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Size Biased Lomax Distribution

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Abstract

Weighted Distributions have very significant place in the mathematical statistics, particularly in the case of Unequal Probability Sampling. Size Biased Distribution is a particular case of Weighted Probability Distribution. In this paper, Size Biased form of Lomax Probability Distribution is introduced. Some basic properties of Size Biased Lomax (SBL) are derived. Reliability measures including the Survival Analysis, Hazard Rate and Vitality functions are computed. Measures of entropy using Shannon's and Renyi's methods are also found. Estimation of its parameters is made through method of Maximum Likelihood and Method of Moments. Lomax Distribution has applications in various fields including size of cities. In this paper, it is showed that SBL fits the data on size of cities of Pakistan in km^2.

Keywords

Lomax Distribution, Survival function, Hazard rate function, Entropy

1. Introduction

Initially, Lomax Distribution was introduced by Lomax (1954) to model the business failure rate. It is also known as Pareto Type-II Distribution.

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Adeyemi and Adebanji (2007) considered Lomax Distribution as a subclass of Burr Type Distribution. Shakil and Ahsanullah (2012) showed that if $X_1 \sim Exp(\lambda)$ and $X_2 \sim Gamma(\sigma)$ then the Distribution of the ratio $X = \frac{X_1}{X_2}$ follows the Lomax Distribution denoted as $Lom(x;\lambda,\sigma)$, where λ is scale and σ is shape parameter. Ashour et al. (2011) discussed its use in modeling the size of cities and other important areas like queuing theory, biomedical, economical and Survival Analysis. It is also used for income and wealth distribution, firm size, size of files on a server. Mahmoud et al. (2013) showed that Lomax Distribution is a mixture of Exponential and Gamma Distribution.
Its probability density function is

$$f(x;\lambda,\alpha) = \frac{\alpha}{\lambda}\left[1+\frac{x}{\lambda}\right]^{-(\alpha+1)} \quad , \quad x \geq 0, \alpha > 0, \lambda > 0 \tag{1.1}$$

Idea of Weighted Distribution was introduced by Fisher (1934). Later on; Rao (1965) extended this idea. Weighted Distributions come up when the observations produced from a stochastic process are not given equal chance of being recorded; instead, they are recorded according to some Weight function. When the Weight function relies on the lengths of the units of interest, the Resultant Distribution is called Length Biased. More generally, when the sampling methodology selects units with probability proportioned to some measure of the unit size, the resulting Distribution is called Size-Biased. Such Distributions arise in life length studies.

Consider a non-negative random variable *X* having the random sample $(x_1, x_2, \ldots, x_n)$ taken from a population, and let the observation *x* of the random variable *X* recorded by the investigator with probability proportional to *w(x)* such as
P (Recording | X= x) = w(x),

where,
w(x) is the Non-negative Weight function, in such situation, as discussed above, it is not possible to have Simple Random Sample. So, the Distribution of the observed random variable will differ by the actual random variable, and in turn it will generate Bias. In order to meet the situation the authors have to use the Moment / Weighted Distribution.

Let *f(x; θ)* be the probability density function of the random variable *X* and *θ* be the unknown parameter, then Weighted Distribution may be expressed as

$$g(x;\theta) = \frac{w(x)\, f(x;\theta)}{E\left[w(x)\right]} \tag{1.2}$$

Patil and Ord (1976) used w(x) = x^m, and they call it Moment Distribution.

$$g(x;\theta) = \frac{x^m f(x;\theta)}{\mu_m^{'}} \tag{1.3}$$

Here, $\mu_m^{'} = \sum x^m \; f(x;\theta)$

For discrete case

$$\mu_m^{'} = \int_{-\infty}^{\infty} x^m \; f(x;\theta)\, dx$$

For continuous case

For m = 1 it's called the Size/Length Biased

For m = 2 it's called the area Biased Distribution

Dara (2011) derived various Moment Distributions and their Size-Biased forms specifically their reliability measures. Nasiri and Hosseini (2012) obtained Maximum Likelihood Estimates (MLE) and Bayesian Estimates under two loss functions of parameters of the Lomax Distribution based on record values. Ahmed et al. (2013) derived a new class of Length-Biased Gamma Distribution and discussed its structural properties. Hasnain (2013) introduced a new family of Distributions named as Exponentiated Moment Exponential Distribution (EMED) and developed its properties. Iqbal, et al. (2013) found a more general class for EMED and built up different properties including characterization through conditional moments.

2. Properties of Size Biased Lomax Distribution

The Size Biased form of Lomax Distribution may be obtained as

$$g(x) = \frac{\alpha(\alpha-1)}{\lambda^2}\, x \left[1+\frac{x}{\lambda}\right]^{-(\alpha+1)} \qquad x>0, \alpha>1, \lambda>0 \tag{2.1}$$

Its cumulative distribution function (c.d.f) is given as

$$G(x) = 1-\left(1+\frac{x}{\lambda}\right)^{-\alpha} \left\{1+\alpha\frac{x}{\lambda}\right\} \qquad x>0, \alpha>1, \lambda>0 \tag{2.2}$$

Graph of the p.d.f given by eq. (2.1) is in Figure 1 and 2.

As value of α increase, peak of the graph becomes sharper, and with increase in value of λ its tail becomes heavy. Graph for Distribution function I given in Figure 3.

2.1 Hazard Rate Function: Hazard Rate Function also known as failure rate may be defined as the conditional probability of failure of an item in the interval $[x, x+h]$ where $h \to 0$ given that certain item is existing till age *x*. it is given as

$$h(x) = \frac{f(x)}{\overline{F}(x)} \tag{2.1.1}$$

It has various names in different areas of study e.g. in actuaries and demography it is called force to mortality and in extreme value theory it is known as Intensity Function. It is useful in various fields of life such as Survival Analysis, biomedical sciences, engineering and in modeling life time data.
Hazard Rate Function for SBL can take the form,

$$h(x) = \frac{\alpha(\alpha-1)}{\lambda}\frac{x}{\lambda}\left[1+\frac{x}{\lambda}\right]^{-1}\left\{1+\alpha\frac{x}{\lambda}\right\}^{-1} \quad x>0, \alpha>1, \lambda>0 \tag{2.1.2}$$

Graph of Hazard Rate Function is given in Figure 4.
Hazard Rate Function of SBL Distribution is upside down (bathtub) shaped.

2.2 Cumulative Hazard Function: Cumulative Hazard Function is given as

$$H(x) = \ln\left\{\frac{\left(1+\frac{x}{\lambda}\right)^{\alpha}}{\left(1+\alpha\frac{x}{\lambda}\right)}\right\} \qquad x>0, \alpha>1, \lambda>0 \tag{2.2.1}$$

2.3 Survival Function: The function $\overline{F}(x) = 1 - F(x) = P(X \geq x)$ is called Survival Function or Reliability Function.

$\overline{F}(x)$ is a non-increasing continuous function with $\overline{F}(0) = 1$ and $\lim_{x\to\infty}\overline{F}(x) = 0$.

In other words, it is complement to c.d.f. and it gives probability of surviving to at least time *t*. It is generally denoted as

$S(x) = P(X > x)$

Survival Function of SBL is as follows:

$$S(x) = 1 - G(x) = \left(1+\frac{x}{\lambda}\right)^{-\alpha}\left\{1+\alpha\frac{x}{\lambda}\right\} \qquad x \geq 0, \alpha>1, \lambda>0 \tag{2.3.1}$$

2.4 Vitality Function: The Vitality Function *v(x)*, of a random variable *X* with an absolutely continuous distribution function F(x) is given as

$$v(x) = E\left[X \,\middle|\, X > x\right] = \frac{1}{\overline{F}(x)} \int_x^{\infty} t dF(t)$$

$$v(x) = \frac{1}{S(x)} \int_x^{\infty} tf(t) dt$$

Vitality Function of SBL Distribution is

$$v(x) = \frac{\lambda}{(\alpha-2)} \left[1 + \frac{\alpha}{\lambda} x\right]^{-1} \left\{ \frac{x^2}{\lambda^2} (\alpha-1)(\alpha-4) - 2\alpha \frac{x}{\lambda} - 2 \right\}, \; x > 0, \alpha > 4, \lambda > 0$$

(2.4.1)

2.5 Mean Residual Function: Mean Residual Life Function (MRLF) m(x), for a random variable *X* with *E(X)* < ∞, is given as

$$e(x) = E\left(X - x \middle| X > x\right).$$

It computes the average lifetime remaining for an item, which has already survived up to time *x*. It is given as

$$e(x) = \frac{1}{\overline{F}(x)} \int_x^{\infty} \overline{F}(t) dt$$

$$e(x) = \frac{\int_x^{\infty} S(t) dt}{S(x)}$$

Mean Residual Function for SBL Distribution is

$$e(x) = \frac{\lambda[1 + x/\lambda] \left[1 + \frac{\alpha}{(\alpha-2)}[1 + x/\lambda]\right]}{\left(1 + \alpha \frac{x}{\lambda}\right)}, \qquad x > 0, \alpha > 2, \lambda > 0 \qquad (2.5.1)$$

2.6 Moments: Moments about zero (Raw moments) for SBL are

$$E(X^r) = \frac{\lambda^r (r+1)!}{(\alpha-2)(\alpha-3)...(\alpha-r-1)} \qquad x \geq 0, \alpha > r+1, \lambda > 0; \qquad (2.6.1)$$

$$\mu_1' = \frac{2\lambda}{(\alpha-2)} \quad ; \qquad \alpha > 2 \qquad (2.6.2)$$

$$\mu_2' = \frac{6\lambda^2}{(\alpha-2)(\alpha-3)} \quad ; \quad \alpha > 3 \tag{2.6.3}$$

$$\mu_3' = \frac{24\lambda^3}{(\alpha-2)(\alpha-3)(\alpha-4)} \quad ; \quad \alpha > 4 \tag{2.6.4}$$

$$\mu_4' = \frac{120\lambda^4}{(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-5)} \quad ; \quad \alpha > 5 \tag{2.6.5}$$

Moments about mean (central moments) are

$$\mu_1 = 0 \tag{2.6.6}$$

$$\mu_2 = \frac{2\alpha\lambda^2}{(\alpha-2)^2(\alpha-3)} \quad ; \quad \alpha > 3 \tag{2.6.7}$$

$$\mu_3 = \frac{4\lambda^3\alpha(a+2)}{(\alpha-2)^3(\alpha-3)(\alpha-4)} \quad ; \quad \alpha > 4 \tag{2.6.8}$$

$$\mu_4 = \frac{24\lambda^4\alpha(\alpha^2+2)}{(\alpha-2)^4(\alpha-3)(\alpha-4)(\alpha-5)} \quad ; \quad \alpha > 5 \tag{2.6.9}$$

Recurrence relationship between raw moments is

$$\mu_r' = \mu_{r-1}' \frac{(r+1)\lambda}{\alpha-r-1}; \quad \alpha > r+1 \tag{2.6.10}$$

2.7 Moment ratios:

$$\beta_1 = \frac{2(\alpha+2)^2(\alpha-3)}{\alpha(\alpha-4)^2} \; ; \quad \alpha > 4 \tag{2.7.1}$$

Measure of Skewness is

$$\sqrt{\beta_1} = \sqrt{\frac{2(\alpha+2)^2(\alpha-3)}{\alpha(\alpha-4)^2}}$$

$$\sqrt{\beta_1} = \sqrt{\frac{2(\alpha-3)}{\alpha}} \frac{\alpha+2}{\alpha-4}$$

It is always positive.

$$\beta_2 = \frac{6(\alpha^2+2)(\alpha-3)}{\alpha(\alpha-4)(\alpha-5)} \; ; \quad \alpha > 5 \tag{2.7.2}$$

As the value of α increase, coefficient of Skewness approaches to zero and measure of Kurtosis approaches to three. Moreover, both Skewness and Kurtosis are free of λ.

2.8 Median: Median of SBL Distribution can be obtained by solving

$G(m) = 0.5$

$$\left(1+\frac{m}{\lambda}\right)^{\alpha} - 2\alpha\frac{m}{\lambda} - 2 = 0 \qquad (2.8.1)$$

where,

G is the c.d.f of SBL Distribution defined in eq. (2.2).

Table 1 represents values of the median for various values of α and λ. Value of the median decreases for increasing α when λ is fixed and opposite results are obtained for increasing λ when α is fixed. This phenomenon is shown in the Figure 9 and 10.

2.9 Geometric Mean:

$E(\ln X) = \ln\lambda + \alpha\{\psi(\alpha) - \psi(\alpha-1)\} - C - \psi(\alpha)$

where,

C is Euler Function. (2.9.1)

$GM = e^{E(\ln X)} = \lambda e^{\alpha[\Psi(\alpha)-\Psi(\alpha-1)]-C-\Psi(\alpha)}$

where,

$$\Psi(x) = \frac{d}{dx}\Gamma(x) \qquad (2.9.2)$$

2.10 Harmonic Mean:

$$E(X^{-1}) = E\left(\frac{1}{X}\right) = \frac{(1-1)!(\alpha+1-2)!}{\lambda(\alpha-2)!} \qquad (2.10.1)$$

$$HM = \frac{1}{E\left(\frac{1}{X}\right)} = \frac{\lambda}{\alpha-1} \qquad (2.10.2)$$

2.11 Mode:

Mode is $\hat{x} = \frac{\lambda}{\alpha}$ (2.11.1)

Table 2 represents values of the mode for various values of α and λ. Value of the Mode decreases for increasing α when λ is fixed and opposite results are obtained for increasing λ when α is fixed. One special property of this Table is that for every value of λ when α is fixed, increment in the value of mode is constant (same), but fixing λ increasing α results in decreasing the value of Mode, and rate of decrease also decreases.

3. Entropy measures

The idea of Entropy in information theory was developed by Shannon (1948). It is a quantitative measure of uncertainty of information related to a random phenomenon. Like measure of dispersion, low Entropy in a Distribution indicates more concentration and more information as compared to high Entropy. Entropy is very useful in Reliability and Survival Analysis problems.

3.1 Shannon's Entropy: Shannon (1948) defined Entropy as

$H\{f(x)\} = E[-\log\{f(x)\}]$

For this, SBL Distribution Shannon's Entropy is calculated as

$$H\{f(x)\} = -\log\left[\alpha(\alpha-1)\right] + \log\lambda + C + (\alpha+1)\psi(\alpha+1) - (\alpha-1)\psi(\alpha) - \psi(\alpha-1) \quad (3.1.1)$$

where,

C = Euler Function, $C = 0.5772...$ and $\Psi(x) = \frac{d}{dx}\Gamma(x)$

***3.2 Renyi's Entropy*:** Renyi's Entropy (1961) purposed Entropy measure as

$$H_q(x) = \frac{1}{1-q}\log\int_{-\infty}^{+\infty}\{f(x)\}^q dx \qquad q \geq 0 \text{ and } q \neq 1$$

For SBL Distribution, Renyi Entropy is calculated as

$$H_q(x) = \frac{1}{1-q}\left\{q\log\alpha + q\log(\alpha-1) - (q-1)\log\lambda + \log q! + \log(q\alpha-2)! - \log(q+q\alpha-1)!\right\} \quad (3.2.1)$$

3.3 Generalized Entropy: Generalized Entropy is defined as

$$I(\beta) = \frac{v_\beta \mu^{-\beta} - 1}{\beta(\beta-1)} \quad \beta \neq 0,1 \text{ (Jenkins,2007)}$$

where,

$$v_\beta = \int_{-\infty}^{\infty} x^\beta f(x)dx \text{ and } \mu = mean$$

For SBL Distribution

$$v_\beta = \frac{\lambda^\beta(\beta+1)!}{(\alpha-2)(\alpha-3)...(\alpha-\beta-3)} \qquad \text{and} \qquad \mu = \frac{2\lambda}{\alpha-2}$$

$$I(\beta)=\frac{1}{\beta(\beta-1)}\left(\frac{\lambda^{\beta}(\beta+1)!}{(\alpha-2)(\alpha-3)...(\alpha-\beta-3)}\left(\frac{2\lambda}{\alpha-2}\right)^{-\beta}-1\right)$$

$$I(\beta)=\frac{1}{\beta(\beta-1)}\left(\frac{(\beta+1)!}{(\alpha-2)(\alpha-3)...(\alpha-\beta-3)}\left(\frac{\alpha-2}{2}\right)^{\beta}-1\right),$$

$$I(\beta)=\frac{(\beta+1)(\beta-2)!}{(\alpha-3)...(\alpha-\beta-3)}\left(\frac{\alpha-2}{2}\right)^{\beta-1}-1,\alpha>\beta-3,\beta>1 \tag{3.3.1}$$

4. Parameter Estimation

4.1 Estimation of parameters by Maximum Likelihood method: Probability distribution function of SBL given by eq. (2.1) is

$$g(x)=\frac{\alpha(\alpha-1)}{\lambda}\frac{x}{\lambda}\left[1+\frac{x}{\lambda}\right]^{-(\alpha+1)}$$

Likelihood Function is

$$L=\left(\frac{\alpha(\alpha-1)}{\lambda^{2}}\right)^{n}\prod_{i=1}^{n}(x_i)\prod_{i=1}^{n}\left[1+\frac{x_i}{\lambda}\right]^{-(\alpha+1)}$$

$$l=n\log\alpha+n\log(\alpha-1)-2n\log\lambda+\sum_{i=1}^{n}\log x_i-(\alpha+1)\sum_{i=1}^{n}\left[1+\frac{x_i}{\lambda}\right]^{-(\alpha+1)}$$

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}+\frac{n}{\alpha-1}-\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\lambda}\right)$$

$$\frac{\partial l}{\partial\lambda}=\frac{-2n}{\lambda}+\frac{\alpha+1}{\lambda}\sum_{i=1}^{n}\left(\frac{x_i}{\lambda+x_i}\right)$$

Putting $\frac{\partial l}{\partial\alpha}$ and $\frac{\partial l}{\partial\lambda}$ equal to 0 .

$$\frac{n}{\alpha}+\frac{n}{\alpha-1}-\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\lambda}\right)=0 \tag{4.1.1}$$

$$\frac{-2n}{\lambda}+\frac{\alpha+1}{\lambda}\sum_{i=1}^{n}\left(\frac{x_i}{\lambda+x_i}\right)=0$$

$$-2n+(\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{\lambda+x_i}\right)=0 \tag{4.1.2}$$

(4.1.1) and (4.1.2) are not in closed form therefore for solution we shall solve them numerically.

4.2 Method of Moments: First sample and population moment about origin are,

$m_1' = \overline{X}$ and $\mu_1' = \dfrac{2\lambda}{\alpha - 2}$

By comparing both raw moment, we get,

$$\overline{X} = \frac{2\lambda}{\alpha - 2} \tag{4.2.1}$$

Second sample and population moment about mean are

$m_2 = S^2$ and $\mu_2 = \dfrac{2\alpha\lambda^2}{(\alpha-2)^2(\alpha-3)}$

By comparing both central moments, we get,

$$S^2 = \frac{2\alpha\lambda^2}{(\alpha-2)^2(\alpha-3)} \tag{4.2.2}$$

Solving eq. (4.2.1) and (4.2.2)

$$\hat{\alpha} = \frac{6S^2}{2S^2 - \overline{X}^2}$$

Put in eq. (4.2.1)

$$\hat{\lambda} = \overline{X}\left(\frac{S^2 + \overline{X}^2}{2S^2 - \overline{X}^2}\right)$$

5. Application

To illustrate the performance of purposed SBL Distribution we consider the data of size of 142 cities (Districts) including Baluchistan (31), Khyber Pakhtunkhwa (26), Punjab (35), Sindh (20) FATA (13), Gilgit-Baltistan (7), Azad Jammun and Kashmir (10) of Pakistan in km^2 given by the website given by the website http://en.wikipedia.org/wiki/List_of_districts_of_Pakistan, area of cities are in Table 3.

R i386 1.3.1 is used to classify, estimate the parameters by the Method of Moments and to find value of Chi-square statistic to compare SBL with Classical Lomax Distribution for the above data.

Parameter estimates obtained from the fitting of the Size Biased Lomax Distribution and Classical Lomax Distribution to the size of cities data in Pakistan by Method of Moments.

Table 4 gives value of Chi-square statistics for SBL is $\chi^2 = 6.052712$ that is less than value of Chi-square statistic for Classical Lomax Distribution $\chi^2 = 9.307393$. It means that SBL best fits the sizes of cities than Classical Lomax Distribution.

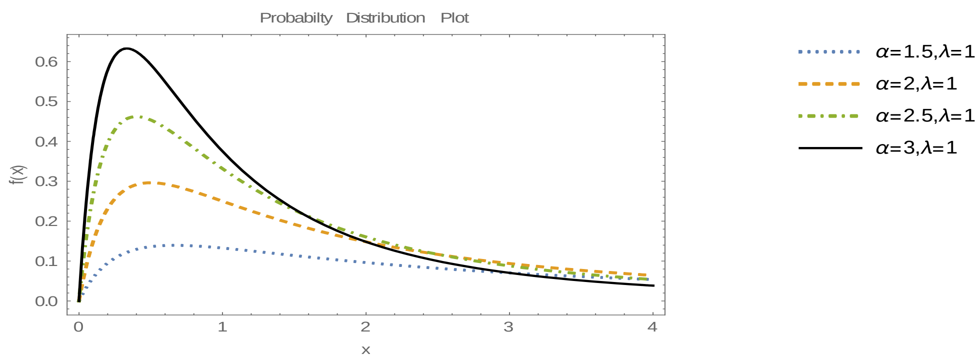

Figure 1: Size Biased Lomax for various values of α keeping λ at 1

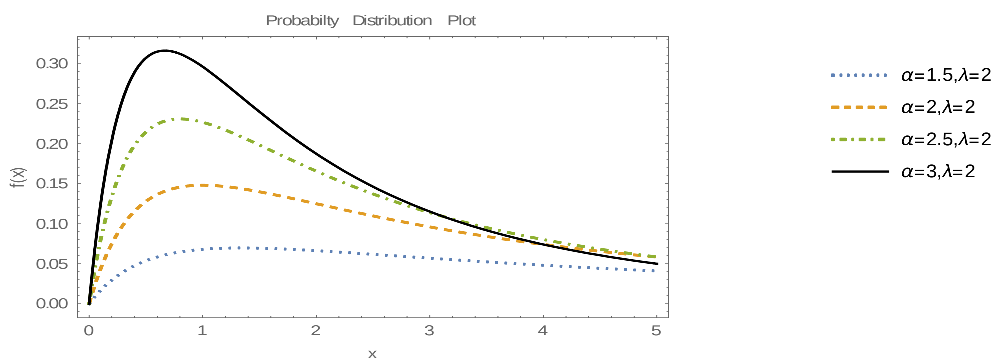

Figure 2: Size Biased Lomax for various values of α keeping λ at 2

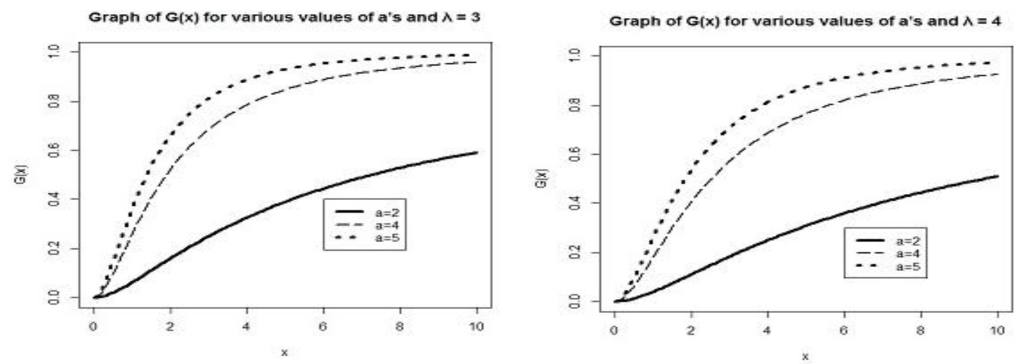

Figure 3: c.d.f of SBL for different values of α keeping λ as 3 and 4.

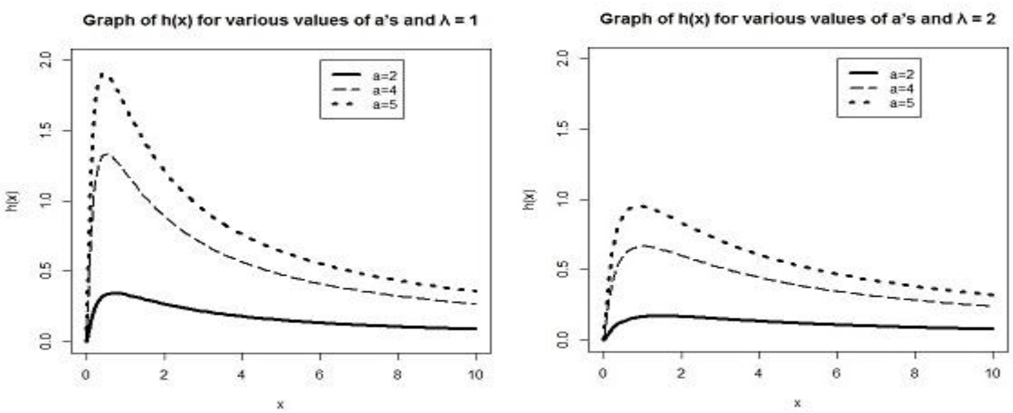

Figure 4: Hazard Rate Function of SBL for various values of α and λ

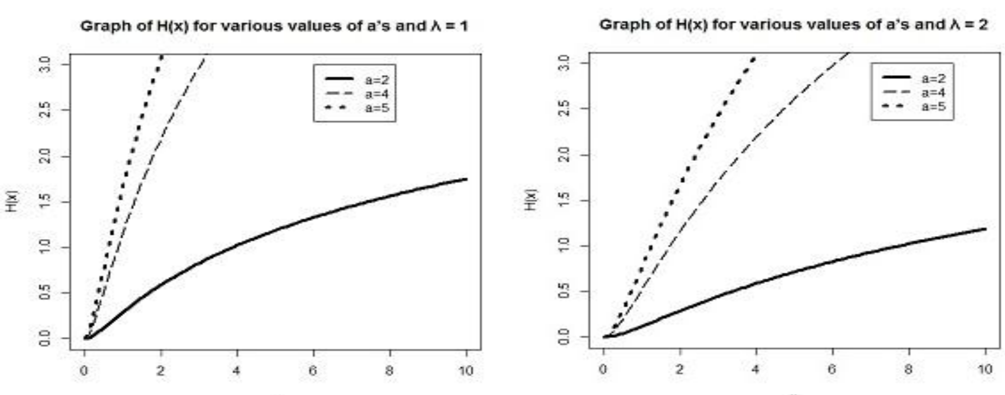

Figure 5: Cumulative Hazard Rate Function of SBL for various values of α and λ

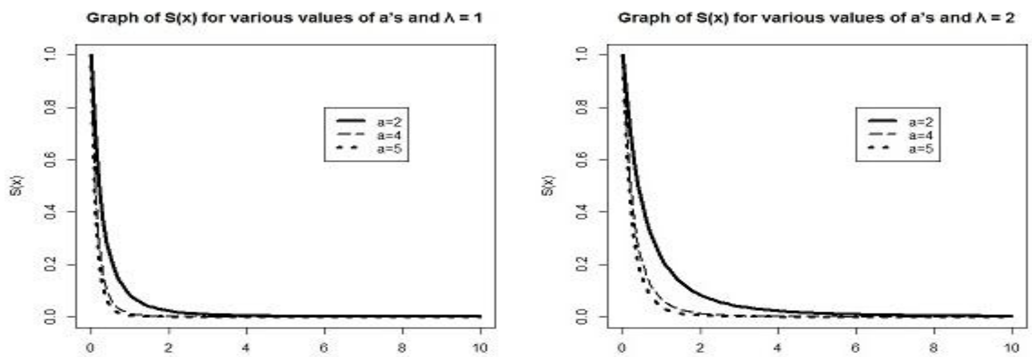

Figure 6: Survival Function of SBL for various values of α and λ

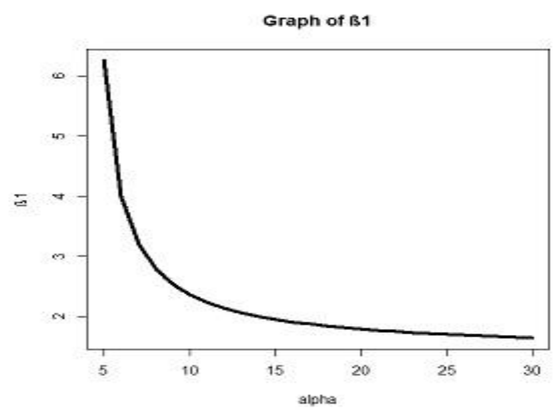

Figure 7: Skewness ($\sqrt{\beta_1}$) of SBL for various values of α and λ

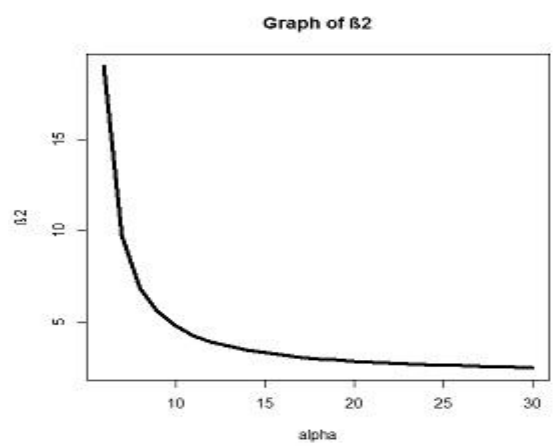

Figure 8: Kurtosis (β_2) of SBL for various values of α and λ

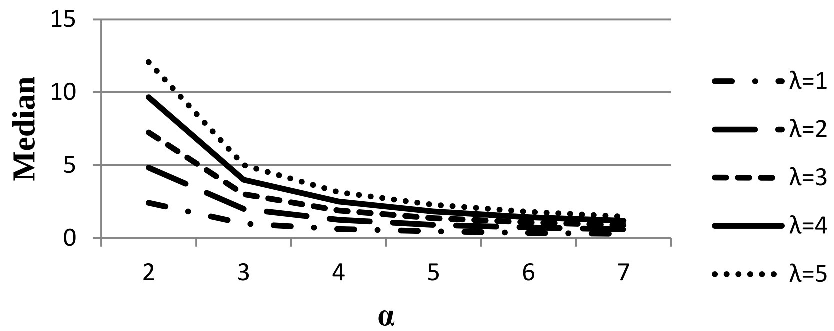

Figure 9: Median for various values of α and λ.

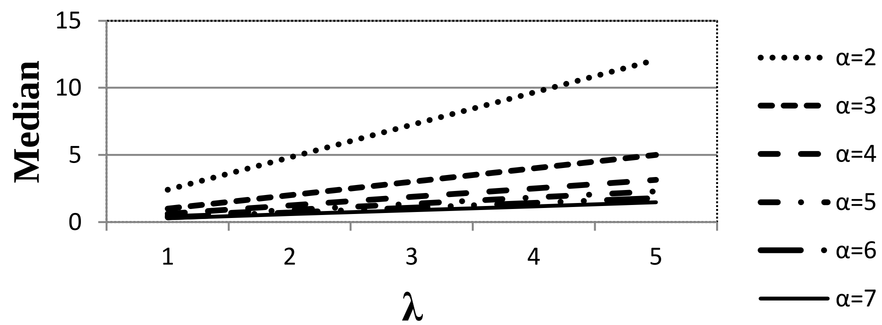

Figure 10: Median for various values of λ and α.

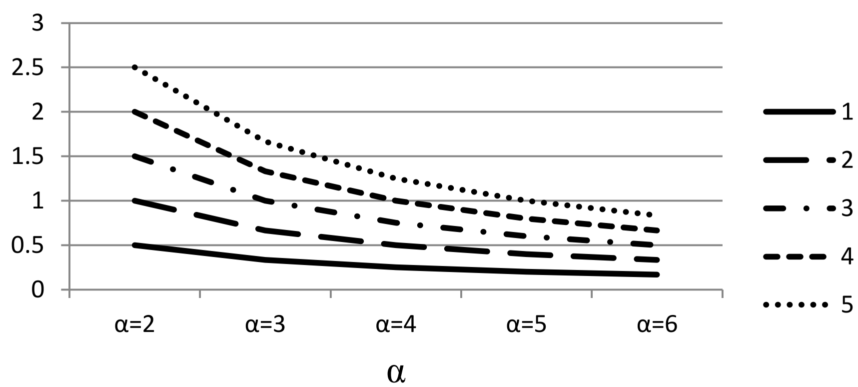

Figure 11: Mode for various values of α and fixing various values of λ

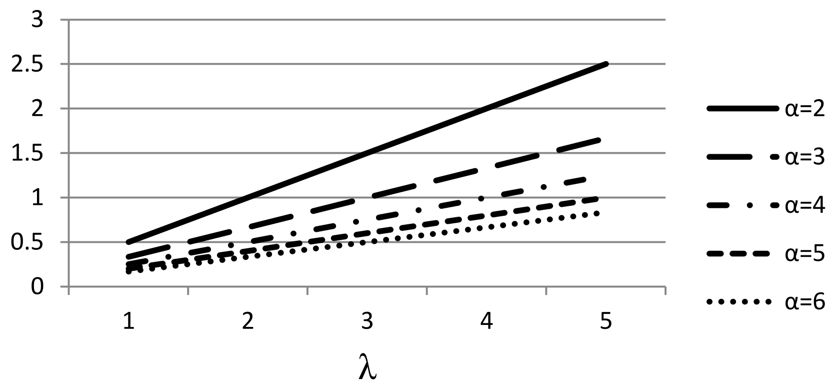

Figure 12: Mode for various values of λ and fixing various values of **α**

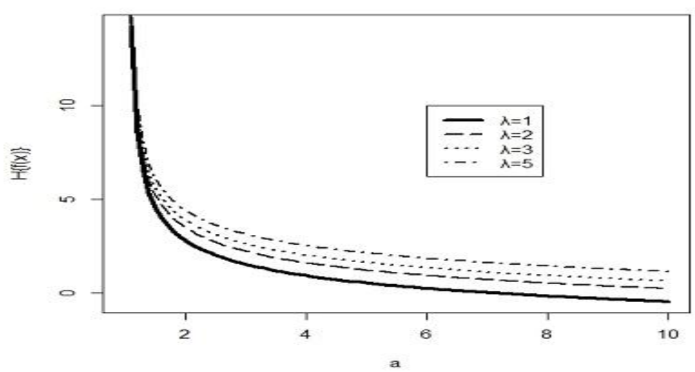

Figure 13: Graph for Shannon's Entropy

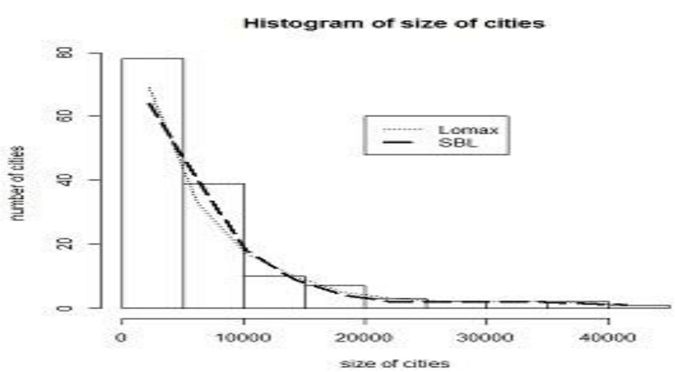

Figure 14: Graph of SBL and Lomax Distributions with histogram of size of cities

Table 1: Table of the values of median for different values of α and β

λ	α					
	2	3	4	5	6	7
1	2.4142	1	0.627942	0.457323	0.359527	0.296159
2	4.8284	2	1.25588	0.914645	0.719054	0.592319
3	7.2426	3	1.88383	1.37197	1.07858	0.888478
4	9.6569	4	2.51177	1.82929	1.43811	1.182464
5	12.071	5	3.13971	2.28661	1.79763	1.48080

Table 2: Mode for various values of α and λ

	λ				
α	1	2	3	4	5
2	0.5	1	1.5	2	2.5
3	0.333	0.667	1	1.333	1.667
4	0.25	0.5	0.75	1	1.25
5	0.2	0.4	0.6	0.8	1
6	0.167	0.333	0.5	0.667	0.833

Table 3: Area of cities

12510	3514	7499	44748	10160	12637	4096	2445	3615	6622	3387
22539	8958	7610	35380	3293	6831	15153	9830	5896	5728	5797
16891	7819	2653	7796	29510	20297	1489	9830	7796	1967	1227
1301	1865	996	14850	7326	1597	1725	3372	2545	7492	3164
1582	952	4579	1632	1748	1257	1586	1543	5337	1679	497
3699	7492	6858	8878	24830	8153	6524	11922	5856	3622	3192
2367	8809	3587	3995	4349	6511	1772	6291	2778	2673	5840
3720	8249	2337	2960	4377	2724	11880	12319	5286	3201	5854
15960	3016	3252	4364	6726	19070	6083	5519	5278	3527	2592
15910	7423	1417	2925	2945	4502	10720	2512	5165	2310	1733
19638	17355	1290	2576	3380	2296	4707	1538	620	745	2008
446	132	261	1221	2496	854	3261	1010	1516	1862	855
1368	598	569	6400	15000	8657	10936	38000	9635	25145	

Table 4: Results and Maximum Likelihood Estimator and Information Criterion

Distribution	$\hat{\alpha}$	$\hat{\lambda}$	Value of χ^2	P-Value	Critical value at α = 0.05
SBL	5.01679	9979.924	6.052712	0.98751605	26.296
Classical Lomax	10.20492	60902.08	9.307393	0.90021664	

Model	**Maximum Likelihood Estimates**		**Information Criterion**			
	α	λ	**W**	**AIC**	**CAIC**	**BIC**
SBL	3.8664(0.6844)	6308.07(1900.52)	2768.4	2772.4	2772.5	2778.3
L	10.6503(9.6916)	63606(63045)	2780.0	2784.0	2784.1	2789.9

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