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A Bivariate Mixture of Chi- Normal Distribution and Bounded Student's t-Distribution

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Abstract

A bivariate mixture of Chi and Normal distribution is introduced by using Jacobian transformation and rescaling the scale, shape parameters of existing Mckay's Bi-variate Gamma distribution which is considered to be the chi-normal distribution and its marginals are univariate chi and normal distribution respectively. Conditional distribution, various generating functions and its constants are shown. Similarly, the authors explored a new Bounded student's t-distribution in the sampling literature based on chi-normal mixture and studied its characteristics, computed the percentage points at 5% and 1% level by using Maple version 16. Three-dimensional probability surfaces are visualized the shape of chi-normal densities and two-dimensional probability curves shown the shape of Bounded student's t density heuristically. Finally, the authors confirmed the limiting distribution of bounded student's t distribution is the standard normal and the application of Bounded student's t distribution was also numerically illustrated.

Keywords: Mckay's Bivariate Gamma distribution, bivariate mixture, chi-normal distribution, Generating functions, Bounded student's t distribution, three dimensional probability surfaces, Two dimensional probability curves, limiting distribution

Mathematical Subject Classification: 62H10

0. Some Preliminaries

Explicit expressions for the PDF of chi-normal distribution, Bounded-t distribution and the Calculation of constants, generating functions involves several special functions (Prudnikov et al. (1986) & Gradshteyn and Ryzhik (2000)) and they are given as follows:

i. The Hyper geometric function of two variables defined by

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$$F_1(a,b,c,d;x,y) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(a)_{k+l}(b)_k(c)_l}{(d)_{k+l}}\left(\frac{x^k y^l}{k!\,l!}\right).$$

ii. The rising factorial or Pochammer symbol is given as
$(a)_k = a(a+1)(a+2)\ldots\ldots(a+k-1)$

iii. The integral representation of the Gamma and the lower incomplete Gamma function are defined by

$$\Gamma(a) = \int_0^{\infty} x^{a-1}e^{-x}\,dx$$

and

$$\gamma(a,x) = \int_0^{x} t^{a-1}e^{-t}\,dt,$$

respectively.

iv. The integral representation of the Beta function, Incomplete Beta function and Regularized Incomplete Beta function are defined as

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt,$$

$$B(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt$$

and

$$I_x(a,b) = \frac{1}{B(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}\,dt,$$

respectively.

v. The integral representation of the Tricomi's Confluent hyper geometric function is defined by

$$U(a,b,x) = \frac{1}{\Gamma(a)}\int_0^{\infty} t^{a-1}(1+t)^{b-a-1}e^{-xt}\,dt$$

vi. The integral representation of the Digamma function due to Gauss is defined by

$$\Psi(z) = \int_0^{\infty}\left(\frac{e^{-x}}{x} - \frac{e^{-zx}}{1-e^{-x}}\right)\,dx$$

vii. The Bessel function of the first Kind is defined by

$$J_\alpha(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(k+\alpha+1)}(x/2)^{2k+\alpha}$$

viii. The Modified Bessel function of the first kind is defined by

$$I_\alpha(x) = i^{-\alpha}\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(k+\alpha+1)}(x/2)^{2k+\alpha}$$

1. Introduction

The finest works of Titterington et. al (1985), McLachlan and Basford (1988), McLachlan and Peel (2000) provided a comprehensive import of finite mixture distributions and the application, inference of finite mixture models. Balakrishnan (2009) examined varied forms of bivariate gamma distributions. Jensen's (1970) generalization of Kibble's distribution resulted in Jensen's Bivariate Gamma Distribution where density function has as a diagonal expansion in terms of Laguerre polynomials and Orthogonal polynomials. McKay (Mckay,1934) proposed the bivariate gamma distribution and its joint pdf given by

$$f_{X',Y'}(x',y') = \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} x'^{p-1}(y'-x')^{q-1}e^{-ay'}; \;\; 0 < x' < y' \;\; a,p,q > 0 \quad (1)$$

From (1) if $x = \sqrt{x'},\; y = \sqrt{y'}\,,\; p = 1/2,\; p + q = v/2 \; and \; a = 1/2$, then using one dimensional Jacobian of transformation, (1) becomes the joint density function of bivariate mixture of chi-half normal distribution. Now change the limits of x as $-y < x < +y$ and then the joint density becomes the bivariate mixture of chi-normal distribution. The introduction of this proposed distribution makes us to explore a new Bounded student's t distribution in the sampling literature and the proposed features of distributions were considered in the following sections.

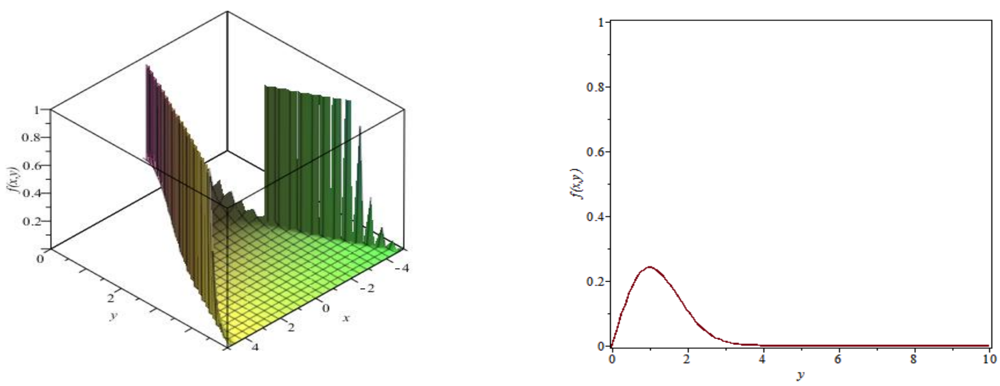
1.1. Bivariate mixture of Chi-normal distribution

Definition 1.1: Let X and Y be the random variables that follow Bivariate Chi-normal distribution with degrees of freedom v, then its density function is defined as

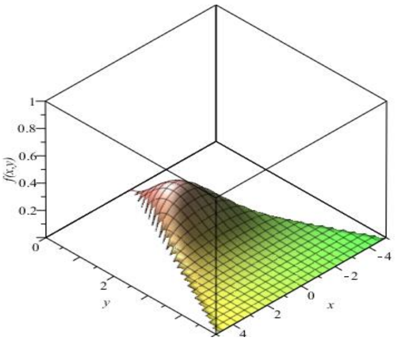
$$f_{X,Y}(x,y) = \frac{2(1/2)^{v/2}}{\sqrt{\pi}\Gamma((v-1)/2)} y(y^2 - x^2)^{((v-1)/2)-1} e^{-y^2/2} \quad (2)$$

where $-y < x < +y$.

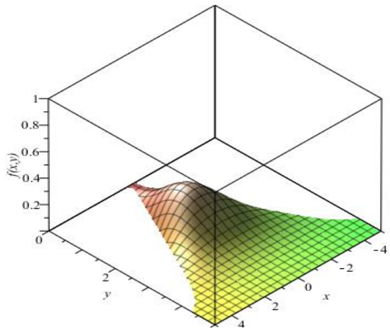
Result 1.2: The following probability surfaces of (2) for the selected values of degrees of freedom v are visualized below:



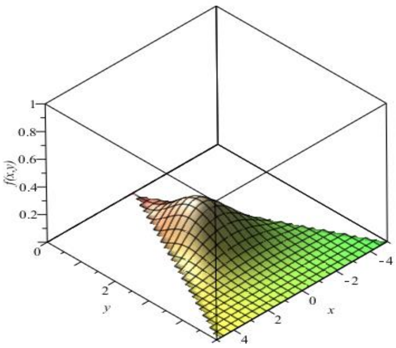
(a) v=2 (b) v=3



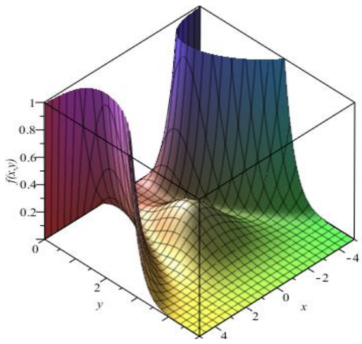
(c) v=4



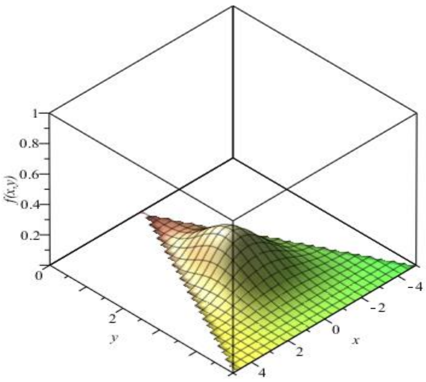
(d) v=5



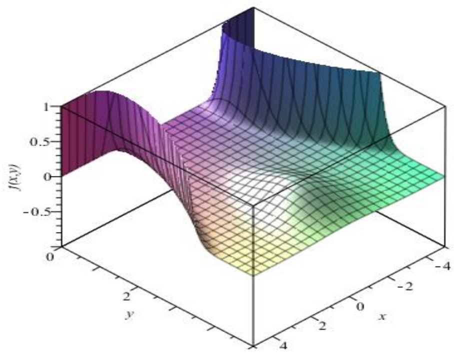
(e) v=6



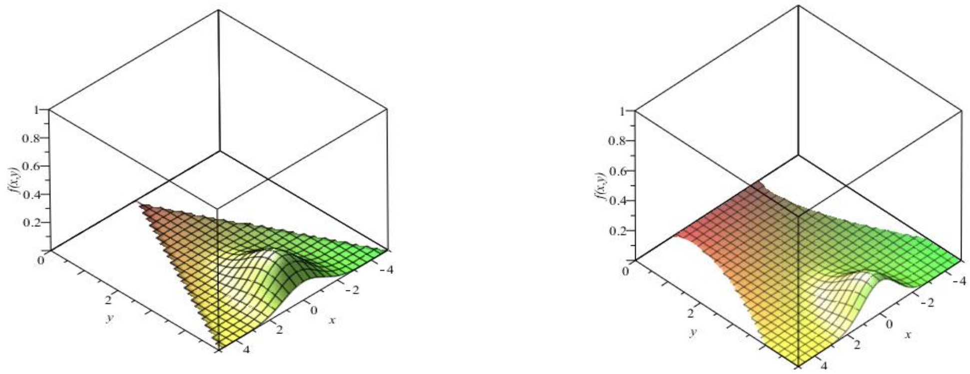
(f) v=7



(g) v=8



(h) v=15



(i) v=20 (j) v=25

Figure 3. probability surfaces of (2) for different values of degrees of freedom v

Theorem 1.3: The cumulative distribution function of (3) is defined by

$$F_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}\Gamma((v-1)/2)} \sum_{k=0}^{((v-1)/2)-1} \binom{((v-1)/2)-1}{k} (-12)^k \times \left(\frac{x^{2k+1}+y^{2k+1}}{2k+1}\right) \gamma\big(((v-1)/2)k, y^2 2\big) \tag{3}$$

where $\gamma(.,.)$ is the lower incomplete Gamma function.

Proof: Let the cumulative distribution function of a bivariate distribution is

$$F_{X,Y}(x,y) = \int_{-y}^{x}\int_{0}^{y} f(U,V)\,dUdV$$

$$F_{X,Y}(x,y) = \frac{2(1/2)^{v/2}}{\sqrt{\pi}\Gamma((v-1)/2)} \int_{-y}^{x}\int_{0}^{y} V(V^2-U^2)^{((v-1)/2)-1} e^{-V^2/2}\,dUdV$$

$$= \frac{2(1/2)^{v/2}}{\sqrt{\pi}\Gamma((v-1)/2)} \int_{-y}^{x}\int_{0}^{y} V^{v-2}(1-U^2/V^2)^{((v-1)/2)-1} e^{-V^2/2}\,dUdV \tag{4}$$

Now using Binomial expansion in (4) for

$$(1-U^2/V^2)^{((v-1)/2)-1} = \sum_{k=0}^{((v-1)/2)-1} \binom{((v-1)/2)-1}{k} (-1)^k (1/V^2)^k (U^2)^k$$

Set $S = V^2/2$ and integrate it, then the final expression of CDF as

$$F_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}\Gamma((v-1)/2)} \sum_{k=0}^{((v-1)/2)-1} \binom{((v-1)/2)-1}{k} \times (-1/2)^k \left(\frac{x^{2k+1}+y^{2k+1}}{2k+1}\right) \gamma\big(((v-1)/2)-k, y^2/2\big)$$

where $\gamma\big(((v-1)/2)-k, y^2/2\big)=\int_0^{y^2/2} S^{((v-1)/2)-k-1}e^{-S}\,dS$ is the lower incomplete gamma function.

2. Marginal and conditional distributions

Theorem 2.1: The MGF of (2) is

$$f_X(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \quad \text{where } -\infty < x < +\infty \tag{5}$$

Proof: The marginal distribution of chi-normal mixture of X exists when the limits of Y is $|x|<y<\infty$. Therefore, marginal distribution of X is derived as

$$=\int_{|x|}^{\infty}\frac{2(1/2)^{v/2}}{\sqrt{\pi}\Gamma((v-1)/2)}y(y^2-x^2)^{((v-1)/2)-1}e^{-y^2/2}\,dy$$

$$=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\int_0^{\infty}\frac{(1/2)^{(v-1)/2}}{\Gamma((v-1)/2)}S^{((v-1)/2)-1}e^{-S/2}\,dS.$$

By setting $S=y^2-x^2$and the result is found to be

$$f_X(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \quad \text{where } -\infty < x < +\infty$$

Theorem 2.2: The marginal distribution of Y of (2) is

$$f_Y(y)=\frac{(1/2)^{(v/2)-1}}{\Gamma(v/2)}y^{v-1}e^{-y/2} \quad \text{where } 0<y<\infty, v>0 \tag{6}$$

Proof: The marginal distribution of chi-normal mixture of Y exists when the limits of X is$-y<x<+y$. Therefore, marginal distribution of Y is computed as

$$f_Y(y)=\int_{-y}^{+y} f_{X,Y}(x,y)\,d$$

$$=\int_{-y}^{+y}\frac{2(1/2)^{v/2}}{\sqrt{\pi}\Gamma((v-1)/2)}y(y^2-x^2)^{((v-1)/2)-1}e^{-y^2/2}$$

$$=\int_{0}^{y}\frac{4(1/2)^{v/2}}{\sqrt{\pi}\Gamma((v-1)/2)}y(y^2-x^2)^{((v-1)/2)-1}e^{-y^2/2}\,dx$$

$$=\frac{(1/2)^{(v/2)-1}}{\Gamma(v/2)}y^{v-1}e^{-y^2/2}\int_0^1\frac{\Gamma(v/2)}{\Gamma(1/2)\Gamma((v-1)/2)}S^{(1/2)-1}(-S)^{((v-1)/2)-1}dS$$

By Setting $S=x^2/y^2$, the final result is found to be

$$f_Y(y)=\frac{(1/2)^{(v/2)-1}}{\Gamma(v/2)}y^{v-1}e^{-y^2/2} \quad \text{where } 0<y<\infty, v>0.$$

Theorem 2.3: The PDF of conditional Chi-normal distribution of X on Y is

$$f_{X/Y}(x/y) = \frac{1}{yB(1/2,(v-1)/2)}(1-x^2/y^2)^{((v-1)/2)-1} \quad (7)$$

where $-y < x < +y$.

Proof: It is derived from $f_{X/Y}(x/y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$

Theorem 2.4: The PDF of Conditional Chi-normal distribution of Y on X is

$$f_{Y/X}(y/x) = \frac{2(1/2)^{(v-1)/2}}{\Gamma((v-1)/2)} y(y^2-x^2)^{((v-1)/2)-1} e^{-(y^2-x^2)/2} \quad (8)$$

where $|x| < y < \infty$.

Proof: It is derived from $f_{Y/X}(y/x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$.

3. Constants of Conditional and Bivariate Chi-normal distribution

Theorem 3.1: The r^{th} odd moments of the conditional bivariate mixture of Chi-normal distribution of X on Y does not exist, and the r^{th} even moments is shown as:

$$E_{X/Y}(x^{2r}/y) = y^{2r}\frac{B(r+(1/2),(v-1)/2)}{B(1/2,(v-1)/2)}, \qquad \text{where } v>1 \quad (9)$$

Proof: The r^{th} even moments of a distribution is

$$E_{X/Y}(x^{2r}/y) = \int_{-y}^{+y} x^{2r} f_{X/Y}(x/y)\,dx$$

$$= \int_{-y}^{+y} \frac{x^{2r}}{yB(1/2,(v-1)/2)} \times (1-x^2/y^2)^{((v-1)/2)-1}\,dx$$

$$= \int_{0}^{y} \frac{2x^{2r}}{yB(1/2,(v-1)/2)} \times (1-x^2/y^2)^{((v-1)/2)-1}\,dx$$

$$E_{X/Y}(x^{2r}/y) = \frac{B(r+(1/2),(v-1)/2)}{B(1/2,(v-1)/2)} y^{2r}, \text{where } v>1.$$

If r=1, then the second moment is $E_{X/Y}(x^2/y) = y^2/v$.

If r=2, then the fourth moment is $E_{X/Y}(x^4/y) = 3y^4/v(v+2)$.

If r=3, then the sixth moment is $E_{X/Y}(x^6/y) = 15y^6/v(v+2)(v+4)$.

Theorem 3.3: The r^{th} conditional moment of the conditional bivariate mixture of Chi-normal distribution of Y on X is

$$E_{Y/X}(y^r/x) = (1/2)^{(v-1)/2} x^{v+r-3} U\big((v-1)/2,(r/2)+1+(v-1)/2,x^2/2\big) \quad (10)$$

where $v>1$ and $U(.,.)$ is the Tricomi's confluent hyper geometric function.

Proof: The r^{th} moment of a distribution is

$$E_{Y/X}(y^r/x) = \int_{|x|}^{\infty} y^r f_{Y/X}(y/x)\,dy$$

$$= \frac{2(1/2)^{(v-1)/2}}{\Gamma\big((v-1)/2\big)} \int_{|x|}^{\infty} y^{r+1}(y^2-x^2)^{((v-1)/2)-1} e^{-(y^2-x^2)/2}\,dy \quad (11)$$

Then set $S = y^2 - x^2$ in (11) and change the limits, then

$$E_{Y/X}(y^r/x) = \frac{(1/2)^{(v-1)/2}}{\Gamma((v-1)/2)} \int_0^\infty (x^2 + S)^{r/2} S^{((v-1)/2)-1} e^{-S/2}\, dS$$
$$= \frac{(1/2)^{(v-1)/2} x^r}{\Gamma((v-1)/2)} \int_0^\infty (1 + S/x^2)^{r/2} S^{((v-1)/2)-1} e^{-S/2}\, dS \qquad (12)$$

Again from (12) set $W = S/x^2$ and integrate it, then the r^{th} order conditional moment of the distribution is given as

$$E_{Y/X}(y^r/x) = x^{r+v-3} \left(\frac{1}{\Gamma((v-1)/2)} \int_0^\infty W^{((v-1)/2)-1} (1 + W)^{(r/2)+1+((v-1)/2)-((v-1)/2)-1} e^{-(x^2/2)W}\, dW \right) \qquad (13)$$

Finally, (13) can be written in terms of Tricomi's Confluent hyper geometric function as

$$E_{Y/X}(y^r/x) = (1/2)^{(v-1)/2} x^{v+r-3} U\big((v-1)/2, (r/2) + 1 + (v-1)/2, x^2/2\big) \qquad (14)$$

where $v > 1$ and

$$U\big((v-1)/2, (r/2) + 1 + (v-1)/2, x^2/2\big) = \frac{1}{\Gamma((v-1)/2)} \int_0^\infty W^{((v-1)/2)-1} (1 + W)^{(r/2)+1+((v-1)/2)-((v-1)/2)-1} e^{-(x^2/2)W} dW$$

is the Tricomi's confluent hyper geometric function.

From (14), if r=1, then the Conditional expectation is

$$E_{Y/X}(y/x) = (1/2)^{(v-1)/2} x^{v-2} U\big((v-1)/2, (v+2)/2, x^2/2\big)$$

If r=2, then the second moment is

$$E_{Y/X}(y^2/x) = (1/2)^{(v-1)/2} x^{v-1} U\big((v-1)/2, (v+3)/2, x^2/2\big)$$

If r=3, then the third moment is

$$E_{Y/X}(y^3/x) = (1/2)^{(v-1)/2} x^{v} U\big((v-1)/2, (v+4)/2, x^2/2\big)$$

Theorem 3.4: If X and Y are jointly distributed according to (2) then the product moment is

$$E(x^m y^n) = 0 \text{ for odd } m \qquad (15)$$

Proof: The results follow on writing

$$E(x^m y^n) = \int_{-y}^{+y} \int_0^\infty x^m y^n \frac{2(1/2)^{v/2}}{\sqrt{\pi}\Gamma((v-1)/2)} y(y^2 - x^2)^{((v-1)/2)-1} e^{-y^2/2} dx dy \qquad (16)$$

From (16), x is symmetric and if m is odd, then the product moments do not exist, hence $E(x^m y^n) = 0$, for odd m.

Corollary-1. According to (2) then the Product moments of the mixture of Chi-normal variables are given as

$$E_{XY}(xy) = 0 \qquad (17)$$
$$COV_{X,Y}(x,y) = 0 \qquad (18)$$
$$\rho_{X,Y}(x,y) = 0 \qquad (19)$$

Therefore, (18) and (19) clearly visualize the Chi-normal variables are uncorrelated but not independent and the variables are jointly dependent. Comparing these results with (2), it is observed that uncorrelatedness does not imply independence. Hence the proposed density function of chi-normal mixture is unique and it's having several notable properties.

Theorem 3.5: The Joint Shannon's differential entropy of (2) is

$$h'_{X,Y}(x,y) = -(\omega_1(v) + \omega_2(v) + \omega_3(v) - v/2) \quad (20)$$

where

$$\omega_1(v) = log\left(2(1/2)^{v/2}/\sqrt{\pi}\Gamma\big((v-1)/2\big)\right)$$

$$\omega_2(v) = \big(log\, 2 + \Psi(v/2)\big)/2$$

$$\omega_3(v) = \big((v-3)/2\big)\left(log\, 2 + \Psi\big((v+1)/2\big) - 2/(v-1)\right)$$

and $\Psi(\ )$ is the di-gamma function, respectively.

Proof: It is found from

$$h'_{X,Y}(x,y) = -\int_{-y}^{+y}\int_{0}^{\infty} f_{X,Y}(x,y)\, log\left(f_{X,Y}(x,y)\right) dxdy$$

$$= -\int_{-y}^{+y}\int_{0}^{\infty} \frac{2(1/2)^{v/2}}{\sqrt{\pi}\Gamma((v-1)/2)} y(y^2 - x^2)^{((v-1)/2)-1} e^{-y^2/2}\ log\left(\frac{2(1/2)^{v/2}}{\sqrt{\pi}\Gamma((v-1)/2)} y(y^2-x^2)^{((v-1)/2)-1} e^{-y^2/2}\right) dxdy$$

$$h'_{X,Y}(x,y) = -(\omega_1(v) + \omega_2(v) + \omega_3(v) - v/2)$$

4. Generating functions

Theorem 4.1: The MGF of the bivariate chi-normal mixture is

$$M_{X,Y}(t_1,t_2) = \frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(t_1\sqrt{2}\right)^{2k}\left(t_2\sqrt{2}\right)^{m}}{2k!\,m!}\left(\frac{\Gamma(k+1/2)\Gamma\big((v+2k+m)/2\big)}{\Gamma\big((v/2)+k\big)}\right) \quad (21)$$

Proof: Let the MGF of a bivariate distribution is given as

$$M_{X,Y}(t_1,t_2) = \int_{-y}^{+y}\int_{0}^{\infty} e^{t_1x+t_2y}\frac{2(1/2)^{vz/2}}{\sqrt{\pi}\Gamma\big((v-1)/2\big)} y(y^2 - x^2)^{((v-1)/2)-1} e^{-y^2/2} dxdy$$

$$= \int_{-y}^{+y}\int_{0}^{\infty} e^{t_1x+t_2y}\frac{2(1/2)^{v/2}}{\sqrt{\pi}\Gamma\big((v-1)/2\big)} y^{v-2}(1 - x^2/y^2)^{((v-1)/2)-1} e^{-y^2/2} dxdy \quad (22)$$

From (22) Set $S = x/y$, expand the exponent into the sum of odd and even power series as $e^{(t_1y)S} = \sum_{k=0}^{\infty}\frac{(t_1y)^{2k+1}S^{2k+1}}{(2k+1)!} + \sum_{k=0}^{\infty}\frac{(t_1y)^{2k}S^{2k}}{2k!}$,break the symmetry, integrate it with respect to S as follows

$$M_{X,Y}(t_1,t_2)=\frac{4(1/2)^{v/2}}{\sqrt{\pi}\Gamma\big((v-1)/2\big)}\sum_{k=0}^{\infty}\frac{(t_1)^{2k}}{2k!}\int_0^1 S^{2k}(1-S^2)^{((v-1)/2)-1}dS\int_0^{\infty}y^{v+2k-1}e^{-y^2/2}e^{t_2y}dy \quad (23)$$

Since the odd powers are vanished from (23), Substitute the results of the integral and integrate with respect to *y*, finally the result is found to be

$$M_{X,Y}(t_1,t_2)=\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\big(t_1\sqrt{2}\big)^{2k}\big(t_2\sqrt{2}\big)^{m}}{2k!\,m!}\left(\frac{\Gamma(k+1/2)\Gamma\big((v+2k+m)/2\big)}{\Gamma\big((v/2)+k\big)}\right)$$

Theorem 4.2: The Cumulant of the bivariate mixture of chi-normal distribution is

$$C_{X,Y}(t_1,t_2)=-(1/2)\log\pi+\log\left(\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\big(t_1\sqrt{2}\big)^{2k}\big(t_2\sqrt{2}\big)^{m}}{2k!\,m!}\left(\frac{\Gamma(k+1/2)\Gamma\big((v+2k+m)/2\big)}{\Gamma\big((v/2)+k\big)}\right)\right) \quad (24)$$

Proof: It is found from $C_{X,Y}(t_1,t_2)=\log M_{X,Y}(t_1,t_2)$

Theorem 4.3: The CF of the bivariate chi-normal mixture is

$$\phi_{X,Y}(t_1,t_2)=\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\big(it_1\sqrt{2}\big)^{2k}\big(it_2\sqrt{2}\big)^{m}}{2k!m!}\left(\frac{\Gamma(k+1/2)\Gamma\big((v+2k+m)/2\big)}{\Gamma\big((v/2)+k\big)}\right) \quad (25)$$

Proof: Let the Cf of a bivariate distribution is given as

$$\phi_{X,Y}(t_1,t_2)=\int_{-y}^{+y}\int_0^{\infty}e^{it_1x+it_2y}\frac{2(1/2)^{v/2}}{\sqrt{\pi}\Gamma\big((v-1)/2\big)}y(y^2-x^2)^{((v-1)/2)-1}e^{-y^2/2}dxdy$$

$$=\int_{-y}^{+y}\int_0^{\infty}e^{it_1x+it_2y}\frac{2(1/2)^{v/2}}{\sqrt{\pi}\Gamma\big((v-1)/2\big)}y^{v-2}(1-x^2/y^2)^{((v-1)/2)-1}e^{-y^2/2}dxdy$$

It is evident from Theorem 4.1, the integration is obvious and expression of final CF is given as

$$\phi_{X,Y}(t_1,t_2)=\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\big(it_1\sqrt{2}\big)^{2k}\big(it_2\sqrt{2}\big)^{m}}{2k!\,m!}\left(\frac{\Gamma(k+1/2)\Gamma\big((v+2k+m)/2\big)}{\Gamma\big((v/2)+k\big)}\right)$$

Theorem 4.4: The survival function of bivariate chi-normal mixture is

$$S_{X,Y}(x,y)=1-\frac{1}{\sqrt{2\pi}\Gamma\big((v-1)/2\big)}\sum_{k=0}^{((v-1)/2)-1}\binom{((v-1)/2)-1}{k}\times(-1/2)^k\left(\frac{x^{2k+1}+y^{2k+1}}{2k+1}\right)\gamma\Big(\big((v-1)/2\big)-k,y^2/2\Big) \quad (26)$$

where $\gamma(.,.)$ is the lower incomplete Gamma function.

Proof: It is found from the following fact

$S_{X,Y}(x,y)=1-F_{X,Y}(x,y)$

Theorem 4.5: The hazard function of the bivariate chi-normal distribution is

$$h_{X,Y}(x,y)=\frac{\frac{2(1/2)^{v/2}}{\sqrt{\pi}\Gamma\big((v-1)/2\big)}y\big(y^2-x^2\big)^{((v-1)/2)-1}e^{-y^2/2}}{1-\frac{1}{\sqrt{2\pi}\Gamma\big((v-1)/2\big)}\sum_{k=0}^{((v-1)/2)-1}\binom{((v-1)/2)-1}{k}(-1/2)^k\left(\frac{x^{2k+1}+y^{2k+1}}{2k+1}\right)\gamma\Big(\big((v-1)/2\big)-k,y^2/2\Big)} \quad (27)$$

Proof: It is found from

$$h_{X,Y}(x,y) = \frac{f_{X,Y}(x,y)}{S_{X,Y}(x,y)}$$

and

$S_{X,Y}(x,y) = 1 - F_{X,Y}(x,y)$.

Theorem 4.6: The Cumulative hazard function of the bivariate chi-normal distribution is

$$H_{X,Y}(x,y) = -\log\left(1 - \frac{1}{\sqrt{2\pi}\Gamma((v-1)/2)}\sum_{k=0}^{((v-1)/2)-1}\binom{((v-1)/2)-1}{k}(-1/2)^k\left(\frac{x^{2k+1}+y^{2k+1}}{2k+1}\right)\gamma\left(((v-1)/2)-k, y^2/2\right)\right) \quad (28)$$

Proof: Let the Cumulative hazard function of a multivariate distribution

$$H_{X,Y}(x,y) = -\log\left(1 - F_{X,Y}(x,y)\right)$$

$$= -\log\left(S_{X,Y}(x,y)\right)$$

$$H_{X,Y}(x,y) = -\log\left(1 - \frac{1}{\sqrt{2\pi}\Gamma((v-1)/2)}\sum_{k=0}^{((v-1)/2)-1}\binom{((v-1)/2)-1}{k}(-1/2)^k\left(\frac{x^{2k+1}+y^{2k+1}}{2k+1}\right)\gamma\left(((v-1)/2)-k, y^2/2\right)\right)$$

5. Some Special Cases

Result 5.1

Table 1: The Special cases of (2) based on Jacobean transformation

Case No	Bivariate Mixtures	Transformation		Parameter
		X'	Y'	v
1	Inverse Chi-normal	-	$1/y$	v
2	Chi-log normal	e^x	-	v
3	inverse chi-log normal	e^x	$1/y$	v
4	Chi-square -normal	-	y^2	v
5	Rayleigh -normal	-	-	2
6	Uncorrelated uniform-Rayleigh	-	-	3

6. Bounded student's t distribution

Theorem 6.1: Bounded student's t ratio (t_b) is defined as the ratio of two uncorrelated, but jointly dependent standard normal variate $\left(x \sim N(0,1)\right)$ and chi-variate $(y \sim \chi_v)$ divided by the square root of its degrees of freedom v,then it's density function is given as

$$f(t_b; v) = \frac{1}{\sqrt{v}B(1/2,(v-1)/2)}\left(1 - \left(t_b/\sqrt{v}\right)^2\right)^{((v-1)/2)-1} \quad (29)$$

where $-\sqrt{v} \le t_b \le +\sqrt{v}$, $v > 1$.

Proof: From (29) and from the above definition, Bounded student's t ratio (t_b) can be written as

$$t_b = \sqrt{v}(x/y) \tag{30}$$

Using two dimensional Jacobian of transformation and change of variable technique set $y = u$ and $x = u(t_b/\sqrt{v})$,then applying the partial derivatives and compute the Jacobian determinant as

$$J = \frac{\partial(x,y)}{\partial(t_b,u)} = \begin{vmatrix} u/\sqrt{v} & t/\sqrt{v} \\ 0 & 1 \end{vmatrix}$$

$$= u/\sqrt{v} \tag{31}$$

Then using the above settings along with Jacobian determinant, the joint density of Bounded student's t ratio (t_b) and u can be given as

$$f(t_b,u) = \frac{2(1/2)^{v/2}}{\sqrt{\pi}\Gamma((v-1)/2)} u \left(u^2 - \left(u(t_b/\sqrt{v})\right)^2\right)^{((v-1)/2)-1} e^{-u^2/2} \times u/\sqrt{v} \tag{32}$$

where $-\sqrt{v} \le t_b \le +\sqrt{v}, 0 \le u < \infty$.

From (32), integrate with respect to u, then

$$f(t_b) = \frac{2(1/2)^{v/2}}{\sqrt{v}\sqrt{\pi}\Gamma((v-1)/2)} \left(1 - (t_b/\sqrt{v})^2\right)^{((v-1)/2)-1} \int_0^{\infty} u^{v-1} e^{-u^2/2}\, du$$

$$= \frac{2(1/2)^{v/2}}{\sqrt{v}\sqrt{\pi}\Gamma((v-1)/2)} \left(1 - (t_b/\sqrt{v})^2\right)^{((v-1)/2)-1} \left(\frac{\Gamma(v/2)}{2^{1-v/2}}\right) \tag{33}$$

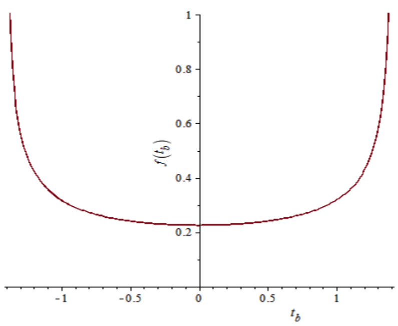
Simplifying (33), the final version of the density function Bounded student's t distribution with v degrees of freedom is given as

$$f(t_b; v) = \frac{1}{\sqrt{v}B(1/2,(v-1)/2)} \left(1 - (t_b/\sqrt{v})^2\right)^{((v-1)/2)-1} \tag{34}$$

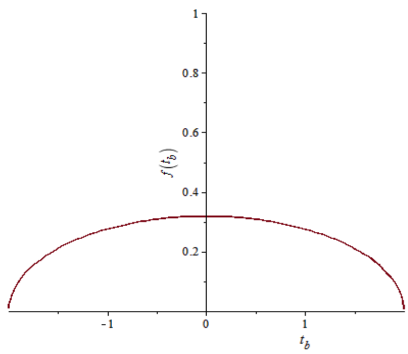
where $-\sqrt{v} \le t_b \le +\sqrt{v}, v > 1$.

From (34), it is the density function of Bounded student's t distribution which is a symmetric beta distribution, and it comes under the Type-II distribution of the Pearsonian system of frequency curves. The distribution is having a shape parameter v (degrees of freedom) and a normalizing constant $B(1/2,(v-1)/2)$ beta function.

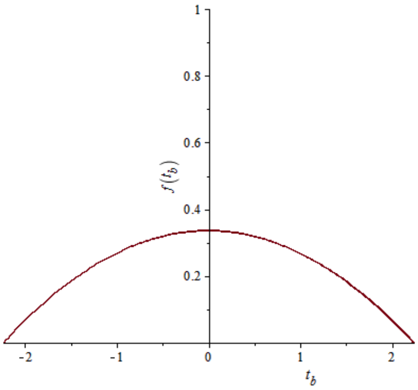
Result 6.2: The following probability curves of the bounded-t-distribution for the selected values of degrees of freedom v are visualized below.



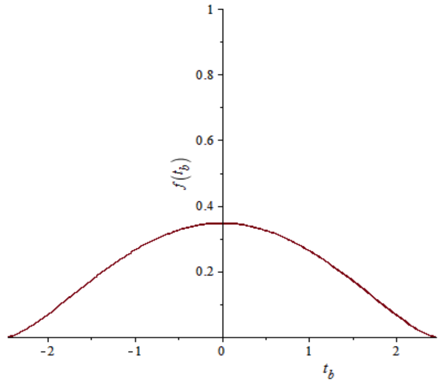
(a). *v=2*



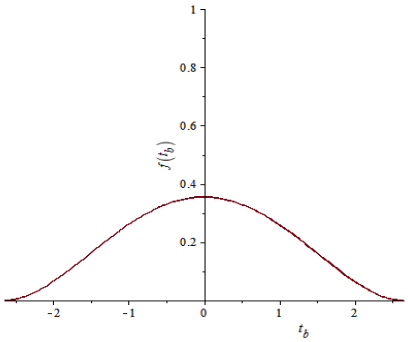
(b). *v=4*



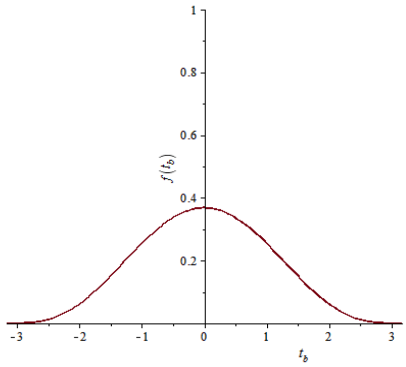
(c). *v=5*



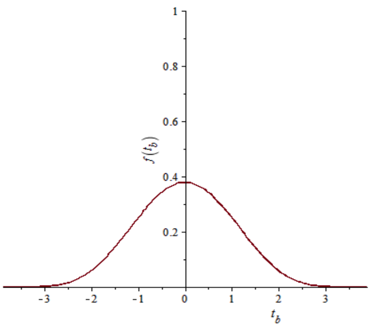
(d). *v=6*



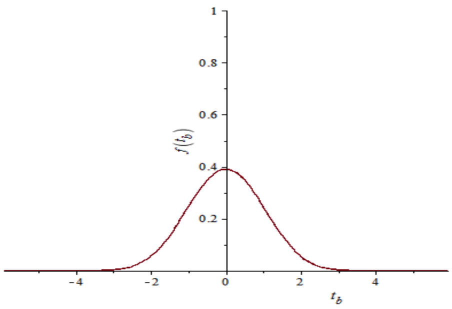
(e) *v=7*



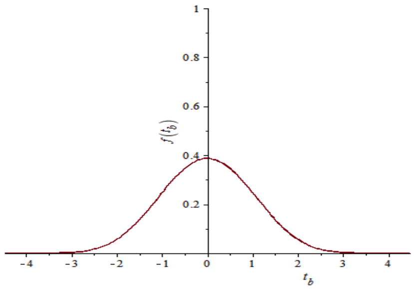
(f) *v=10*



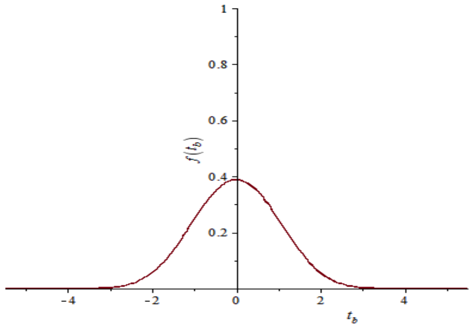
(g) v=15



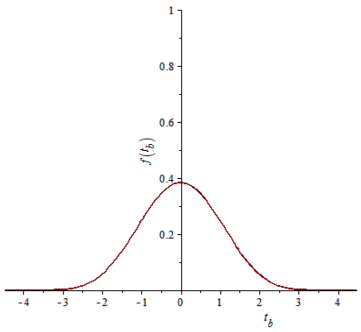
(h) v=20



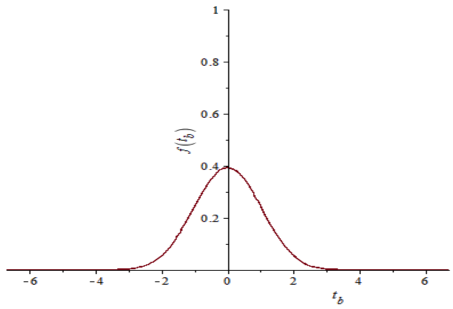
(i) *v=25*



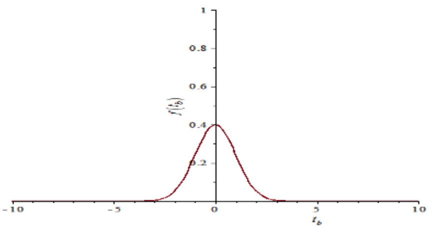
(j) *v=30*



(k) *v=35*



(l) *v=45*



(m) *v=∞*

Figure 2. Probability curves of the bounded-t-distribution for different values of degrees of freedom v

Theorem 6.3: The CDF of the Bounded student's t distribution is defined by

$$F(t_b) = \left(I_{t_b{}^2/v}(1/2,(v-1)/2) - 1\right)/2 \tag{35}$$

Where $I_{t_b{}^2/v}(1/2,(v-1)/2) = B(t_b{}^2/v; 1/2,(v-1)/2)/B(1/2,(v-1)/2)$ and $B(t^2/v; 1/2, v)$ are the regularized and incomplete beta function respectively.

Proof: Let the CDF of a distribution is

$$F(t_b) = \int_{-\sqrt{v}}^{t} f(S)dS$$

$$= \frac{1}{\sqrt{v}B(1/2,(v-1)/2)} \int_{-\sqrt{v}}^{t_b} \left(1 - \left(S/\sqrt{v}\right)^2\right)^{((v-1)/2)-1} dS$$

By setting $S^2/v = w$ and integrate with respect to w,the final result is

$$F(t_b) = \left(I_{t_b{}^2/v}(1/2, v) - 1\right)/2$$

Where $I_{t^2/v}(1/2, v) = \frac{1}{B(1/2,(v-1)/2)} \int_0^{t^2/v} w^{(1/2)-1}(1-w)^{((v-1)/2)-1} dw$ is the regularized beta function.

7. Constants of bounded t distribution

Theorem 7.1: The r^{th} odd order moment of the Bounded student's t-distribution with v degrees of freedom does not exist, then the r^{th} even order moment is given as

$$E(t^{2r}) = v^r \frac{B(r+(1/2),(v-1)/2)}{B(1/2,(v-1)/2)} \qquad \text{where } v > 1 \tag{36}$$

Proof: The r^{th} even order moment of the distribution is

$$E(t^{2r}) = \int_{-\sqrt{v}}^{+\sqrt{v}} t^{2r} f(t)\, dt$$

$$= \int_{-\sqrt{v}}^{+\sqrt{v}} \frac{t^{2r}}{\sqrt{v}B(1/2,(v-1)/2)} \left(1-\left(t/\sqrt{v}\right)^2\right)^{((v-1)/2)-1} dt$$

$$= \int_{0}^{\sqrt{v}} \frac{2t^{2r}}{\sqrt{v}B(1/2,(v-1)/2)} \left(1-\left(t/\sqrt{v}\right)^2\right)^{((v-1)/2)-1} dt$$

- (37)

From (37) by setting ${t_b}^2/v = u$ and integrate with respect to u ,the final result is found to be

$$E(t^{2r}) = v^r \frac{B(r+(1/2),(v-1)/2)}{B(1/2,(v-1)/2)} \qquad \text{where } v > 1 \tag{38}$$

If r=1, then the second moment is $E({t_b}^2) = 1$
If r=2, then the fourth moment is $E({t_b}^4) = 3v/(v+2)$
If r=3, then the sixth moment is $E({t_b}^6) = 15v^2/(v+2)(v+4)$ and so on.

Theorem 7.2: The Shannon's differential entropy of the Bounded student's- t - distribution is given as

$$h'(t_b) = log\left(\sqrt{v}B(1/2,(v-1)/2)\right) + \frac{(v-3)\Gamma\left((v+1)/2\right)}{(v-1)\Gamma\left((v-1)/2\right)} \times \left(\Psi(v/2) - \Psi\left((v+1)/2\right) + 2/(v-1)\right) \tag{39}$$

where $\Psi(.,\ .)$ is the di-gamma function.

Proof: It is found from

$$h'(t_b) = -\int_{-\sqrt{v}}^{+\sqrt{v}} f(t_b)\, log\, f(t_b) dt_b$$

$$= -\int_{-\sqrt{v}}^{+\sqrt{v}} \frac{1}{\sqrt{v}B(1/2,(v-1)/2)} \left(1-\left(t_b/\sqrt{v}\right)^2\right)^{((v-1)/2)-1} log\left(\frac{\left(1-\left(t_b/\sqrt{v}\right)^2\right)^{((v-1)/2)-1}}{\sqrt{v}B(1/2,(v-1)/2)}\right) dt_b \tag{40}$$

By setting $S = t_b/\sqrt{v}$ in (40) and using the change of variable technique, the expression will be

$$= log\left(\sqrt{v}B(1/2,(v-1)/2)\right)$$
$$-\int_{-1}^{+1}\frac{(v-3)/2}{B(1/2,(v-1)/2)}(1-S^2)^{((v-1)/2)-1}\,log(1-S^2)\,dS \quad (41)$$

Now integrate (41) with respect to S and simplify, then the final result is found to be

$$h'(t_b) = log\left(\sqrt{v}B(1/2,(v-1)/2)\right) + \frac{(v-3)\Gamma((v+1)/2)}{(v-1)\Gamma((v-1)/2)}\left(\Psi(v/2) - \Psi((v+1)/2) + 2/(v-1)\right)$$

8. Generating functions

Theorem 8.1: The MGF of the Bounded student's t distribution is given as

$$M_{t_b}(t) = \frac{\Gamma(v/2)}{(t\sqrt{v}/2)^{(v/2)-1}} I_{(v/2)-1}(t\sqrt{v}) \quad (42)$$

where $I(\;)$ is the modified Bessel function of Kind-1.

Proof: Let the MGF of a distribution is given as

$$M_{t_b}(t) = \int_{-\sqrt{v}}^{+\sqrt{v}} \frac{e^{t(t_b)}}{\sqrt{v}B(1/2,(v-1)/2)}\left(1-(t_b/\sqrt{v})^2\right)^{((v-1)/2)-1} dt_b$$

By setting $S = t_b/\sqrt{v}$ and using the change of variable technique, the expression will be

$$= \int_{-1}^{+1}\frac{e^{t(S\sqrt{v})}}{B(1/2,(v-1)/2)}(1-S^2)^{((v-1)/2)-1}dS \quad (43)$$

Now from (43) expand the exponent $\left(e^{t(S\sqrt{v})} = \sum_{k=0}^{\infty}\left(t^{2k+1}(S\sqrt{v})^{2k+1}/(2k+1)!\right) + \sum_{k=0}^{\infty}\left(t^{2k}(S\sqrt{v})^{2k}/2k!\right)\right)$ into odd and even power series, substitute it and integrate with respect S, then the odd terms of the power series are vanished and the final result is found to be

$$M_{t_b}(t) = \frac{\Gamma(v/2)}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(t\sqrt{v})^{2k}}{2k!}\frac{\Gamma(k+1/2)}{\Gamma(k+v/2)}$$
$$M_{t_b}(t) = \frac{\Gamma(v/2)}{(t\sqrt{v}/2)^{(v/2)-1}} I_{(v/2)-1}(t\sqrt{v})$$

Theorem 8.2: The Cumulant of the Bounded student's- t -distribution is given as

$$C_{t_b}(t) = log(\Gamma(v/2)) - ((v/2)-1)\,log(t\sqrt{v}/2) + log\left(I_{(v/2)-1}(t\sqrt{v})\right) \quad (44)$$

Proof: It is found from $C_{t_b}(t) = log\,M_{t_b}(t)$

Theorem 8.3: The Cf of the Bounded student's t distribution is given as

$$\phi_{t_b}(t) = \frac{\Gamma(v/2)}{(it\sqrt{v}/2)^{(v/2)-1}} i^{-((v/2)-1)} J_{(v/2)-1}(it\sqrt{v}) \quad (45)$$

where $J(\;)$ is the Bessel function of Kind-1.

Proof: Let the Cf of a distribution is given as

$$\phi_{t_b}(t) = \int_{-\sqrt{v}}^{+\sqrt{v}} \frac{e^{it(t_b)}}{\sqrt{v}B(1/2,(v-1)/2)}\left(1-(t_b/\sqrt{v})^2\right)^{((v-1)/2)-1} dt_b$$

By setting $S = t_b / \sqrt{v}$ and using the change of variable technique, the expression will be

$$= \int_{-1}^{+1} \frac{e^{it(S\sqrt{v})}}{B(1/2,(v-1)/2)} (1 - S^2)^{((v-1)/2)-1} dS \tag{46}$$

Now from (46) expand the exponent with complex argument $\left(e^{it(S\sqrt{v})} = \sum_{k=0}^{\infty}\left((it)^{2k+1}\left(S\sqrt{v}\right)^{2k+1}/(2k+1)!\right) + \sum_{k=0}^{\infty}\left((it)^{2k}\left(S\sqrt{v}\right)^{2k}/2k!\right)\right)$ into odd and even power series, substitute it and integrate with respect S, then the odd terms of the power series are vanished and the final result is found to be

$$\phi_{t_b}(t) = \frac{\Gamma(v/2)}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\left(it\sqrt{v}\right)^{2k}}{2k!} \frac{\Gamma(k+1/2)}{\Gamma(k+v/2)}$$

$$\phi_{t_b}(t) = \frac{\Gamma(v/2)}{\left(it\sqrt{v}/2\right)^{(v/2)-1}} i^{-((v/2)-1)} J_{(v/2)-1}\left(it\sqrt{v}\right)$$

9. Special cases and percentage points

Theorem 9.1: From (29), the Limiting distribution of the Bounded student's- t – distribution when $v \to \infty$ is the standard normal distribution $N(0,1)$ and it given as

$$f(t_b) = \frac{1}{\sqrt{2\pi}} e^{-t_b^2/2} \; where \; -\infty < t_b < +\infty \tag{47}$$

Proof: It is found from

$$\lim_{v\to\infty} f(t_b; v) = \lim_{v\to\infty} \frac{1}{\sqrt{v}B(1/2,(v-1)/2)} \times \lim_{v\to\infty} \left(1 - \left(t_b/\sqrt{v}\right)^2\right)^{(v-3)/2}$$

$$= \lim_{v\to\infty} \frac{\Gamma(v/2)}{\sqrt{v}\Gamma(1/2)\Gamma((v-1)/2)} \times \lim_{v\to\infty} \left(\left(1 - \left(t_b/\sqrt{v}\right)^2\right)^{v}\right)^{1/2}$$

$$\times \lim_{v\to\infty} \left(1 - \left(t_b/\sqrt{v}\right)^2\right)^{-3/2} \tag{48}$$

From (48), the limits can be separately applied for each term in the product, and it is given as

$$\lim_{v\to\infty} \frac{1}{\Gamma((v-1)/2)/\Gamma(v/2)} = (v/2)^{1/2} \tag{49}$$

$$\lim_{v\to\infty} \left(\left(1 - \left(t_b/\sqrt{v}\right)^2\right)^{v}\right)^{1/2} = \left(e^{-t_b^2}\right)^{1/2} \tag{50}$$

$$\lim_{v\to\infty} \left(1 - \left(t_b/\sqrt{v}\right)^2\right)^{-3/2} = 1 \tag{51}$$

Now substitute (49), (50) and (51) in (48), then the result is found to be

$$f(t_b) = \frac{1}{\sqrt{2\pi}} e^{-t_b^2/2} \qquad \text{where } -\infty < t_b < +\infty .$$

Result 9.2: The two-sided significant percentage points of Bounded-t-distribution are also computed with the help of Maple version 16 shown in Table 3.

Table 2: The Special cases of (29)

Case No	Distribution	Parameters			Transformation
		Location	Scale	shape	Jacobean
1	Scaled Bounded student's-t	$\bar{x}$	s	v	$\bar{x}+st_b$
2	symmetric arcsine distribution	-	-	2	$t_b/\sqrt{2}$
3	Uniform	-	-	3	-
4	Wigner's Semi-circle	-	-	4	-
5	Wigner's unit semi-circle	-	-	4	$t_b/\sqrt{4}$

Table 3: Significant Two tail Percentage points of Bounded-t-distribution $P(|t_{b(v)}| > t_{b(v)}(\alpha)) = \alpha$

$df(v)$	$t_{b(v)}(\alpha)$						
	$t_{b(v)}(0.01)$	$t_{b(v)}(0.05)$	$t_{b(v)}(0.1)$	$t_{b(v)}(0.2)$	$t_{b(v)}(0.3)$	$t_{b(v)}(0.4)$	$t_{b(v)}(0.5)$
2	1.4142	1.4131	1.4099	1.3968	1.3751	1.3450	1.3066
3	1.7234	1.6887	1.6454	1.5588	1.4722	1.3856	1.2990
4	1.9481	1.8474	1.7567	1.6108	1.4865	1.3741	1.2694
5	2.1057	1.9408	1.8143	1.6308	1.4858	1.3604	1.2474
6	2.2182	2.0002	1.8481	1.6398	1.4821	1.3493	1.2318
7	2.3011	2.0408	1.8698	1.6443	1.4782	1.3407	1.2203
8	2.3643	2.0699	1.8848	1.6467	1.4747	1.3339	1.2116
9	2.4138	2.0919	1.8957	1.6481	1.4716	1.3284	1.2048
10	2.4536	2.1089	1.9039	1.6488	1.4690	1.3239	1.1993
11	2.4862	2.1226	1.9103	1.6492	1.4667	1.3202	1.1949
12	2.5133	2.1337	1.9154	1.6495	1.4648	1.3171	1.1911
13	2.5363	2.1429	1.9196	1.6495	1.4630	1.3144	1.1880
14	2.5560	2.1507	1.9231	1.6496	1.4615	1.3122	1.1852
15	2.5730	2.1573	1.9261	1.6495	1.4602	1.3102	1.1829
16	2.5879	2.1631	1.9286	1.6494	1.4590	1.3084	1.1809
17	2.6010	2.1681	1.9308	1.6494	1.4580	1.3069	1.1790
18	2.6126	2.1725	1.9327	1.6493	1.4570	1.3055	1.1774
19	2.6230	2.1765	1.9343	1.6491	1.4562	1.3042	1.1760
20	2.6323	2.1800	1.9358	1.6490	1.4554	1.3031	1.1747
21	2.6408	2.1831	1.9371	1.6489	1.4547	1.3021	1.1735
22	2.6485	2.1859	1.9383	1.6488	1.4540	1.3012	1.1725
23	2.6555	2.1885	1.9394	1.6487	1.4534	1.3003	1.1715
24	2.6619	2.1909	1.9403	1.6486	1.4529	1.2996	1.1706
25	2.6678	2.1930	1.9412	1.6485	1.4524	1.2988	1.1698
26	2.6732	2.1950	1.9420	1.6484	1.4519	1.2982	1.1690
27	2.6782	2.1968	1.9428	1.6483	1.4514	1.2976	1.1684

28	2.6829	2.1985	1.9435	1.6482	1.4510	1.2970	1.1677
29	2.6872	2.2000	1.9441	1.6482	1.4507	1.2965	1.1671
30	2.6913	2.2015	1.9447	1.6481	1.4503	1.2960	1.1665
40	2.7205	2.2119	1.9488	1.6474	1.4477	1.2924	1.1625
50	2.7379	2.2180	1.9512	1.6470	1.4461	1.2902	1.1600
60	2.7495	2.2220	1.9527	1.6467	1.4450	1.2888	1.1584
70	2.7578	2.2248	1.9538	1.6464	1.4442	1.2878	1.1573
80	2.7640	2.2269	1.9546	1.6463	1.4437	1.2870	1.1564
90	2.7688	2.2286	1.9552	1.6461	1.4432	1.2864	1.1557
100	2.7726	2.2299	1.9557	1.6460	1.4428	1.2859	1.1552
∞	2.5758	1.9600	1.6449	1.2816	1.0364	0.8416	0.6745

10. Numerical illustration and discussion

The application of the proposed Bounded student's t distribution was explained with the help of the Fisher's Iris Plants Database. The best-known database to be found in the pattern recognition literature. Fisher's paper is a classic in the field and is referenced frequently until this day. (See Duda & Hart, for example.) The data set consists 4 different characteristics of Iris plants in centimeters (Sepal length (X_1), Sepal width(X_2), petal length(X_3), petal width (X_4)), 3 classes(Iris Setosa , Iris Versicolour , Iris Virginica) of 50 instances each, where each class refers to a type of iris plant. Out of 150 instances, the authors randomly select 30 instances for giving a numerical illustration. From (29) the classical and Bounded student's t ratio are similar in their computation, hence at first standard t-scores are computed to find the univariate outliers. Secondly 4 different regression models (X_1on X_2, X_3, X_4, X_2on X_1, X_3, X_4, X_3on X_1, X_2, X_4 and X_4on X_1, X_2, X_3)are fitted based on 4 characteristics of iris plants and then Jackknife residuals are computed to identify the outliers in the Y-space. The Comparative results of Classical and Bounded student's t ratio are given in Table 3 and Table 4 along with discussion.

Table 4: Comparative results of Classical t-ratio, Bounded student's t-ratio and Identification of Univariate Outliers

Observ-ation	Sepal length X_1	Sepal width X_2	petal length X_3	petal width X_4	$\lvert t_{b1}\rvert$	$\lvert t_{b2}\rvert$	$\lvert t_{b3}\rvert$	$\lvert t_{b4}\rvert$
1	5.10	3.50	1.40	.20	.197	.143	.395	.463
2	4.90	3.00	1.40	.20	.340	1.285	.395	.463
3	4.70	3.20	1.30	.20	.878	.714	.934	.463
4	4.60	3.10	1.50	.20	1.14	1.000	.144	.463
5	5.00	3.60	1.40	.20	.072	.428	.395	.463
6	5.40	3.90	1.70	.40	1.00	1.285	1.22	1.521
7	4.60	3.40	1.40	.30	1.14	.143	.395	.529
8	5.00	3.40	1.50	.20	.072	.143	.144	.463
9	4.40	2.90	1.40	.20	1.68	1.571	.395	.463
10	4.90	3.10	1.50	.10	.340	1.000	.144	1.455
11	5.40	3.70	1.50	.20	1.00	.714	.144	.463

12	4.80	3.40	1.60	.20	.609	.143	.683	.463
13	4.80	3.00	1.40	.10	.609	1.285	.395	1.455
14	4.30	3.00	1.10	.10	1.95	1.285	2.01	1.455
15	5.80	4.00	1.20	.20	**2.07**	1.571	1.47	.463
16	5.70	4.40	1.50	.40	1.80	**2.713**[a],	.144	1.521
17	5.40	3.90	1.30	.40	1.00	1.285	.934	1.521
18	5.10	3.50	1.40	.30	.197	.143	.395	.529
19	5.70	3.80	1.70	.30	1.80	1.000	1.22	.529
20	5.10	3.80	1.50	.30	.197	1.000	.144	.529
21	5.40	3.40	1.70	.20	1.00	.143	1.22	.463
22	5.10	3.70	1.50	.40	.197	.714	.144	1.521
23	4.60	3.60	1.00	.20	1.14	.428	**2.55**	.463
24	5.10	3.30	1.70	.50	.197	.428	1.22	**2.513**
25	4.80	3.40	1.90	.20	.609	.143	**2.29**	.463
26	5.00	3.00	1.60	.20	.072	1.285	.683	.463
27	5.00	3.40	1.60	.40	.072	.143	.683	1.521
28	5.20	3.50	1.50	.20	.466	.143	.144	.463
29	5.20	3.40	1.40	.20	.466	.143	.395	.463
30	4.70	3.20	1.60	.20	.878	.714	.683	.463

[b]**Critical** $|t_{b(\alpha=0.05,\nu=29)}| = 2.20$, [a]**Critical** $|t_{b(\alpha=0.01,\nu=29)}| = 2.6872$ $df\,(v=29)$

[d]**Critical** $|t_{(\alpha=0.05,\nu=29)}| = 2.045$, [c]**Critical** $|t_{(\alpha=0.01,\nu=29)}| = 2.756$

Table 5: Comparative results of Classical Jackknife, Bounded Jackknife residuals in linear regression analysis and identification of outliers in Y space

Observ -ation	Sepal length X_1	Sepal width X_2	petal length X_3	petal width X_4	$\|t_{b1}\|$	$\|t_{b2}\|$	$\|t_{b3}\|$	$\|t_{b4}\|$
1	5.10	3.50	1.40	.20	.221	.097	.250	.4759
2	4.90	3.00	1.40	.20	1.375	1.903	1.015	.6525
3	4.70	3.20	1.30	.20	.138	.339	.782	.2198
4	4.60	3.10	1.50	.20	.686	.010	.406	.0625
5	5.00	3.60	1.40	.20	.622	.962	.144	.7991
6	5.40	3.90	1.70	.40	.394	.754	.993	.5378
7	4.60	3.40	1.40	.30	1.495	.732	.029	.6882
8	5.00	3.40	1.50	.20	.074	.172	.317	.5244
9	4.40	2.90	1.40	.20	.565	.639	.282	.5493
10	4.90	3.10	1.50	.10	.491	.461	.337	1.127
11	5.40	3.70	1.50	.20	.542	.366	.244	1.020
12	4.80	3.40	1.60	.20	1.253	1.143	1.329	.9243
13	4.80	3.00	1.40	.10	.693	.905	.325	.7077
14	4.30	3.00	1.10	.10	.931	.092	1.368	.3715

15	5.80	4.00	1.20	.20	**2.278**	.257	2.020	.9496
16	5.70	4.40	1.50	.40	.594	2.043	.301	.0685
17	5.40	3.90	1.30	.40	.655	.311	1.599	1.547
18	5.10	3.50	1.40	.30	.409	.489	.717	.7499
19	5.70	3.80	1.70	.30	1.249	.251	.696	.2871
20	5.10	3.80	1.50	.30	1.041	1.339	.541	.2227
21	5.40	3.40	1.70	.20	1.300	.733	.857	.7625
22	5.10	3.70	1.50	.40	.452	.213	.156	1.301
23	4.60	3.60	1.00	.20	1.699	1.570	1.851	.1765
24	5.10	3.30	1.70	.50	1.078	**2.506**[b,]	.339	**3.887**
25	4.80	3.40	1.90	.20	**2.475**	**2.363**[b,]	**3.942**[a,c]	2.011
26	5.00	3.00	1.60	.20	1.332	1.736	.073	.2398
27	5.00	3.40	1.60	.40	.057	.791	.008	1.794
28	5.20	3.50	1.50	.20	.424	.008	.172	.6392
29	5.20	3.40	1.40	.20	1.075	.767	.653	.1672
30	4.70	3.20	1.60	.20	.888	.424	1.059	.4787

[b]**Critical** $|t_{b(\alpha=0.05,\nu=26)}| = 2.1950$, [a]**Critical** $|t_{b(\alpha=0.01,\nu=26)}| = 2.6732$ $df\,(\nu = 26)$

[d]**Critical** $|t_{(\alpha=0.05,\nu=26)}| = 2.056$, [c]**Critical** $|t_{(\alpha=0.01,\nu=26)}| = 2.779$

Table 1 exhibits the result of bounded student's t-ratio and the computed standard student t-scores are similar. The standardized student t-scores of the variable sepal length (X_1) is having a single outlier (observation 15) at 5% significance level for 29 degrees of freedom based on classical student's t-statistic and the bounded student's t-statistic failed to identify the remote observation in the same variable. Likewise observation 16 is an outlier in the variable sepal width(X_2) at 5 % level based on classical statistic and 1% significance level based on Bounded t- statistic. In the same manner, in Petal width (X_3), observations 23, 25 and in Petal width(X_4), observations 24 are identified as outliers by both test statistics at 5% significance level. Table 3 visualizes the comparative results of Classical and Bounded Jackknife residuals in linear regression analysis. These residual helps to identify the outliers in the Y-space (response space) and it exactly follows the classical student's t-distribution for $n - p - 1$ degrees of freedom, where p is the number of regressors in the regression model. Since the computation of the residuals on both the test statistics are similar, but distributional assumptions are different. The absolute Jackknife residual of the variable sepal length (X_1)shows the observations 15, 25 and in sepal width(X_2), Observations 24, 25 are outliers at 5% significance level based on both the test statistic. Similarly in Petal length, observation 25 is the only outlier identified based on Bounded student's statistic at 5%, 1% significance level. Finally in variable Petal width(X_4), Observation 24 is the only outlier identified by both the statistic at 1% significance level. From the above discussion, authors came to know, both the distributional assumption of the test statistic are giving similar results in a small and finite sample and the introduction of Bounded student's t distribution can be used as a proxy to student's t distribution when we accept the relationship assumption between the standard normal and chi-variate are uncorrelated, but not independent.

11. Conclusion

This paper proposed a new bivariate mixture of chi and normal distribution, which is said to be chi-normal distribution. The most significant property of the distribution is the correlation between the standard normal variate and the chi-variables is zero and it denotes both the random variables are not independent but uncorrelated. The classical student's t distribution is the pioneering work proposed by William Sealy Gosset under the pen name 'student' Likewise, the authors explored a new sampling distribution to the literature based on the proposed chi-normal distribution which is declared to be the Bounded student's t distribution. The properties of the Bounded student's t distribution are scrutinized, and the limiting form of the distribution becomes the standard normal distribution when the degree of freedom is larger. This distribution creates an alternate path to the sampling literature and the Bounded student's t statistic can also use to test the significance of Means and difference between two means from the normal population. The multivariate extension of the proposed Bounded student's t distribution can also be left it for future research, and it can also open the way to introduce new distance metrics useful to identify the multivariate outliers in a multivariate data matrix. Finally, the proposed Bounded student's t statistic gave approximately similar results when it compares with the classical student's t statistic, and this confirms the Bounded student's t distribution can also be used as an alternate in a small sample.

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