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Construction of Partially Balanced Incomplete Block Designs in Blocks of Size Three using Cyclic Shifts

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Abstract

Partially Balanced Incomplete Block (PBIB) designs play important role in design of experiments, especially in field experiments where Balanced Incomplete Block Designs required a large number of blocks. In this article, PBIB designs are constructed in blocks of size three using method of cyclic shifts. Some standard properties of a design can be expressed just through the sets of shifts; therefore, this method has an edge over existing methods.

Keywords

Balanced Incomplete Block Designs, Binary block, Block designs, Partially Balanced Incomplete Block Designs

1. Introduction

Block designs are used in experimental planning with the purpose of maximizing the information extracted from a given number of experiments. If homogeneous blocks of size k are available to accommodate all the k treatments, a Randomized Complete Block Design is preferred. In many situations, the experimenter is not able to run all the treatment combinations in each block because of shortage of experimental apparatus or facilities and/or the physical size of block.

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Then an Incomplete Block Design is preferred where one would use smaller blocks which did not contain all of the varieties. When all treatments comparisons are equally important, Balanced Incomplete Block (BIB) Designs are used. Since, the number of replications and block size must be kept within practical limits, it is possible to arrange BIB designs for only specific number of varieties, otherwise PBIB designs should be preferred (see Yasmin et al. 2015). A PBIB design is obtained by identifying the *v* treatments with the *v* objects of an association scheme and arranging them into b blocks satisfying the following conditions:

- Each block contains *k* treatments.
- Each treatment occurs in *r* blocks.
- If two treatments are i^{th} associates, they occur together in λ_i blocks.

An associate class is a set of treatment pairs where each pair from the set occur together the same number of times, λ_i. In this study, we constructed PBIB designs with two-class association scheme. Some of the references for PBIB designs are Bose (1963), Bose (1939), Bose and Nair (1939), Bose and Shimamoto (1952), Cheng and Wu (1981), Kumar (2007), Nair and Rao (1942) and Wallis (1996) established the relation between PBIBDs and strongly regular graphs. Bose (1963) has shown that strongly regular graph emerges from PBIB design with 2-association scheme. Yasmin et al. (2015) constructed BIB designs using method of cyclic shifts.

2. Method of Cyclic Shifts

Method of cyclic shifts introduced by Iqbal (1991) is simplified here only for Balanced Incomplete Block Designs. In this construction, *v* treatments are labeled as 0, 1, 2, … , *v*-1 and here $[q_1, \ldots, q_{k-1}](a/v)$ means take only first ‘a’ blocks among the *v* blocks which are generated by the set of shifts $[q_1, \ldots, q_{k-1}]$. $[q_1, \ldots, q_{k-2}]t(a/v)$ means take only first ‘a’ blocks among the *v*-1 blocks which are generated by the set of shifts $[q_1, \ldots, q_{k-2}]t$.

Rule I: Let $\underline{S}_j = [q_{j1}, q_{j2}, \ldots, q_{j(k-1)}]$ be a set of shifts where $1 \le q_{ji} \le v-1$. A design is partially Balanced Incomplete Block with two associate scheme if each element of $\underline{S}_j^*$ along with its complement contains all elements 1, 2, …, *v*-1 either λ_1 or λ_2 times.

where, $\lambda_2 = \lambda_1+1$ and $\underline{S}_j^*$ = [q_{j1}, q_{j2}, …, $q_{j(k-1)}$, ($q_{j1}+q_{j2}$), ($q_{j2}+q_{j3}$),…, ($q_{j(k-2)}+q_{j(k-1)}$), ($q_{j1}+q_{j2}+q_{j3}$), ($q_{j2}+q_{j3}+q_{j4}$), …, ($q_{j(k-3)}+q_{j(k-2)}+q_{j(k-1)}$),…, ($q_{j1}+q_{j2}+ \ldots + q_{j(k-1)}$)], here complement of q_i is v-q_i.

Rule II: Let $\underline{S}_j$ = [q_{j1}, q_{j2}, …, $q_{j(k-2)}$] be a set of shifts where $1 \le q_{ji} \le v-2$.

A design is Partially Balanced Incomplete Block with two associate scheme if each element of $\underline{S}_j^*$ along with its complement contains all elements 1, 2, …, v-2 either λ_1 or λ_2 times.

where, $\lambda_2 = \lambda_1+1$ and $\underline{S}_j^*$ = [q_{j1}, q_{j2}, …, $q_{j(k-2)}$, ($q_{j1}+q_{j2}$), ($q_{j2}+q_{j3}$),…, ($q_{j(k-3)}+q_{j(k-2)}$), ($q_{j1}+q_{j2}+q_{j3}$), ($q_{j2}+q_{j3}+q_{j4}$), …, ($q_{j(k-4)}+q_{j(k-3)}+q_{j(k-2)}$), …, ($q_{j1}+q_{j2}+ \ldots + q_{j(k-2)}$)], here complement of q_i is v-1-q_i.

3. Construction of PBIB Designs

3.1 PBIB designs for $v = 6i+3$ and $k = 3$ with $\lambda_1 = 0$ and $\lambda_2 = 1$: PBIB designs for $v = 6i+3$ and k = 3 with $\lambda_1 = 0$ and $\lambda_2 = 1$ are constructed where any of the pairs (0, 1), (1, 2), …, (v-2, v-1), (0, v-1) do not appear together in same block while each of the remaining pairs appears once.

Example 3.1: Following is PBIB design constructed for v = 9 in block of size three from a set of shifts [3, 2].

B_1	**B_2**	**B_3**	**B_4**	**B_5**	**B_6**	**B_7**	**B_8**	**B_9**
0	1	2	3	4	5	6	7	8
3	4	5	6	7	8	0	1	2
5	6	7	8	0	1	2	3	4

Here any of the pairs (0, 1), (1, 2), …, (7, 8), (0, 8) does not appear together in same block. Each of the remaining pairs appears once. Following is the catalog of such designs for v < 50.

v	**k**	**Set(s) of Shifts**
9	3	[3,2]
15	3	[2,4]+[3,7]
21	3	[2,7]+[3,5]+[4,6]
27	3	[2,5]+[3,8]+[4,10]+[6,9]
33	3	[2,5]+[3,8]+[4,13]+[6,12]+[9,10]
39	3	[2,17]+[3,13]+[4,10]+[5,6]+[7,8]+[9,12]

45	3	[2,10]+[3,14]+[4,9]+[5,15]+[6,18]+[7,16]+[8,11]

3.2 PBIB designs for $v = 6i+1$; $i > 1$ and $k = 3$ with $\lambda_1 = 0$ and $\lambda_2 = 1$: PBIB designs for $v = 6i+1$; $i > 1$ and k = 3 with $\lambda_1 = 0$ and $\lambda_2 = 1$ are constructed where two pairs (0, v-1) and (0, 3) do not appear together in same block while each of the remaining pairs appears once.

Example 3.2: Following is PBIB design constructed for v = 13 in block of size three from two sets of shifts [5, 2] + [1, 3](12/13). Here we obtain first 13 blocks from [5, 2]. [1, 3] will also provide 13 blocks but we will take the first 12 blocks.

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}
0	1	2	3	4	5	6	7	8	9	10	11	12
5	6	7	8	9	10	11	12	0	1	2	3	4
7	8	9	10	11	12	0	1	2	3	4	5	6

B_{14}	B_{15}	B_{16}	B_{17}	B_{18}	B_{19}	B_{20}	B_{21}	B_{22}	B_{23}	B_{24}	B_{25}
0	1	2	3	4	5	6	7	8	9	10	11
1	2	3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	0	1	2

Here (0, 3) do not appear together in same block, similarly (0,12) also do not appear together in same block. Each of the remaining pairs appears once.
Following is the catalog of such designs for v < 50.

v	k	Set(s) of Shifts
13	3	[5,2]+ [1,3](12/13)
19	3	[5,6]+[2,7]+[1,3](18/19)
25	3	[5,9]+[2,8]+[6,12]+[1,3](24/25)
31	3	[7,9]+[2,10]+[5,18]+[6,14]+[1,3](30/31)
37	3	[9,11]+[2,12]+[5,13]+[7,22]+[6,10]+[1,3](36/37)
43	3	[10,12]+[2,15]+[5,25]+[6,23]+[7,9]+[8,11]+[1,3](42/43)
49	3	[2,29]+[5,16]+[7,27]+[8,9]+[11,12]+[6,13]+[10,14]+[1,3](48/49)

3.3 PBIB designs for $v = 6i+4$ and $k = 3$ with $\lambda_1 = 0$ and $\lambda_2 = 1$: PBIB designs for $v = 6i+4$; and k = 3 with $\lambda_1 = 0$ and $\lambda_2 = 1$ are constructed where the pairs (0, v/2), (1, (v+2)/2), …, ((v-4)/2, v-2) and ((v-2)/2, v-2) do not appear together in same block while each of the remaining pairs appears once.

Example 3.3: Following is PBIB design constructed for v = 10 in block of size three from two sets of shifts [1, 2] + [4]t(4/9). Here we obtain first 10 blocks from [1, 2]. [4]t will provide 9 blocks but we will take the first 4 blocks.

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}
0	1	2	3	4	5	6	7	8	0	1	2	3
1	2	3	4	5	6	7	8	0	4	5	6	7
3	4	5	6	7	8	0	1	2	9	9	9	9

Here each of the pairs (0, 5), (1, 6), (2, 7), (3, 8) and (4, 8) do not appear together in same block. Each of the remaining pairs appears once. Following is the catalog of such designs for v < 50.

v	k	Set(s) of Shifts
10	3	[1,2]+[4]t(4/9)
16	3	[1,2]+[4,5]+[7]t(7/15)
22	3	[1,2]+[4,5]+[6,7]+[10]t(10/21)
28	3	[1,2]+[4,5]+[6,10]+[7,8]+[13]t(13/27)
34	3	[1,8]+[2,3]+[6,7]+[10,11]+[4,14]+[16]t(16/33)
40	3	[1,10]+[2,16]+[3,12]+[4,5]+[6,7]+[8,14]+[19]t(19/39)
46	3	]t(22/45)

3.4 PBIB designs for v = 6i-1 and k = 3 with λ_1 = 1 and λ_2 = 2: PBIB designs for $v = 6i$-1 and k = 3 with $\lambda_1 = 1$ and $\lambda_2 = 2$ are constructed where each of the pairs (0, 1), (1, 2), …, (v-2, v-1), (0, v-1) appears together in two blocks while each of the remaining pairs appears together in one block.

Example 3.4: Following is PBIB design constructed for v = 11 in block of size three from two sets of shifts [1, 2] + [5, 7].

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}
0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
3	4	5	6	7	8	9	10	0	1	2

B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}	B_{18}	B_{19}	B_{20}	B_{21}	B_{22}
0	1	2	3	4	5	6	7	8	9	10
5	6	7	8	9	10	0	1	2	3	4
1	2	3	4	5	6	7	8	9	10	0

Here each of the pairs (0, 1), (1, 2), …, (9, 10), (0, 10) appears together in two blocks. Each of the remaining pairs appears together in one block. Following is the catalog of such designs for $v < 50$.

v	**k**	**Set(s) of Shifts**
5	3	[2,4]
11	3	[1,2]+[5,7]
17	3	[1,2]+[4,5]+[7,11]
23	3	[1,5]+[3,4]+[2,10]+[9,15]
29	3	[1,8]+[2,10]+[3,4]+[5,6]+[14,16]
35	3	[1.9]+[3,4]+[5,12]+[6,8]+[13,33]+[16,20]
41	3	[1,9]+[1,16]+[3,18]+[4,15]+[6,8]+[12,36]+[13,39]
47	3	[1,14]+[2,18]+[3,19]+[5,43]+[6,7]+[8,9]+[10,16]+[11,12]

3.5 PBIB designs for v = 6i+d (d = 2, 4) and k = 3 with λ_1 = 1 and λ_2 = 2: PBIB designs for $v = 6i$+d (d = 2, 4) and k = 3 with $\lambda_1 = 1$ and $\lambda_2 = 2$ are constructed where some of the pairs appear together in two blocks while each of the remaining pairs appears together in one block.

Example 3.5: Following is PBIB design constructed for $v = 10$ in block of size three from two sets of shifts [1, 2] + [5, 4].

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}
0	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	0
3	4	5	6	7	8	9	0	1	2

B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}	B_{18}	B_{19}	B_{20}
0	1	2	3	4	5	6	7	8	9
5	6	7	8	9	0	1	2	3	4
9	0	1	2	3	4	5	6	7	8

Here each of the pairs (0, 1), (1, 2), …, (8, 9), (0, 5), (1, 6), …, (4, 9), (0,9) appears together in two blocks. Each of the remaining pairs appears together in one block. Following is the catalog of such designs for $v \leq 50$.

v	**k**	**Set(s) of Shifts**	**Remarks**
8	3	[1,2]+[4,5]	(0,1),(1,2), …, (6,7),(0,7),(0,3),(1,4), …, (4,7),(0,5),(1,6),(2,7),(0,4),(1,5),(2,6), (3,7) appear twice

10	3	[1,2]+[4,5]	(0,1),(1,2), …, (8,9),(0,9),(0,5),(1,6), …, (4,9), appear twice
14	3	[1,2]+[4,5]+[6,7]	(0,1),(1,2), …, (12,13),(0,13),(0,5),(1,6), …, (8,13),(0,9),(1,10),…,(4,13),(0,7), (1,8), …, (6,13), appear twice
16	3	[1,2]+[4,5]+[6,8]	(13,15),(0,14),(1,15),(0,8),(1,9), …, (7,15), appear twice
20	3	[1,2]+[4,5]+[4,6]+[7,8]	(0,4),(1,5), …,(15,19),(0,16),(1,17), …, (3,19),(0,5),(1,6), …,(14,19),(0,15), (1,16),…,(4,19),(0,10),(1,11), …, (9,19), appear twice
22	3	[1,2]+[4,7]+[5,9]+[6,10]	(0,6),(1,7), …, (15,21),(0,16),(1,17),…, (5,21), (0,11),(1,12), …, (10,21), appear twice
26	3	[1,2]+[4,5]+[6,7]+[8,10]+[11,12]	(0,3),(1,4), …, (22,25),(0,23),(1,24), (2,25),(0,8),(1,9), …, (17,25),(0,18), (1,19),…,(7,25),(0,13),(1,14), …, (12,25), appear twice
28	3	[1,11]+[2,5]+[3,7]+[4,9]+[6,8]	(0,7),(1,8), …, (20,27),(0,21),(1,22),…, (6,27), (0,14),(1,15), …, (13,27), appear twice
32	3	[1,2]+[1,11]+[3,13]+ [4,6]+[5,9]+[7,8]	(0,1),(1,2), …, (30,31),(0,31),(0,2),(1,3), …, (29,31),(0,30),(1,31),(0,16),(1,17), …, (15,31), appear twice
34	3	[1,11]+[2,13]+[3,14]+ [4,6]+[5,8]+[7,9]	(0,13),(1,14), …, (20,33),(0,21),(1,22),…, (12,33), (0,17),(1,18), …, (16,33), appear twice
38	3	[1,13]+[2,15]+[3,16]+ [4,8]+[5,6]+[7,9]+[8,10]	(0,8),(1,9), …, (29,37),(0,30), (1,31), …, (7,37),(0,16),(1,17), …, (21,37),(0,22), (1,23),…,(15,37),(0,19),(1,20), …, (18,37), appear twice
40	3	[1,13]+[2,6]+[2,17]+ [3,9]+[4,16]+[5,10]+ [7,11]	(0,2),(1,3), …, (37,39),(0,38),(1,39), (0,20),(1,21), …, (19,39), appear twice
44	3	[1,13]+[1,16]+[2,18]+ [3,19]+[4,7]+[5,10]+ [6,8]+[9,12]	(0,1),(1,2), …, (42,43),(0,43),(0,14), (1,15), …, (29,43),(0,30), (1,31),…, (13,43),(0,22),(1,23), …, (21,43), appear twice
46	3	[1,16]+[2,19]+[3,20]+ [4,9]+[4,14]+[5,6]+ [7,8]+[10,12]	(0,4),(1,5), …, (41,45),(0,42),(1,43),…, (3,45), (0,23),(1,24), …, (22,45), appear twice
50	3	[1,9]+[1,22]+[2,18]+ [3,21]+[4,15]+[5,11]+ [6,8]+[7,10]+[12,13]	(0,1),(1,2), …, (48,49),(0,49),(0,10), (1,11), …, (39,49),(0,40),(1,41),…,(9,49), (0,25),(1,26), …, (24,49), appear twice

3.6 PBIB designs for $v = 6i$ and $k = 3$ with $\lambda_1 = 1$ and $\lambda_2 = 2$: PBIB designs for $v = 6i$ and k = 3 with $\lambda_1 = 1$ and $\lambda_2 = 2$ are constructed where some of the pairs appear together in two blocks while each of the remaining pairs appears together in one block.

Example 3.6: Following is PBIB design constructed for v = 12 in block of size three from the sets of shifts [1, 5] + [2, 3] + [4, 4](1/3). Here we obtain first 12 blocks from [1, 5], next 12 from [2, 3]. [4, 4] will also provide 12 blocks but we will take first (1/3) of them i.e. only first 4 blocks.

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}
0	1	2	3	4	5	6	7	8	9	10	11	0	1
1	2	3	4	5	6	7	8	9	10	11	0	2	3
6	7	8	9	10	11	0	1	2	3	4	5	5	6

B_{15}	B_{16}	B_{17}	B_{18}	B_{19}	B_{20}	B_{21}	B_{22}	B_{23}	B_{24}	B_{25}	B_{26}	B_{27}	B_{28}
2	3	4	5	6	7	8	9	10	11	0	1	2	3
4	5	6	7	8	9	10	11	0	1	4	5	6	7
7	8	9	10	11	0	1	2	3	4	8	9	10	11

Here each of the pairs (0, 5), (1, 6), …, (6, 11), (0, 6), (1, 7), …, (5, 11), (0, 7), (1, 8), …, (4, 11) appears together in two blocks. Each of the remaining pairs appears together in one block. Following is the catalog of such designs for $v \leq 50$.

v	k	Set(s) of Shifts	Remarks
6	3	[1, 2]	(0,3),(1,4), (2,5) appear twice
12	3	[1,5]+[2,3]+[4,4](1/3)	(0,5),(1,6), …,(6,11),(0,7),(1,8), …, (4,11),(0,6),(1,7), …,(5,11) appear twice
18	3	[1,8]+[2,3]+[4,7]+ [6,6](1/3)	(0,7),(1,8), …, (10,17),(0,11),(1,12), …, (6,17),(0,9),(1,10), …,(8,17) appear twice
24	3	[1,11]+[2,3]+[4,6]+ [7,9]+[8,8](1/3)	(0,8),(1,9), …, (15,23),(0,16),(1,17), …, (7,23),(0,12),(1,13),…,(11,23) appear twice
30	3	[1,14]+[2,3]+[4,7]+ [6,12]+[8,9]+ [10,10](1/3)	(0,12),(1,13), …, (17,29),(0,18),(1,19), …, (11,29),(0,15),(1,14), …,(14,29) appear twice
36	3	[1,17]+[2,3]+[4,6]+ [7,8]+9,11]+[13,14]+ [12,12](1/3)	(0,9),(1,10), …, (26,35),(0,27),(1,28), …, (8,35),(0,18),(1,19), …,(17,35) appear twice
42	3	[1,20]+[2,9]+[3,15]+ [4,6]+[5,8]+[7,12]+ [16,17+[14,14](1/3)	(0,9),(1,10), …, (32,41),(0,33),(1,34), …, (8,41),(0,21),(1,22), …,(20,41) appear twice
48	3	[1,23]+[2,20]+[3,9]+ [4,17]+[5,14]+[6,7]+ [8,10]+[11,15]+ [16,16](1/3)	(0,22),(1,23),…, (4,47),(0,26),(1,27), …, (21,47),(0,24),(1,25), …,(23,47) appear twice

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