.ji}!\,a_{o.ji}!)$.

Now for the Bayesian estimation of the worth parameters, the home-ground advantage and the tie parameters both at home grounds and away-from-home, we need the posterior distribution of all the unknown parameters, which combines the likelihood function L based on the sample data $\mathbf{X}$ and a prior distribution via the Bayes theorem. So, the likelihood function L is defined as

$$L(\boldsymbol{X}|\boldsymbol{\theta}) = \prod_{i \neq j}^{t} P_{ij,} \qquad \text{for all } i(\neq j) = 1,2,\dots,t,$$

where $\boldsymbol{\theta} = (\theta_i^H, \theta_j^A, \tau_i^H, \tau_j^A)$ for all $i(\neq j) = 1,2,\dots,t$. For the Bayesian analysis of the model, we may use some distribution with well-behaved form as an informative prior. We may analyze the data using the Jeffreys' prior as well as the uniform prior. However, Aslam (2002) shows that the Bayesian estimates using the uniform prior and the Jeffreys' prior significantly agree. Hence, we use the uniform distribution as the non-informative, defuse or flat prior which is proportional to a constant not depending on the value of the unknown parameters $\boldsymbol{\theta}$ and may be written as $\pi(\boldsymbol{\theta}) \propto 1$, for all $-\infty \leq \underline{\theta} \leq \infty$.

The posterior distribution $P(\boldsymbol{\theta}\,|\,\mathbf{X})$ under this uniform prior U (- ∞, ∞) is given by

$$P(\boldsymbol{\theta}|\boldsymbol{X}) \propto \pi(\boldsymbol{\theta}).L(\boldsymbol{X},\boldsymbol{\theta})$$
or
$$P(\boldsymbol{\theta}|\boldsymbol{X}) = c\pi(\boldsymbol{\theta})(\boldsymbol{X},\boldsymbol{\theta}) \tag{1}$$

where c is known as the normalizing constant and is independent of the unknown parameters $\boldsymbol{\theta}$. Now, the consequent analysis of the data will be carried out on the basis of the posterior distribution given in (1). The computational algorithm may be described as follows for the ease of the readers:

(i) Collect dataset from multinomial distribution that meets the needs of the study.
(ii) Define and derive likelihood function.
(iii) Define suitable prior distribution.
(iv) Merge likelihood function with prior distribution via the Bayes theorem to derive posterior distribution.
(v) Differentiate the posterior distribution with respect to the unknown parameter(s) and equate them to zero to get as many equations as the number of parameters.
(vi) Solve the equations for the unknowns to get the desired estimates.

4. An illustrative numerical example

Here five cricket teams, namely Australia, England, India, Pakistan and New Zealand, have been compared with regards to their strength/worth both at their home grounds and away-from-home. We have used the current real dataset for the aforesaid five teams for the years 2000 onwards for the analysis, which can be accessed through the website www.howstat.com. The data are given in Table 1.

Table 1: Home wins, losses and ties data for five cricket teams

Pairs(i, j)	a_{ij}	$a_{i.ij}$	$a_{j.ij}$	$a_{o.ij}$
(1, 2)	6	6	0	0
(2, 1	11	2	7	2
(1, 3)	10	9	1	0
(3, 1)	9	3	6	0
(1, 4)	6	2	4	0
(4, 1)	13	2	10	1
(1, 5)	14	10	4	0
(5, 1)	0	0	0	0
(2, 3)	7	3	3	1
(3, 2)	12	7	5	0
(2, 4)	2	0	2	0
(4, 2)	5	3	2	0
(2, 5)	6	2	4	0
(5, 2)	8	5	3	0
(3, 4)	3	1	2	0
(4,3)	7	5	2	0
(3, 5)	7	2	5	0
(5, 3)	10	3	7	0
(4, 5)	9	6	3	0
(5, 4)	8	8	0	0

In Bayesian inference we usually use the mathematical expectation or mode of the posterior distribution assuming respectively the squared error loss and the absolute error loss functions. We may find/use the means as the estimates of the unknown parameters, but it is accomplished through the quadrature method of cumbersome numerical integration which involves evaluation of multiple integrals of very high dimensionality (18-dimensional integration is needed for the current study with two constraints on the parametric values). So, we resort to finding the posterior modes which involves the solution of 20 simultaneous equations obtained by equating to zero the first derivatives about all unknown parameters of the logarithm of the posterior distribution. This is accomplished through developing a computer program in SAS software using the PROC SYSNLIN command[5] to get the Bayesian estimates in the form of posterior modes along with their standard errors. The resulting estimates, along with their sstandard errors (in brackets), are reported in Table 2.

Table 2: The model estimates of the parameters

Teams	$\theta_i{}^H$	$\theta_j{}^A$	$\tau_i{}^H$	$\tau_j{}^A$	Home-ground Effect
Australia	0.139942 (0.43090)	0.206024 (0.52015)	-1.233382 (2.67487)	-1.122180 (2.04689)	-0.066082
England	-0.121743 (0.42288)	-0.111402 (0.40652)	-1.002964 (3.10129)	-1.168759 (1.95400)	-0.010341
India	-0.025917 (0.17262)	-0.055597 (0.29894)	-1.288520 (1.83401)	-1.352645 (1.88204)	0.081514
New Zealand	0.028055 (0.25092)	0.006530 (0.02695)	-1.193209 (2.14401)	-1.0966531 (1.83186)	0.021525
Pakistan	0.146668 (0.85235)	-0.018695 (0.11009)	-1.95936 (1.56207)	-1.302672 (1.67892)	0.165363

From these results, it becomes quite obvious that the five teams may be ranked as:

Australia → Pakistan → New Zealand → India → England,

where the symbol '→' may be read as 'precedes'. Moreover, the results shown in Table 2 also assign the same ranks to Australia and Pakistan and render the rest of the teams as almost equal in worth.

5. Conclusions

From the results shown in Table 2, it becomes well-evident that the five teams may be ranked on the basis of their strength/worth/merit as Australia being the number one, Pakistan the second, New Zealand the third, India the forth, and England being the fifth one. It is interesting to note that same the order of ranks is exhibited both at home grounds and the away-from-home. As far as the nature of the home-ground effect is concerned, the last column of Table 2 makes it quite evident that the home ground advantage has different adverse/favourable effects on the teams. The home ground is advantageous for the last three teams, i.e., India, New Zealand, and Pakistan, which means that the home grounds have added to their worth and they have done well in these grounds as compared to that in the away-from-home grounds. Pakistan is the highest in getting benefit of this factor and India and New Zealand being respectively the second and third ones. From the current data, it is

[5] The SAS codes are not provided here to save space but may be had from the author on request.

also noticed that of the five competing teams, Australia is the highly affected by the home-ground effect and England with second in suffering from this factor. It is clear from the results that all the tie parameters at home and away-from-home are negative. Perhaps, it might be due to the fact that there is very small probability of drawing a one-day-match and the dataset under study shows that only four matches are drawn out of total 153 matches. So, the negative values of the tie parameter may indicate that there is no significance of tie parameters for the current dataset, and it may be declared as the nuisance parameter. The sums of the tie parameters for the home grounds as well as the away-from-home grounds are approximately equal as assumed by Kuk (1995).

We have conducted the Bayesian analysis using just the non-informative prior. However, the informative priors may use to incorporate the expert opinions in the analysis. Further dimensions of the important factors affecting the strength and tie parameters may also be modelled via the Kuk (1995) model or the other models existing in the literatures.

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