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New Product-Type and Ratio-Type Exponential Estimators of the Population Mean Using Auxiliary Information in Sample Surveys

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Abstract

This paper addresses the problem of estimating the population mean of the study variable *y* using information on an auxiliary variable *x*. A class of Exponential-Type Estimators has been suggested along with its properties under large sample approximation. It is identified that the usual Unbiased Estimator, Product-Type and Ratio-Type Exponential Estimators are members of the proposed class of Exponential-Type Estimators. It has been shown that the proposed class of Exponential-Type Estimators is more efficient than the usual Unbiased Estimator and some existing Estimators. An empirical study is carried out in support of the present study.

Keywords

Auxiliary variable, Study variable, Bias, Mean squared error

1. Introduction

It is common to use the auxiliary information at the estimation stage in order to obtain improved estimates of the population mean $\bar{Y}$ of the study variate y . Out of many, Ratio, Product and Regression methods are good examples in this context. When the auxiliary variable x is positively (high) correlated with the study variable y , the Ratio method of estimation is quite effective. On the other hand, if this correlation is negative, the Product method of estimation can be employed.

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Consider a finite population $U=(u_1,u_2,...,u_N)$ of size *N*. Let (y,x) be the study and auxiliary variables, respectively, taking values (y_i,x_i) on the i^{th} unit $U_i\,(i=1,2,...,N)$ of the population. Let $(\bar{Y},\bar{X})$ be the population means of (y,x), respectively. It is assumed that the population mean $\bar{X}$ of the auxiliary variable x is known. For estimating the population mean $\bar{Y}$ of the study variable y, a simple random sample of size n is selected without replacement from the population *U*.

Then the Classical Ratio Estimator for the population mean $\bar{Y}$ is defined by

$$t_R = \hat{R}\bar{X} \tag{1.1}$$

where,

$\hat{R}=\dfrac{\bar{y}}{\bar{x}},\ \bar{x}\neq 0$ is the estimate of the Ratio *R* of the population means.

$\bar{y}=\left(\dfrac{1}{n}\right)\sum_{i=1}^{n} y_i$ and $\bar{x}=\left(\dfrac{1}{n}\right)\sum_{i=1}^{n} x_i$ are the un-weighted sample means of y and *x*, respectively. This Estimator is only efficient if the variable *y* and *x* are strongly positively correlated. The ordinary Product Estimator for $\bar{Y}$ is defined by

$$t_P = \frac{\hat{P}}{\bar{X}} \tag{1.2}$$

where, $\hat{P}=\bar{y}.\bar{x}$ is the estimate of the Product P of the population means, will often be used if the two variables are supposed to be strongly negatively correlated.

It is to be noted that the Estimator t_P is due to Robson (1957) and revisited by Murthy (1964). Bahl and Tuteja (1991) suggested Ratio and Product-Type Exponential Estimators for the population mean $\bar{Y}$ respectively as

$$t_{\mathrm{Re}} = \bar{y}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{1.3}$$

and

$$t_{Pe} = \bar{y}\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right) \tag{1.4}$$

The simple Expansion Estimator for the population mean $\bar{Y}$ is

$$t_0 = \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i \tag{1.5}$$

is used otherwise.

Murthy (1967, p. 370) pointed out that the variability of the sample mean $\bar{x}$ is usually less than that of sample mean $\bar{y}$. If $C(\bar{x})$ denotes the coefficient of variation of $\bar{x}$ likewise $C(\bar{y})$ that of $\bar{y}$, then

$$C^2(\bar{x}) = \frac{(1-f)}{n} C_x^2 \text{ and } C^2(\bar{y}) = \frac{(1-f)}{n} C_y^2$$

where,

$f = n/N$ is the sampling fraction, $C_x = S_x / \bar{X}$ and $C_y = S_y / \bar{Y}$ are the population coefficients of variation for the two variables.

$$S_x^2 = (N-1)^{-1} \sum_{i=1}^{N} (x_i - \bar{X})^2 \text{ and } S_y^2 = (N-1)^{-1} \sum_{i=1}^{N} (y_i - \bar{Y})^2.$$

It follows that if $C_x = aC_y$, we have,

$$C(\bar{x}) = aC(\bar{y}) \;;\; 0 < a \le 1 \tag{1.6}$$

We suppose that the observation on y and x are all non-negative, so that the sample and population means are all positive. Murthy (1964) suggested the use of

$$t_R \quad \text{if} \quad \frac{\rho}{a} > \frac{1}{2}, \tag{1.7}$$

$$\bar{y} \quad \text{if} \quad -\frac{1}{2} \le \frac{\rho}{a} \le \frac{1}{2}, \tag{1.8}$$

$$t_P \quad \text{if} \quad \frac{\rho}{a} < -\frac{1}{2}, \tag{1.9}$$

where,

$$\rho = \frac{S_{xy}}{S_x S_y}$$

is the correlation coefficient between the study variable y and the auxiliary variable x and $S_{xy} = (N-1)^{-1} \sum_{i=1}^{N} (x_i - \bar{X})(y_i - \bar{Y})$.

Here, of course $\frac{\rho}{a} \ge -\frac{1}{a}$

and

$\frac{\rho}{a} \le \frac{1}{a}$ as $|\rho| \le 1$

In this paper motivated by Sahai (1979), we have suggested a variant of the Product and Ratio-Type Exponential Estimators, with the intention to improve their efficiency. We have obtained the Bias and Mean Squared Error (MSE) of the Proposed Estimator to the first degree of approximation and compared with those of the well known methods Simple Expansion, Ratio and Product. An empirical study is given in support of the present study.

2. The Suggested Estimator

Motivated by Sahai (1979), we derive the following modified Exponential-Type Estimator for the population mean $\bar{Y}$ as

$$t_{Me} = \bar{y}\exp\left\{\frac{(\bar{x}+\theta\bar{X})-(X+\theta\bar{x})}{(\bar{x}+\theta\bar{X})+(X+\theta\bar{x})}\right\}$$

$$= \bar{y}\exp\left\{\frac{\bar{x}(1-\theta)+(\theta-1)\bar{X}}{\bar{x}(1+\theta)+(\theta+1)\bar{X}}\right\}$$

$$= \bar{y}\exp\left\{\frac{(\theta-1)\bar{X}-(\theta-1)\bar{x}}{(\theta+1)\bar{X}+(\theta+1)\bar{x}}\right\}$$

$$t_{Me} = \bar{y}\exp\left\{\frac{(\theta-1)(\bar{X}-\bar{x})}{(\theta+1)(\bar{X}+\bar{x})}\right\} \quad (2.1)$$

where, θ is a scalar used as a design parameter. It is worth mention that, for $\theta=1$ $t_{Me}=\bar{y}$ and that for $\theta=0$, $t_{Me}=t_{Pe}$.

Moreover, if θ is very large t_{Me} is almost the same as the t_{Re}

i.e. $$\lim_{\theta\to\infty} t_{Me} = \lim_{\theta\to\infty}\bar{y}\exp\left\{\frac{(\theta-1)(\bar{X}-\bar{x})}{(\theta+1)(\bar{X}+\bar{x})}\right\}$$

$$\cong t_{\mathrm{Re}} = \bar{y}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right).$$

Sometimes, a good guess of the value of ρ/a is available from a pilot sample past data, experience or otherwise. In other practical situations, the value of ρ/a may be known or guessed to be in certain interval. Using such knowledge, one can give a suitable value to θ, the design parameter in order that the Proposed Modified Estimator will have a smaller Mean Squared Error (MSE) than the usual Ratio, Product, Exponential-Type Ratio, Exponential-Type Product and Simple Expansion Estimator, respectively.

2.1 Sampling Bias and Mean Squared Error of the Estimators: Following Murthy (1967), we write,

$\bar{y}=\bar{Y}(1+e_0)$, $\bar{x}=\bar{X}(1+e_1)$

such that

$E(e_0)=E(e_1)=0$ and

$$E(e_0^2)=\frac{(1-f)}{n}C_y^2\ ,E(e_1^2)=\frac{(1-f)}{n}C_x^2\ ,E(e_0e_1)=\frac{(1-f)}{n}\rho C_yC_x\ .$$

We can reasonably suppose that the sample size is large to make $|e_0|$ and $|e_1|$<1.

Further, to validate the first order large sample approximation we are going to obtain, we suppose that the sample size is large enough to obtain $|e_0|$ and $|e_1|$ small so that the terms involving e_0 and / or e_1 in a degree greater than two will be negligible; an assumption which is usually not unrealistic.

Now expressing eq. (2.1) in terms of e_0 and e_1, we have,

$$t_{Me}=\bar{Y}(1+e_0)\exp\left\{\frac{-(\theta-1)e_1}{(\theta+1)(2+e_1)}\right\}$$

$$=\bar{Y}(1+e_0)\exp\left\{\frac{(1-\theta)}{(1+\theta)}\frac{e_1}{2}\left(1+\frac{e_1}{2}\right)^{-1}\right\}$$

$$=\bar{Y}(1+e_0)\exp\left\{\frac{Ge_1}{2}\left(1+\frac{e_1}{2}\right)^{-1}\right\} \qquad (2.1.1)$$

where, $G=\frac{(1-\theta)}{(1+\theta)}$.

Expanding the right hand side of eq. (2.1.1), multiplying and neglecting terms of e’s having power greater than two, we have,

$$t_{Me}\cong\bar{Y}\left[1+e_0+\frac{Ge_1}{2}+\frac{Ge_0e_1}{2}+\frac{G(G-2)}{8}e_1^2\right]$$

and

$$\left(t_{Me}-\bar{Y}\right)\cong\bar{Y}\left[e_0+\frac{Ge_1}{2}+\frac{Ge_0e_1}{2}+\frac{G(G-2)}{8}e_1^2\right] \qquad (2.1.2)$$

Taking expectation on both sides of eq. (2.1.2), we get the Bias of the first degree of approximation as

$$B(t_{Me}) = \bar{Y}\,\frac{(1-f)}{n}\left[\frac{G\rho C_y C_x}{2} + \frac{G(G-2)C_x^2}{8}\right] = B_0\left(\frac{G}{2}\right)\left[\rho + \frac{(G-2)}{4a}\right] \tag{2.1.3}$$

where, $B_0 = \frac{a(1-f)\bar{Y}C_y^2}{n}$ and $G = \frac{(1-\theta)}{(1+\theta)}$.

Squaring both sides of eq. (2.1.2) and neglecting terms of e's having power greater than two, we have,

$$\left(t_{Me} - \bar{Y}\right)^2 \cong \bar{Y}^2\left[e_0^2 + \frac{G^2 e_1^2}{4} + Ge_0 e_1\right]. \tag{2.1.4}$$

Taking expectation of both sides of eq. (2.1.4), we get the MSE of t_{Me} to the first degree of approximation as

$$MSE(t_{Me}) = MSE(t_{Me})_I = \frac{(1-f)}{n}\bar{Y}^2\left[C_y^2 + G\rho C_y C_x + \frac{\left(G^2 C_x^2\right)}{4}\right]$$

$$= V_0\left(1 + \frac{G^2 a^2}{4} + aG\rho\right) \tag{2.1.5}$$

where,

$$V_0 = \frac{(1-f)}{n}S_y^2 = Var(\bar{y}) \tag{2.1.6}$$

The $MSE\,(t_{Me})$ is minimum when

$$G_0 = -2\left(\frac{\rho}{a}\right) \tag{2.1.7}$$

Thus, the resulting minimum MSE of t_{Me} is given by

$$\min .MSE(t_{Me}) = V_0\left(1-\rho^2\right) \tag{2.1.8}$$

which equals to the approximate variance/MSE of the usual Regression Estimator

$$\bar{y}_{lr} = \bar{y} + \hat{\beta}\left(\bar{X} - \bar{x}\right). \tag{2.1.9}$$

2. Efficiency Comparison

***3.1 When the Scalar $G(or\,\theta)$ Coincides with its True Optimum Value*:** It is well known under Simple Random Sampling WithOut Replacement (SRSWOR) scheme that

$$Var(\bar{y}) = MSE(\bar{y}) = \left(\frac{1-f}{n}\right)S_y^2 = \left(\frac{1-f}{n}\right)\bar{Y}^2 C_y^2 = V_0 \quad (3.1.1)$$

To the first degree of approximation, Biases and Mean Squared Errors of the Estimators, t_R, t_P, t_{Re} and t_{Pe} are respectively given by

$$B(t_R) = B_0(a-\rho) \quad (3.1.2)$$

$$B(t_P) = B_0\rho \quad (3.1.3)$$

$$B(t_{Re}) = \left(\frac{B_0}{8}\right)(3a-4\rho) \quad (3.1.4)$$

$$B(t_{Pe}) = \left(\frac{B_0}{8}\right)(4\rho-a) \quad (3.1.5)$$

$$MSE(t_R) = V_0(1+a^2-2\rho a) \quad (3.1.6)$$

$$MSE(t_P) = V_0(1+a^2+2\rho a) \quad (3.1.7)$$

$$MSE(t_{Re}) = V_0\left[1+\left(\frac{a}{4}\right)(a-4\rho)\right] \quad 3.1.8)$$

$$MSE(t_{Pe}) = V_0\left[1+\left(\frac{a}{4}\right)(a+4\rho)\right] \quad (3.1.9)$$

where,

$$B_0 = \frac{a(1-f)}{n}\bar{Y}C_y^2$$

From eq. (2.1.8), (3.1.1), (3.1.5), (3.1.6), (3.1.7) and (3.1.8), we have,

$$MSE(\bar{y}) - \min.MSE(t_{Me}) = V_0\rho^2 \geq 0 \quad (3.1.10)$$

$$MSE(t_R) - \min.MSE(t_{Me}) = V_0(a-\rho)^2 \geq 0 \quad (3.1.11)$$

$$MSE(t_P) - \min.MSE(t_{Me}) = V_0(a+\rho)^2 \geq 0 \quad (3.1.12)$$

$$MSE(t_{Re}) - \min.MSE(t_{Me}) = V_0\left(\frac{a}{2}-\rho\right)^2 \geq 0 \quad (3.1.13)$$

$$MSE(t_{Pe}) - \min.MSE(t_{Me}) = V_0\left(\frac{a}{2}+\rho\right)^2 \geq 0 \quad (3.1.14)$$

It is observed from eq. (3.1.10) to eq. (3.1.14) that the Proposed class of Estimators t_{Me} is more efficient than

- The usual Unbiased Estimator $\bar{y}$ unless the correlation between the study variable y and the auxiliary variable x is zero . We note that when $\rho = 0$

(i.e. the two variables y and x are uncorrelated) both the Estimators $\bar{y}$ and t_{Me} are equally efficient.

- The usual Ratio Estimator t_R except when $a = \rho$, the case where both the Estimators t_R and t_{Me} are equally efficient.
- The usual Product Estimator t_P except when $a = -\rho$, the case where both the Estimators t_P and t_{Me} are equally efficient.
- The Bahl and Tuteja (1991) Ratio-Type Exponential Estimator t_{Re} except when $a = 2\rho$, the case where both the Estimators t_{Re} and t_{Me} are equally efficient.
- The Bahl and Tuteja (1991) Product-Type Exponential Estimator t_{Pe} except when $a = -2\rho$, the case where both the Estimators t_{Pe} and t_{Me} are equally efficient.

3.2 When the Scalar $G(or\,\theta)$ does not Coincide with Its True Optimum Value:

In consequence of formula eq. (2.6), we have,

$$\begin{aligned} MSE(t_{Me}) - Var(\bar{y}) &= V_0 a^2 \left(\frac{G^2}{4} + \frac{G\rho}{a} \right) \\ &= V_0 a^2 \left(\frac{G^2}{4} - \frac{GG_0}{2} \right) \because G_0 = -2\left(\frac{\rho}{a} \right) \\ &= \frac{V_0 a^2}{4} \left(G^2 - 2GG_0 \right) \\ &= \frac{V_0 a^2}{4} \left(G^2 - 2GG_0 + G_0^2 - G_0^2 \right) \\ &= \frac{V_0 a^2}{4} \left\{ (G - G_0)^2 - G_0^2 \right\} \end{aligned}$$

which is less than 'zero' if $(G - G_0)^2 < G_0^2$

i.e. if $|G - G_0| < |G_0|$ (3.2.1)

or equivalently $\min.\left(0, -\frac{4\rho}{a} \right) < G < \max.\left(0, -\frac{4\rho}{a} \right)$

From eq. (2.1.5) and eq. (3.1.6), we have,

$$MSE(t_{Me}) - MSE(t_R) = V_0 a^2 \left(\frac{G^2}{4} + \frac{G\rho}{a} - 1 + \frac{2\rho}{a} \right)$$

$$= V_0 a^2 \left(\frac{G^2}{4} - \frac{GG_0}{2} - 1 - G_0 \right) \because G_0 = -2\left(\frac{\rho}{a}\right)$$

$$= \frac{V_0 a^2}{4}\left(G^2 - 2GG_0 - 4 - 4G_0\right)$$

$$= \frac{V_0 a^2}{4}\left(G^2 - 2GG_0 + G_0^2 - 4 - 4G_0 - G_0^2\right)$$

$$= \frac{V_0 a^2}{4}\left\{(G - G_0)^2 - (2 + G_0)^2\right\}$$

which is less than 'zero' if $(G - G_0)^2 < (2 + G_0)^2$

i.e. if $|G - G_0| < |2 + G_0|$ (3.2.2)

or equivalently $\min.\left\{-2, 2\left(1 - \frac{2\rho}{a}\right)\right\} < G < \max.\left\{-2, 2\left(1 - \frac{2\rho}{a}\right)\right\}$

From eq. (2.1.5) and eq. (3.1.7), we have,

$$MSE(t_{Me}) - MSE(t_p) = V_0 a^2 \left(\frac{G^2}{4} + \frac{G\rho}{a} - 1 - \frac{2\rho}{a} \right)$$

$$= V_0 a^2 \left(\frac{G^2}{4} - \frac{GG_0}{2} - 1 + G_0 \right) \because G_0 = -2\left(\frac{\rho}{a}\right)$$

$$= \frac{V_0 a^2}{4}\left(G^2 - 2GG_0 - 4 + 4G_0\right)$$

$$= \frac{V_0 a^2}{4}\left(G^2 - 2GG_0 + G_0^2 - 4 + 4G_0 - G_0^2\right)$$

$$= \frac{V_0 a^2}{4}\left\{(G - G_0)^2 - (2 - G_0)^2\right\}$$

which is less than 'zero' if $(G - G_0)^2 < (2 - G_0)^2$

i.e. if $|G - G_0| < |2 - G_0|$ (3.2.3)

or equivalently $\min.\left\{2, -2\left(1 + \frac{2\rho}{a}\right)\right\} < G < \max.\left\{2, -2\left(1 + \frac{2\rho}{a}\right)\right\}$

From eq. (2.1.5) and eq. (3.1.8), we have,

$$MSE(t_{Me})-MSE(t_{Re})=V_0a^2\left(\frac{G^2}{4}+\frac{G\rho}{a}-\frac{1}{4}+\frac{\rho}{a}\right)$$

$$=V_0a^2\left(\frac{G^2}{4}-\frac{GG_0}{2}-\frac{1}{4}-\frac{G_0}{2}\right)\because G_0=-2\left(\frac{\rho}{a}\right)$$

$$=\frac{V_0a^2}{4}\left(G^2-2GG_0-1-2G_0\right)$$

$$=\frac{V_0a^2}{4}\left(G^2-2GG_0+G_0^2-1-2G_0-G_0^2\right)$$

$$=\frac{V_0a^2}{4}\left\{(G-G_0)^2-(1+G_0)^2\right\}$$

which is less than 'zero' if $(G-G_0)^2<(1+G_0)^2$

i.e. if $|G-G_0|<|1+G_0|$ (3.2.4)

or equivalently $\min.\left\{-1,\left(1-\frac{4\rho}{a}\right)\right\}<G<\max.\left\{-1,\left(1-\frac{4\rho}{a}\right)\right\}$

From eq. (2.1.5) and eq. (3.1.9), we have,

$$MSE(t_{Me})-MSE(t_{Pe})=V_0a^2\left(\frac{G^2}{4}+\frac{G\rho}{a}-\frac{1}{4}-\frac{\rho}{a}\right)$$

$$=V_0a^2\left(\frac{G^2}{4}-\frac{GG_0}{2}-\frac{1}{4}+\frac{G_0}{2}\right)\because G_0=-2\left(\frac{\rho}{a}\right)$$

$$=\frac{V_0a^2}{4}\left(G^2-2GG_0-1+2G_0\right)$$

$$=\frac{V_0a^2}{4}\left(G^2-2GG_0+G_0^2-1+2G_0-G_0^2\right)$$

$$=\frac{V_0a^2}{4}\left\{(G-G_0)^2-(1-G_0)^2\right\}$$

which is less than 'zero' if $(G-G_0)^2<(1-G_0)^2$

i.e. if $|G-G_0|<|1-G_0|$ (3.2.5)

or equivalently $\min.\left\{1,-\left(1+\frac{4\rho}{a}\right)\right\}<G<\max.\left\{1,-\left(1+\frac{4\rho}{a}\right)\right\}$

It follows from eq. (3.2.1), (3.2.2). (3.2.3), (3.2.4) and (3.2.5) that the Proposed Modified Exponential Estimator t_{Me} is more efficient than

- The usual Unbiased Estimator $\bar{y}$ if $|G-G_0|<|G_0|$ or equivalently
  $$\min.\left(0,-\frac{4\rho}{a}\right)<G<\max.\left(0,-\frac{4\rho}{a}\right)$$
- The usual Ratio Estimator t_R if $|G-G_0|<|2+G_0|$ or equivalently
  $$\min.\left\{-2,2\left(1-\frac{2\rho}{a}\right)\right\}<G<\max.\left\{-2,2\left(1-\frac{2\rho}{a}\right)\right\}$$
- The usual Product Estimator t_P if $|G-G_0|<|2-G_0|$ or equivalently
  $$\min.\left\{2,-2\left(1+\frac{2\rho}{a}\right)\right\}<G<\max.\left\{2,-2\left(1+\frac{2\rho}{a}\right)\right\}$$
- The Ratio-Type Exponential Estimator t_{Re} due to Bahl and Tuteja (1991) if $|G-G_0|<|1+G_0|$ or equivalently
  $$\min.\left\{-1,\left(1-\frac{4\rho}{a}\right)\right\}<G<\max.\left\{-1,\left(1-\frac{4\rho}{a}\right)\right\}$$
- The Product-Type Exponential Estimator t_{Pe} due to Bahl and Tuteja (1991) if $|G-G_0|<|1-G_0|$ or equivalently
  $$\min.\left\{1,-\left(1+\frac{4\rho}{a}\right)\right\}<G<\max.\left\{1,-\left(1+\frac{4\rho}{a}\right)\right\}$$

4. Accuracy of First Order Approximations to MSE's

We have already compared the MSEs of the Proposed Estimator and the other Estimators, subject to the first order of approximations. Here, we intend to examine the accuracy of these approximations by obtaining the second order approximations to the MSEs. We assume that $C_x=C_y=C$, say $(a=1)$ and that the sample comes from a large Bivariate Normal population as otherwise very complicated expressions are obtained, see Sahai (1979, p.33). For this case, we have,

$$\mu_{30}=\mu_{03}=\mu_{12}=\mu_{21}=0$$
$$\mu_{04}=\mu_{40}=3C^4$$

$$\mu_{31} = \mu_{13} = 3\rho C^4$$
$$\mu_{22} = (1 + 2\rho^2)C^4$$

where,

$E(e_0^i e_1^j) \approx \dfrac{\mu_{ij}}{n^b}$, $i, j = 0,1,2,4$ and $b = 2$ for $i + j = 4$

Thus, to the second order of approximations, we have,

$$t_{Me} = \bar{Y}(1 + e_0)\exp\left\{\frac{Ge_1}{2}\left(1 + \frac{e_1}{2}\right)^{-1}\right\}$$
$$= \bar{Y}(1 + e_0)\left[1 + \frac{Ge_1}{2} + \frac{G(G-2)e_1^2}{8} + \frac{G(G^2 - 6G + 6)e_1^3}{48} + \frac{G(G^3 - 12G^2 + 36G - 24)e_1^4}{384}\right]$$

We denote by $a_1 = \dfrac{G}{2}$, $a_2 = \dfrac{G(G-2)}{8}$,

$a_3 = \dfrac{G(G^2 - 6G + 6)}{48}$ and $a_4 = \dfrac{G(G^3 - 12G^2 + 36G - 24)}{384}$.

then

$$t_{Me} = \bar{Y}(1 + e_0)\{1 + a_1 e_1 + a_2 e_1^2 + a_3 e_1^3 + a_4 e_1^4 + ...\}$$
$$= \bar{Y}\{1 + e_0 + a_1 e_1 + a_1 e_0 e_1 + a_2 e_1^2 + a_2 e_0 e_1^2 + a_3 e_1^3 + a_3 e_0 e_1^3 + a_4 e_1^4 + ...\}$$

Neglecting term of $e's$ having power greater than four, we have,

$$t_{Me} \cong \bar{Y}[1 + e_0 + a_1(e_1 + e_0 e_1) + a_2(e_1^2 + e_0 e_1^2) + a_3(e_1^3 + e_0 e_1^3) + a_4 e_1^4]$$

or

$$(t_{Me} - \bar{Y}) \cong \bar{Y}[e_0 + a_1(e_1 + e_0 e_1) + a_2(e_1^2 + e_0 e_1^2) + a_3(e_1^3 + e_0 e_1^3) + a_4 e_1^4] \quad (4.1)$$

Taking expectation of both sides of eq. (4.1), we get the bias of t_{Me} as

$$B(t_{Me}) = \frac{\bar{Y}C^2}{n}\left[(\rho a_1 + a_2) + \frac{3C^2}{n}(a_3\rho + a_4)\right] \quad (4.2)$$

Squaring both sides of eq. (4.1) and neglecting term of $e's$ having power greater than two, we have,

$$(t_{Me} - \bar{Y})^2 = \bar{Y}^2\begin{bmatrix} e_0^2 + 2a_1 e_0 e_1 + a_1^2 e_1^2 + 2(a_1^2 + a_2)e_0 e_1^2 + 2a_1 e_0^2 e_1 + 2a_1 a_2 e_1^3 \\ + (a_1^2 + 2a_2)e_0^2 e_1^2 + 2(a_3 + 2a_1 a_2)e_0 e_1^3 + (a_2^2 + 2a_1 a_3)e_1^4 \end{bmatrix} \quad (4.3)$$

Taking expectation of both sides of eq. (4.3), we get the Mean Squared Error to the second order of approximation (retaining the term up to fourth degree in e_0 and/or e_1), we have,

$$MSE(t_{Me})_{II} = \left[MSE(t_{Me})_I + \left(\frac{C^4\overline{Y}^2}{n^2}\right)\left\{(a_1^2 + 2a_2)(1+2\rho^2) + 6(a_3 + 2a_1a_2)\rho + 3(a_2^2 + 2a_1a_3)\right\}\right]$$

$$= MSE(t_{Me})_I + \frac{C^4\overline{Y}^2}{64n^2}\left[7G^4 + 4G^3(14\rho - 9) + 4G^2(16\rho^2 - 36\rho + 17) - 16G(4\rho^2 - 3\rho + 2)\right] \quad (4.4)$$

In case a good guess of G_0 is available, $G \cong G_0 = -2\rho$ (with a =1). Then putting $G = -2\rho$ in eq. (4.4), we have,

$$MSE(t_{Me})_{II} = MSE(t_{Me})_I + \frac{C^4\overline{Y}^2}{4n^2}\rho(4 + 11\rho - 10\rho^2 - 5\rho^3) \quad (4.5)$$

$$= MSE(t_{Me})_I\left[1 + \frac{C^2}{4n}\rho\frac{(4 + 15\rho + 5\rho^2)}{(1+\rho)}\right] \quad (4.6)$$

$$= \left(\frac{C^2\overline{Y}^2}{n}\right)\left[(1-\rho^2) + \frac{C^2}{4n}\rho(1-\rho)(4 + 15\rho + 5\rho^2)\right] \quad (4.7)$$

Here, we note that

$$MSE(t_{Me})_I = MSE(\bar{y}_{lr})_I = \frac{C^2\overline{Y}^2}{n}(1-\rho^2) \quad (4.8)$$

Inserting $G = -1$ and $G = 1$ in eq. (4.4), we get the MSE expressions for Ratio-Type and Product-Type Exponential Estimators to the second order of approximation, respectively as

$$MSE(t_{\mathrm{Re}})_{II} = MSE(t_{\mathrm{Re}})_I\left[1 + \frac{C^2}{16n}\frac{(143 - 248\rho + 128\rho^2)}{(5 - 4\rho)}\right]$$

$$= \frac{C^2\overline{Y}^2}{n}\left[\left(\frac{5}{4} - \rho\right) + \frac{C^2}{64n}(143 - 248\rho + 128\rho^2)\right] \quad (4.9)$$

And $MSE(t_{Pe})_{II} = MSE(t_{Pe})_I\left[1 + \dfrac{C^2}{16n}\dfrac{(7 - 40\rho)}{(5 + 4\rho)}\right]$

$$= \frac{C^2\overline{Y}^2}{n}\left[\left(\frac{5}{4} + \rho\right) + \frac{C^2}{64n}(7 - 40\rho)\right] \quad (4.10)$$

where,

$$MSE(t_{\mathrm{Re}})_I = \frac{C^2\bar{Y}^2}{4n}(5-4\rho) \tag{4.11}$$

and

$$MSE(t_{Pe})_I = \frac{C^2\bar{Y}^2}{4n}(5+4\rho) \tag{4.12}$$

From eq. (4.6) and eq. (4.9), we have,

$$MSE(t_{\mathrm{Re}})_{II} - MSE(t_{Me})_{II} = \frac{C^2\bar{Y}^2}{n}\left[\left(\frac{1}{2}-\rho\right)^2 + \frac{C^2}{64n}(143-312\rho-48\rho^2+160\rho^3+80\rho^4)\right]$$

which is positive if

$$\left[\left(\frac{1}{2}-\rho\right)^2 + \frac{C^2}{64n}(143-312\rho-48\rho^2+160\rho^3+80\rho^4)\right] > 0 \tag{4.13}$$

Further, from eq. (4.7) and eq. (4.10), we have,

$$MSE(t_{Pe})_{II} - MSE(t_{Me})_{II} = \frac{C^2\bar{Y}^2}{n}\left[\left(\frac{1}{2}+\rho\right)^2 + \frac{C^2}{64n}\left(7-104\rho-176\rho^2+160\rho^3+80\rho^4\right)\right]$$

which is positive if

$$\left[\left(\frac{1}{2}+\rho\right)^2 + \frac{C^2}{64n}\left(7-104\rho-176\rho^2+160\rho^3+80\rho^4\right)\right] > 0 \tag{4.14}$$

To compare t_{Me} with regression Estimator $\bar{y}_{lr}$ we have from eq. (2.9) and eq. (4.7) that for large sample the former are approximately as efficient as the latter, if a good guess G_0 is available. In case the sample (not very large) is drawn from a large Bivariate Normal population, the Mean Squared Error of the usual Regression Estimator $\bar{y}_{lr}$; to the second order approximation(i.e. terms up to n^{-2}) .[Cochran (1967), 7.4, p.198, $(\gamma_1 = 0, \gamma_2 = 0)$ for Normal Distribution] given as

$$MSE(\bar{y}_{lr})_{II} = C^2\bar{Y}^2(1-\rho^2)\left(\frac{1}{n}+\frac{1}{n^2}\right) \tag{4.15}$$

Also, for this case, if a good guess of G_0 is available, we have from eq. (4.6) and eq. (4.15)

$$MSE(\bar{y}_{lr})_{II} - MSE(t_{Me})_{II} = \frac{C^2\bar{Y}^2}{n^2}\left[\left(1-\rho^2\right) - \frac{C^2}{4}\rho(1-\rho)\left(4+15\rho+5\rho^2\right)\right] > 0$$

if $$\left[\left(1-\rho^2\right)-\frac{C^2}{4}\rho\left(4+15\rho+5\rho^2\right)\right]>0 \tag{4.16}$$

Thus the supposition $C_x = C_y = C$; say (a =1) and that the sample comes from a large Bivariate Normal population, the Mean Squared Error of the usual Ratio Estimator $t_R = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)$ and the Product Estimator $t_P = \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right)$ to the second degree of approximation are respectively given by

$$MSE(t_R)_{II} = MSE(t_R)_I\left[1+\frac{C^2}{n}(6-3\rho)\right]$$

$$=\left[\frac{C^2\bar{Y}^2}{n}\left(2(1-\rho)+\frac{6C^2}{n}\left(2-3\rho+\rho^2\right)\right)\right] \tag{4.17}$$

and

$$MSE(t_P)_{II} = MSE(t_P)_I\left[1+\frac{C^2}{2n}\frac{\left(1+2\rho^2\right)}{(1+\rho)}\right]$$

$$=\left(\frac{C^2\bar{Y}^2}{n}\right)\left[2(1+\rho)+\frac{C^2}{n}\left(1+2\rho^2\right)\right] \tag{4.18}$$

From eq. (4.9) and eq. (4.17), we have,

$$MSE(t_R)_{II} - MSE(t_{\mathrm{Re}})_{II} = \left(\frac{C^2\bar{Y}^2}{n}\right)\left[\left(\frac{3}{4}-\rho\right)+\frac{C^2}{n}\left\{6(1-\rho)(2-\rho)-\frac{1}{64}\left(143-248\rho+128\rho^2\right)\right\}\right]$$

which is greater than 'zero' if

$$\left[\left(\frac{3}{4}-\rho\right)+\frac{C^2}{n}\left\{6(1-\rho)(2-\rho)-\frac{1}{64}\left(143-248\rho+128\rho^2\right)\right\}\right]>0 \tag{4.19}$$

Next, from eq. (4.9) and eq. (4.18), we have,

$$MSE(t_P)_{II} - MSE(t_{Pe})_{II} = \left(\frac{C^2\bar{Y}^2}{n}\right)\left[\left(\frac{3}{4}+\rho\right)+\frac{C^2}{64n}(57+168\rho)\right]$$

which is greater than 'zero' if

$$\left[\left(\frac{3}{4}+\rho\right)+\frac{C^2}{64n}(57+168\rho)\right]>0 \tag{4.20}$$

Thus the Ratio-Type (t_{Re}) and Product-Type (t_{Pe}) Exponential Estimators are better than the usual Ratio (t_R) and Product (t_P) Estimators, respectively, as long as the conditions eq. (4.19) and eq. (4.20) are satisfied.

5. Empirical Study

To examine the merits of the suggested Estimator we have considered five natural population data sets. The descriptions of the population are given below

Population 1: [Murthy (1967)]
y : Output,
x : Fixed Capital,
$N = 80$, $n = 20$, $\bar{Y} = 51.8264$, $\bar{X} = 11.2646$, $C_y = 0.3542$,
$C_x = 0.7507$, $\rho = 0.9413$, $C = 0.4441$, $f = 0.25$.

Population 2: [Murthy (1967)]
y : Output,
x : Number of workers,
$N = 80$, $n = 20$, $\bar{Y} = 51.8264$, $\bar{X} = 2.8513$, $C_y = 0.3542$,
$C_x = 0.9484$, $\rho = 0.9150$, $C = 0.3417$, $f = 0.25$.

Population 3: [Das (1988)]
y : Number of agricultural laborers for 1971,
x : Number of agricultural laborers for 1961,
$N = 278$, $n = 30$, $\bar{Y} = 39.0680$, $\bar{X} = 25.1110$, $C_y = 1.4451$,
$C_x = 1.6198$, $\rho = 0.7213$, $C = 0.6435$, $f = 0.1079$.

Population 4: [Steel and Torrie (1960)]
y : Log of leaf burn in secs,
x : Chlorine percentage,
$N = 30$, $n = 6$, $\bar{Y} = 0.6860$, $\bar{X} = 0.8077$, $C_y = 0.700123$,
$C_x = 0.7493$, $\rho = -0.4996$, $C = -0.3203$, $f = 0.20$.

Population 5: [Maddala (1977)]
y : Consumption per capita,
x : Deflated price of veal,
$N = 16$, $n = 4$, $\bar{Y} = 7.6375$, $\bar{X} = 75.4313$, $C_y = 0.2278$,
$C_x = 0.0986$, $\rho = -0.6823$, $C = -1.5761$, $f = 0.25$.

We have computed the range of G and percent relative efficiencies of different Estimators $(\bar{y}, t_R, t_P, t_{\mathrm{Re}}, t_{Pe}$ and $t_{Me})$ of the population mean $\bar{Y}$ with respect to $\bar{y}$. Findings are compiled in Tables 1, 2 and 3.

It is observed from Table 1, 2 and 3 that there is enough scope of selecting the value of scalar $G(or\theta)$ in order to get Estimators better than sample mean $\bar{y}$, t_R, t_P, t_{Re} and t_{Pe}. We also note that the Proposed class of Estimators t_{Me} is better than Conventional Estimators $\bar{y}$ even if the scalar $G(or\theta)$ slider away from the true optimum value of $G(or\theta)$. Larger gain in efficiency is observed if the scalar $G(or\theta)$ moves around the vicinity of the true optimum value.

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Table 1: Range of G in which Proposed Estimator t_{Me} is better than $\bar{y}$, t_R, t_P, t_{Re} and t_{Pe}.

Population	Range of G in which Proposed Estimator t_{Me} is better than					Common range of G in which t_{Me} is better than $\bar{y}$, t_R, t_P, t_{Re} and t_{Pe}
	$\bar{y}$	t_R	t_P	t_{Re}	t_{Pe}	
1	(-1.7765,0)	(-2,0.2235)	(-3.7765,2)	(-1,-0.7765)	(-2.7765,1)	(-1,-0.7765)
2	(-1.3669,0)	(-2,0.6331)	(-3.3669,2)	(-1,-0.3669)	(-2.3669,1)	(-1,-0.3669)
3	(-2.5742,0)	(-2,-0.5742)	(-4.5742,2)	(-1.5742,-1)	(-3.5742,1)	(-1.5742,-0.5740)
4	(0,1.8673)	(-2,3.8673)	(-0.1328,2)	(-1,2.8673)	(0.8673,1)	(0.8673,1)
5	(0,6.3059)	(-2,8.3059)	(2,4.3054)	(-1,7.3059)	(1,5.3059)	(-1,4.3054)

Table 2: PREs of Estimators $\bar{y}$, t_R , t_P , t_{Re} and t_{Pe} with respect to $\bar{y}$.

Estimator	**PRE** $(\bullet, \bar{y})$				
	Population				
	1	**2**	**3**	**4**	**5**
$\bar{y}$	100.0000	100.0000	100.0000	100.0000	100.0000
t_R	66.5810	30.5860	156.3967	31.1061	56.2431
t_P	10.5463	7.6514	25.8171	92.9342	167.5887
t_{Re}	781.3982	292.0779	197.7846	54.9135	74.5067
t_{Pe}	24.2836	19.0754	47.1121	133.0386	133.0649

Table 3: PREs of t_{Me} with respect to $\bar{y}$ for different values of G .

G	θ	$PRE(t_{Me}, \bar{y})$				
		Population				
		1	**2**	**3**	**4**	**5**
-2.0000	-3.0000	66.5810	30.5860	156.3967	31.1061	56.2431
-1.7500	-3.6667	105.4984	45.4209	182.7966	35.5534	60.2317
-1.5000	-5.0000	187.1938	73.6467	202.4393	40.8775	64.5841
-1.2500	-9.0000	383.2829	135.4864	208.2654	47.2636	69.3319
-1.0000	0.0000	781.3982	292.0779	197.7846	54.9135	74.5067
-0.7500	7.0000	738.4366	585.7795	175.3442	64.0167	80.1386
-0.5000	3.0000	353.0569	448.2367	148.3074	74.6863	86.2537
-0.2500	1.6667	174.9985	200.1900	122.3232	86.8380	92.8714
0.0000	1.0000	100.0000	100.0000	100.0000	100.0000	100.0000
0.2500	0.6000	63.7373	57.9872	81.8494	113.0935	107.6315
0.5000	0.3333	43.8933	37.4100	67.4411	124.3406	115.7344
0.7500	0.1429	31.9699	26.0031	56.0835	131.5695	124.2463
1.0000	0.0000	24.2836	19.0754	47.1121	133.0386	133.0649
1.2500	-0.1111	19.0533	14.5708	39.9776	128.3594	142.0399
1.5000	-0.2000	15.3392	11.4840	34.2528	118.7285	150.9669
1.7500	-0.2727	12.6097	9.2794	29.6138	106.2421	159.5862
2.0000	-0.3333	10.5463	7.6514	25.8171	92.9342	167.5887

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