ISSN 1684-8403
Journal of Statistics
Volume 22, 2015. pp. 121-138

Bayesian Analysis of Kumaraswamy Mixture Distribution under Different Loss Functions

Tabassum Naz Sindhu[1], Navid Feroze[2] and Muhammad Aslam[3]

Abstract

This paper develops a Bayesian Analysis in the context of new improved informative Prior for the shape parameter of the mixture of Kumaraswamy Distribution using the censored data. The objective of this study is to mingle both the informative and non-informative Priors for the improvement of the Prior information for the unknown parameter of the considered Distribution. We modeled the heterogeneous population using two components mixture of the Kumaraswamy Distribution. A comprehensive simulation scheme has been carried out to highlight the properties and behavior of the Estimates in terms of sample size, corresponding risks and the mixing weights. A censored mixture data is simulated by probabilistic mixing for the computational purpose. The Bayes Estimators of the said parameters have been derived under the assumption of informative and non-informative Priors using different Loss functions.

Keywords

Bayes estimators, Posterior risks, Probabilistic mixing, Credible Intervals

1. Introduction

Kumaraswamy (1980) has introduced the Kumaraswamy Distribution.

[1] Department of Statistics, Quaid-i-Azam University, Islamabad 44000, Pakistan
Email: sindhuqau@gmail.com
[2] Department of Mathematics and Statistics, Allama Iqbal Open University, Islamabad, Pakistan
Email: navidferoz@gmail.com
[3] Department of Statistics, Quaid-i-Azam University, Islamabad 44000, Pakistan

This Distribution is applicable to many natural phenomena whose outcomes have lower and upper bounds, such as the height of individuals, scores obtained on a test, atmospheric temperatures, hydrological data, such as daily rain fall, daily stream flow, etc. Kumaraswamy (1980) and Ponnambalam et al. (2001) have pointed out that depending on the choice of the parameter α and β Kumaraswamy's Distribution can be used to approximate many Distributions, such as Uniform, Triangular, or almost any single model Distribution and can also reproduce results of Beta Distribution. Nadarajah (2008) has argued that Kumaraswamy's Distribution is a special case of the three parameter Beta Distribution. The basic properties of the Distribution have been given by Jones (2009). Garg (2009) discussed Generalized Order Statistics from Kumaraswamy Distribution. Gholizadeh et al. (2011) have carried out the classical analysis of the Kumaraswamy Distribution under Progressively Type II censored data. Sindhu et al. (2013) considered the Posterior Analysis of the Kumaraswamy Distribution under Type II censored samples.

In recent years, the mixture models have received a considerable attention in the area of Survival Analysis and Reliability. Mixtures of Lifetime Distributions occur when two different causes of failure are present, each with the same parametric form of Lifetime Distributions. Demidenko (2004), Landsay (1995), Mclachlan and Peel (2000), Mcculloch and Searle (2001) and Titterington et al. (1985) are amongst the authors considering the Analysis of the mixture models. The characterizations of mixtures has been studied by Ismail and Khodary (2001), Nassar (1988) and Nassar and Mahmoud (1985). Ahmad et al. (1997) deduced approximate Bayes Estimation for mixtures of two Weibull Distributions under Type II censoring. Further, a mixture of two Inverse Weibull Distributions has been studied by Sultan et al. (2007).The authors dealing with Bayesian Analysis of mixture models include: Feroze and Aslam (2012), Feroze and Aslam (2013), Majeed and Aslam (2012), Saleem et al. (2010), Saleem and Aslam (2008a), Saleem and Aslam (2008b) and Saleem and Irfan (2010).

The article is outlined as follows. In Section 2, we defined the mixture model of Kumaraswamy. The sampling and Likelihood function are presented in the Sections 3 and 4, respectively. In Sections 5, the expressions for the Posterior Distributions have been presented. The Section 6 contains the derivation of the Estimators and corresponding Posterior risks. Method of Elicitation of hyper-parameter for the mixture of Kumaraswamy Distribution via PriorPredictive approach is discussed in the Section 7. Predictive Distributions, Predictive Intervals and Credible Intervals are derived in the Sections 8 and 9, respectively.

A simulation study is performed in the 10. Some concluding remarks are given in the Section 11.

2. The population and the model

A population is postulated to be composed of two sub-populations with specified parameters. The sub-populations are mixed in proportion *w*, *(1-w)*, where *0<w<1*. A Finite Mixture Distribution function with the two component densities of specified parametric form (but with unknown parameters, β_1 and β_2 and with unknown mixing weights, *w* and *(1-w)* is,

$$F(x) = wF_1(x) + (1-w)F_2(x),\ 0 < w < 1$$

with the two component Distribution functions of specified parametric (Kumaraswamy) form

$$F_1(x) = 1-(1-x^{\alpha})^{\beta_1} \text{ and } F_2(x) = 1-(1-x^{\alpha})^{\beta_2}$$

Throughout we assume that $\alpha_1 = \alpha_2 = \alpha$ .The corresponding Finite Mixture density function has its probability density function as:

$$p(x \mid \beta_1, \beta_2, w) = w\alpha\beta_1 x^{\alpha-1}(1-x^{\alpha})^{\beta_1-1} + (1-w)\alpha\beta_2 x^{\alpha-1}(1-x^{\alpha})^{\beta_2-1}$$

$$\beta_i > 0, i = 1,2 \qquad 0 < x < 1 \tag{2.1}$$

3. Sampling

A random sample of *n* units from the above Mixture model is operating to a life testing experiment. The test is terminated at a fixed time *T*. Let, the test to be conducted and it is observed that out of *n*, *r* units have lifetime in the interval [0, *T*] and (*n*-*r*) units are still functioning when the test termination time *T* is over. Hence *(n-r)* units that have not failed by the time *T* are censored objects and yield no information. According to Mendenhall and Hader (1958), in many real life situations only the failed objects can easily be identified as member of either sub-population 1 or sub-population 2. So, depending upon the causes of failure it may be observed that r_1 and r_2 objects are identified as members of the first sub-population and the second population, respectively. Obviously, $r = r_1 + r_2$ and remaining *(n-r)* units provide no information about the sub-population to which they belong. Furthermore, let x_{ij} as the failure time of the j^{th} unit to the i^{th} sub-population, where $j = 1,2,...,r_i, i = 1,2; 0 \le x_{1j}, x_{2j} \le T$.

4. The Maximum Likelihood function

The Likelihood function for a two-component mixture with n items under study, the probability that r_1 will fail due to cause 1, r_2 will fail due to cause 2 and remaining $(n-r_1-r_2)$ will survive at time T when test is terminated is given as:

$$L(\beta_1,\beta_2,w\,|\,\mathrm{x}) \propto \prod_{j=1}^{r_1} wf_1\left(x_{1j}\right)\prod_{j=1}^{r_2} wf_2\left(x_{2j}\right)\left\{1-F\left(T\right)\right\}^{n-r}$$

$$L\left(\beta_1,\beta_2,w\,|\,\mathrm{x}\right) \propto \sum_{k=0}^{n-r}\binom{n-r}{k} w^{n-r_2-k}\left(1-w\right)^{r_2+k}\beta_1^{r_1}\beta_2^{r_2} e^{\beta_1\left\{\sum_{j=1}^{r_1}\ln(1-x_{1j}{}^{\alpha})+(n-r-k)\ln(1-T^{\alpha})\right\}} e^{\beta_2\left\{\sum_{j=1}^{r_2}\ln(1-x_{1j}{}^{\alpha})+k\ln(1-T^{\alpha})\right\}}$$

$$L\left(\beta_1,\beta_2,w\,|\,\mathrm{x}\right) \propto \sum_{k=0}^{n-r}\binom{n-r}{k} w^{n-r_2-k}\left(1-w\right)^{r_2+k}\beta_1^{r_1}\beta_2^{r_2} e^{-\beta_1\left\{\xi_{1j}\left(x_{1j}\right)\right\}} e^{-\beta_2\left\{\xi_{2j}\left(x_{2j}\right)\right\}} \qquad (4.1)$$

where

$\mathbf{x} = (\mathbf{x_1},\mathbf{x_2}) = (x_{11},x_{12},...,x_{1r_1},x_{21},x_{22},...,x_{2r_2})$ is data,

$$\xi_{1j}\left(x_{1j}\right) = -\left\{\sum_{j=1}^{r_1}\ln(1-x_{1j}{}^{\alpha})+(n-r-k)\ln(1-T^{\alpha})\right\}$$

And

$$\xi_{2j}\left(x_{2j}\right) = -\left\{\sum_{j=1}^{r_2}\ln(1-x_{2j}{}^{\alpha})+k\ln(1-T^{\alpha})\right\}$$

5. Prior and Posterior Distributions

The Posterior Distributions under the assumption of Uniform, Jeffrey's and Jeffreys Gamma Priors have been derived and presented in the following.

5.1 Bayesian Estimation under Uniform Prior: Let us assume a state of ignorance that β_1,β_2 are uniformly distributed over $(0,\infty)$ and w is uniformly distributed over $(0,1)$.

Hence, $p_1(\beta_1)=k_1,\ p_2(\beta_2)=k_2,\beta_i>0$ and $p_3(w)=1, 0<w<1$

Assuming independence, we have an improper joint Prior that is proportional to a constant. The Joint Prior is incorporated with the Likelihood equation (4.1) to yield a proper joint Posterior Distribution of β_1, β_2 and w. The joint Posterior Distribution of β_1, β_2 and w is,

$$p(\beta_1, \beta_2, w \mid \mathbf{x}) = \frac{\sum_{k=0}^{n-r} \binom{n-r}{k} w^{n-r_2-k} (1-w)^{r_2+k} \beta_1^{r_1} \beta_2^{r_2} e^{-\beta_1\{\xi_{1j}(x_{1j})\}} e^{-\beta_2\{\xi_{2j}(x_{2j})\}}}{\sum_{k=0}^{n-r} \binom{n-r}{k} B(\vartheta_1, \vartheta_2) \frac{\Gamma(r_1+1)\Gamma(r_2+1)}{\{\xi_{1j}(x_{1j})\}^{r_1+1} \{\xi_{2j}(x_{2j})\}^{r_2+1}}} \quad (5.1.1)$$

where

$\vartheta_1 = n - r_2 - k + 1$, $\vartheta_2 = r_2 + k + 1$ and $B(\vartheta_1, \vartheta_2)$ is Standard Beta function.

5.2 Bayesian Estimation under Jeffrey's Prior: Jeffrey's Prior is locally Uniform and hence non-informative. An appealing property of Jeffrey's Prior is that it is invariant with respect to one-to-one transformations. For the Kumaraswamy model given in section 2, the Jeffrey's Priors are

$p_1(\beta_1) = \beta_1^{-1}, 0 < \beta_1 < \infty,\ p_2(\beta_2) = \beta_2^{-1}, 0 < \beta_2 < \infty$ and $p_3(w) = 1, 0 < w < 1$.

Assuming independence, the joint Prior $g(\beta_1, \beta_2, w) \propto (\beta_1\beta_2)^{-1}$ is incorporated with the Likelihood equation (4.1) to yield a proper joint Posterior Distribution of β_1, β_2 and w.The joint Posterior Distribution under Jeffrey's Prior is:

$$p(\beta_1, \beta_2, w \mid \mathbf{x}) = \frac{\sum_{k=0}^{n-r} \binom{n-r}{k} w^{n-r_2-k} (1-w)^{r_2+k} \beta_1^{r_1-1} \beta_2^{r_2-1} e^{-\beta_1\{\xi_{1j}(x_{1j})\}} e^{-\beta_2\{\xi_{2j}(x_{2j})\}}}{\sum_{k=0}^{n-r} \binom{n-r}{k} B(\vartheta_1, \vartheta_2) \frac{\Gamma(r_1)\Gamma(r_2)}{\{\xi_{1j}(x_{1j})\}^{r_1} \{\xi_{2j}(x_{2j})\}^{r_2}}} \quad (5.2.1)$$

5.3 Bayesian Estimation under the Jeffrey's-Gamma Prior: For the Kumaraswamy Model the Jeffrey's Prior is,

$g(\beta_1, \beta_2) \propto (\beta_1\beta_2)^{-1}$

Now, let us suppose $\beta_1 \sim JGamma(a_1, b_1), \beta_2 \sim JGamma(a_2, b_2)$ and $w \sim (0,1)$, assuming independence, the joint Prior $g(\beta_1, \beta_2, w) \propto \beta_1^{(a_1-2)} \beta_2^{(a_2-2)} e^{-(b_1\beta_1 + b_2\beta_2)}$ is incorporated with the Likelihood equation (4.1) to have the Posterior Distribution. The Joint Posterior Distribution under the Jeffrey's-Gamma Prior

$$p(\beta_1,\beta_2,w\,|\,\mathbf{x})=\frac{\sum_{k=0}^{n-r}\binom{n-r}{k}w^{n-r_2-k}\left(1-w\right)^{r_2+k}\beta_1^{(\gamma_{1k})-1}\beta_2^{(\gamma_{2k})-1}e^{-\beta_1\{b\psi_{1j}\}}e^{-\beta_2\{\psi_{2j}\}}}{\sum_{k=0}^{n-r}\binom{n-r}{k}B(\vartheta_1,\vartheta_2)\frac{\Gamma(\gamma_{1k})\Gamma(\gamma_{2k})}{\left\{\psi_{1j}\right\}^{(\gamma_{1k})}\left\{\psi_{2j}\right\}^{(\gamma_{2k})}}} \quad (5.3.1)$$

where

$\gamma_{1k}=r_1+a_1-1,\ \gamma_{2k}=r_2+a_2-1,\ \psi_{1k}=b_1+\xi_{1j}(x_{1j})$ and $\psi_{2k}=b_2+\xi_{2j}(x_{2j})$.

6. Bayesian Estimator and Posterior risk

The Posterior Distributions in equations (5.1.1), (5.2.1) and (5.3.1) have been used to derive the Bayes Estimators and Posterior risks of β_1,β_2 and w under Weighted Loss Function (WLF), Quadratic Loss Function (QLF) and Squared Error Loss Function (SELF). The definitions of these Loss functions along with formulas for Bayes Estimators and Posterior risks are as under:

Weighted Loss Function can be defined as:

$$L\left(\beta_{WLF},\beta\right)=\beta^{-1}\left(\beta_{WLF}-\beta\right)^2$$

$$\theta_{WLF}=\left\{\mathrm{E}\left(\theta^{-1}\right)\right\}^{-1} \quad ; \quad \rho\left(\theta_{WLF}\right)=\mathrm{E}\left(\theta\right)-\theta_{WLF}$$

The Squared Error Loss Function is defined as:

$$L\left(\beta,\beta_{SELF}\right)=\left(\beta-\beta_{SELF}\right)^2.$$

$$\theta_{SELF}=\mathrm{E}\left(\theta\right) \quad ; \quad \rho\left(\theta_{SELF}\right)=\mathrm{E}\left(\theta^2\right)-\left\{\mathrm{E}\left(\theta\right)\right\}^2$$

The Quadratic Loss Function can be defined as:

$$L\left(\beta,\beta_{QLF}\right)=\left(\beta^{-1}\left(\beta-\beta_{QLF}\right)\right)^2$$

$$\theta_{QLF}=\frac{\mathrm{E}\left(\theta^{-1}\right)}{\mathrm{E}\left(\theta^{-2}\right)} \quad ; \quad \rho\left(\theta_{QLF}\right)=1-\frac{\left\{\mathrm{E}\left(\theta^{-1}\right)\right\}^2}{\mathrm{E}\left(\theta^{-2}\right)}$$

6.1 The expressions for the Bayes Estimators and Posterior risks under Uniform Prior using WLF: For convenience following notation has been used.

$$v_{r_1\pm i,r_2\pm i}\left(\vartheta_1\pm i,\vartheta_2\right)=\sum_{k=0}^{n-r}\binom{n-r}{k}\frac{B(\vartheta_1\pm i,\vartheta_2)\Gamma(r_1\pm i)\Gamma(r_2\pm i)}{\left\{\xi_{1j}(x_{1j})\right\}^{r_1\pm i}\left\{\xi_{2j}(x_{2j})\right\}^{r_2\pm i}}$$

where $i = 0,1,2$

The Bayes Estimators and Posterior risks are evaluated under Weighted Loss Function are provided as:

$$\hat{\beta}_1 = \frac{v_{r_1+1,r_2+1}(\vartheta_1,\vartheta_2)}{v_{r_1,r_2+1}(\vartheta_1,\vartheta_2)} \text{ and } \rho(\hat{\beta}_1) = \frac{v_{r_1+2,r_2+1}(\vartheta_1,\vartheta_2)}{v_{r_1+1,r_2+1}(\vartheta_1,\vartheta_2)} - \frac{v_{r_1+1,r_2+1}(\vartheta_1,\vartheta_2)}{v_{r_1,r_2+1}(\vartheta_1,\vartheta_2)}$$

$$\hat{\beta}_2 = \frac{v_{r_1+1,r_2+1}(\vartheta_1,\vartheta_2)}{v_{r_1+1,r_2}(\vartheta_1,\vartheta_2)} \text{ and } \rho(\hat{\beta}_2) = \frac{v_{r_1+1,r_2+2}(\vartheta_1,\vartheta_2)}{v_{r_1+1,r_2+1}(\vartheta_1,\vartheta_2)} - \frac{v_{r_1+1,r_2+1}(\vartheta_1,\vartheta_2)}{v_{r_1+1,r_2}(\vartheta_1,\vartheta_2)}$$

$$\hat{w} = \frac{v_{r_1+1,r_2+1}(\vartheta_1,\vartheta_2)}{v_{r_1+1,r_2+1}(\vartheta_1-1,\vartheta_2)} \text{ and } \rho(\hat{w}) = \frac{v_{r_1+1,r_2+1}(\vartheta_1+1,\vartheta_2)}{v_{r_1+1,r_2+1}(\vartheta_1,\vartheta_2)} - \frac{v_{r_1+1,r_2+1}(\vartheta_1,\vartheta_2)}{v_{r_1+1,r_2+1}(\vartheta_1-1,\vartheta_2)}$$

6.2 The expressions for the Bayes Estimators and Posterior risks under Uniform Prior using QLF: The Bayes Estimators and Posterior risks are evaluated under Quadratic Loss Function are provided as:

$$\hat{\beta}_1 = \frac{v_{r_1,r_2+1}(\vartheta_1,\vartheta_2)}{v_{r_1-1,r_2+1}(\vartheta_1,\vartheta_2)} \text{ and } \rho(\hat{\beta}_1) = 1 - \frac{\left\{v_{r_1,r_2+1}(\vartheta_1,\vartheta_2)\right\}^2}{\left\{v_{r_1+1,r_2+1}(\vartheta_1,\vartheta_2)\right\}\left\{v_{r_1-1,r_2+1}(\vartheta_1,\vartheta_2)\right\}}$$

$$\hat{\beta}_2 = \frac{v_{r_1+1,r_2}(\vartheta_1,\vartheta_2)}{v_{r_1+1,r_2-1}(\vartheta_1,\vartheta_2)} \text{ and } \rho(\hat{\beta}_2) = 1 - \frac{\left\{v_{r_1+1,r_2}(\vartheta_1,\vartheta_2)\right\}^2}{\left\{v_{r_1+1,r_2+1}(\vartheta_1,\vartheta_2)\right\}\left\{v_{r_1+1,r_2-1}(\vartheta_1,\vartheta_2)\right\}}$$

$$\hat{w} = \frac{v_{r_1+1,r_2+1}(\vartheta_1-1,\vartheta_2)}{v_{r_1+1,r_2+1}(\vartheta_1-2,\vartheta_2)} \text{ and } \rho(w) = 1 - \frac{\left\{v_{r_1+1,r_2} + (\vartheta_1-1,\vartheta_2)\right\}^2}{\left\{v_{r_1+1,r_2+1}(\vartheta_1,\vartheta_2)\right\}\left\{v_{r_1+1,r_2+1}(\vartheta_1-2,\vartheta_2)\right\}}$$

***6.3 The expressions for the Bayes Estimators and Posterior risks under Uniform Prior using SELF*:** The Bayes Estimators and Posterior risks are evaluated under Squared Error Loss Function are provided as:

$$\hat{\beta}_1 = \frac{v_{r_1+2,r_2+1}(\vartheta_1,\vartheta_2)}{v_{r_1+1,r_2+1}(\vartheta_1,\vartheta_2)} \text{ and } \rho(\hat{\beta}_1) = \frac{v_{r_1+3,r_2+1}(\vartheta_1,\vartheta_2)}{v_{r_1+1,r_2+1}(\vartheta_1,\vartheta_2)} - \left\{\frac{v_{r_1+2,r_2+1}(\vartheta_1,\vartheta_2)}{v_{r_1+1,r_2+1}(\vartheta_1,\vartheta_2)}\right\}^2$$

$$\hat{\beta}_2 = \frac{v_{r_1+1,r_2+2}\left(\vartheta_1,\vartheta_2\right)}{v_{r_1+1,r_2+1}\left(\vartheta_1,\vartheta_2\right)} \text{ and } \rho(\hat{\beta}_2) = \frac{v_{r_1+1,r_2+3}\left(\vartheta_1,\vartheta_2\right)}{v_{r_1+1,r_2+1}\left(\vartheta_1,\vartheta_2\right)} - \left\{\frac{v_{r_1+1,r_2+2}\left(\vartheta_1,\vartheta_2\right)}{v_{r_1+1,r_2+1}\left(\vartheta_1,\vartheta_2\right)}\right\}^2$$

$$\hat{w} = \frac{v_{r_1+1,r_2+1}\left(\vartheta_1+1,\vartheta_2\right)}{v_{r_1+1,r_2+1}\left(\vartheta_1,\vartheta_2\right)} \text{ and } \rho(\hat{w}) = \frac{v_{r_1+1,r_2+1}\left(\vartheta_1+2,\vartheta_2\right)}{v_{r_1+1,r_2+1}\left(\vartheta_1,\vartheta_2\right)} - \left\{\frac{v_{r_1+1,r_2+1}\left(\vartheta_1+1,\vartheta_2\right)}{v_{r_1+1,r_2+1}\left(\vartheta_1,\vartheta_2\right)}\right\}^2$$

The expressions for Bayes Estimators and Posterior risks under remaining Priors can be derived with little modifications.

7. Elicitation of hyper-parameters of Jeffrey's-Gamma Prior through Prior Predictive probabilities

To elicit a Prior density, Aslam (2003) forms some new methods base on the Prior Predictive Distribution. For the elicitation of hyper-parameters, he considers Prior Predictive probabilities, Predictive mode and confidence level. In this study, the method of Prior Predictive probabilities is used for obtaining the hyper-parameters of the considered informative Prior. The equation of Prior Predictive under the Jeffrey's-Gamma Prior is given as:

$$p(y) = \int_0^\infty \int_0^\infty \int_0^1 p(\beta_1,\beta_2,w)\, p(y \mid \beta_1,\beta_2,w) \qquad (7.1)$$

where

$$p(y \mid \beta_1,\beta_2,w) = w\alpha\beta_1 y^{\alpha-1}(1-y^\alpha)^{\beta_1-1} + (1-w)\alpha\beta_2 y^{\alpha-1}(1-y^\alpha)^{\beta_2-1}$$

and

$$p(\beta_1,\beta_2,w) = \frac{b_1^{a_1}}{\Gamma(a_1)}\beta_1^{a_1-2}e^{-\beta_1 b_1}\,\frac{b_2^{a_2}}{\Gamma(a_2)}\beta_2^{a_2-2}e^{-\beta_2 b_2}.$$

Throughout we assume $\alpha_1 = \alpha_2 = \alpha$.

After some algebra,

$$p(y) = \frac{\alpha y^{\alpha-1}}{2(1-y^\alpha)}\left[\frac{b_2 b_1^{a_1}}{(a_2-1)\left\{b_1-\ln(1-y^\alpha)\right\}^{a_1}} - \frac{b_1 b_2^{a_2}}{(a_1-1)\left\{b_2-\ln(1-y^\alpha)\right\}^{a_2}}\right] \qquad (7.2)$$

As we have to elicit four hyper-parameters so we have to consider four integrals. The set of hyper-parameters with minimum values has been chosen to be the elicited values of the hyper-parameters. By considering the Prior Predictive

Distribution in equation (7.2), we have assumed the expert's probabilities to be 0.20 for each integral. We considered the following integrals:

$$\int_{0.00}^{0.15} g(y)\,dy = 0.20, \quad \int_{0.15}^{0.30} g(y)dy = 0.20, \quad \int_{0.30}^{0.40} g(y)\,dy = 0.20 \text{ and } \int_{0.45}^{0.60} g(y)\,dy = 0.20.$$

Now, these integrals have been simultaneously solved through a program written in SAS package using the "PROC SYSLIN" command and the elicited values of the hyper-parameters have been found to be $a_1 = 4.001269$, $b_1 = 2.401326$ and $a_2 = 0.001596, b_2 = 1.234217$.

8. Predictive Distribution

The Predictive Distribution contains the information about the independent future random observation given preceding observations. The more details can see from Bolstad (2004) and Bansal (2007).

***8.1 Posterior Predictive Distribution and Predictive Intervals*:** The Posterior Predictive Distribution of the future observation $y = x_{n+1}$ is,

$$p(y \mid \mathbf{x}) = \int_0^\infty \int_0^\infty \int_0^1 p(\beta_1, \beta_2, w \mid \mathrm{x})\, p(y \mid \beta_1, \beta_2, w)\, dw\, d\beta_1 d\beta_2 \tag{8.1.1}$$

where

$$p(y \mid \beta_1, \beta_2, w) = w\alpha\beta_1 y^{\alpha-1}\left(1 - y^{\alpha}\right)^{\beta_1 - 1} + (1-w)\alpha\beta_2 y^{\alpha-1}\left(1 - y^{\alpha}\right)^{\beta_2 - 1}$$

is the future observation density and $p\left(\beta_1, \beta_2, w \mid \mathbf{x}\right)$ is the joint Posterior Distribution obtained by incorporating the Likelihood with the respective Prior distributions. A (1-*k*) 100% Bayesian Predictive Interval (*L*, *U*) can be obtained by solving the following two equations simultaneously.

$$\int_{-\infty}^{L} p(y \mid \mathbf{x})dy = \frac{k}{2} = \int_{U}^{\infty} p(y \mid \mathbf{x})dy$$

8.1.1 Posterior Predictive Distribution and Predictive Intervals assuming Uniform Prior: The Posterior Predictive Distribution of the future observation $y = x_{n+1}$ is,

$$p(y\,|\,\mathbf{x}) = \frac{\alpha y^{(\alpha-1)} \sum_{k=0}^{n-r} \binom{n-r}{k}}{A_{1k}(1-y^{\alpha})} \left\{ \begin{array}{l} \dfrac{B(\vartheta_1+1,\vartheta_2)\Gamma(r_2+1)\Gamma(r_1+2)}{\left\{\xi_{1j}(x_{1j})-\ln(1-y^{\alpha})\right\}^{r_1+2}\left\{\xi_{2j}(x_{2j})\right\}^{r_2+1}} \\ +\dfrac{B(\vartheta_1,\vartheta_2+1)\Gamma(r_2+2)\Gamma(r_1+1)}{\left\{\xi_{1j}(x_{1j})\right\}^{r_1+1}\left\{\xi_{2j}(x_{2j})-\ln(1-y^{\alpha})\right\}^{r_2+2}} \end{array} \right\} \tag{8.1.1a}$$

The lower and upper limits of the Posterior Predictive Interval (L,U) can be obtained by the simultaneous solution of the following two equations.

$$\frac{k}{2} = \frac{\Gamma(r_2+1)\Gamma(r_1+1)\sum_{k=0}^{n-r}\binom{n-r}{k}}{A_{1k}} \left[\begin{array}{l} \dfrac{B(\vartheta_1+1,\vartheta_2)}{\left\{\xi_{2j}(x_{2j})\right\}^{r_2+1}} \left\{ \dfrac{1}{\left\{\xi_{1j}(x_{1j})\right\}^{r_1+1}} - \dfrac{1}{\left\{\xi_{1j}(x_{1j})-\ln(1-L^{\alpha})\right\}^{r_1+1}} \right\} \\ +\dfrac{B(\vartheta_1,\vartheta_2+1)}{\left\{\xi_{1j}(x_{1j})\right\}^{r_1+1}} \left\{ \dfrac{1}{\left\{\xi_{2j}(x_{2j})\right\}^{r_2+1}} - \dfrac{1}{\left\{\xi_{2j}(x_{2j})-\ln(1-L^{\alpha})\right\}^{r_2+1}} \right\} \end{array} \right]$$

$$\frac{k}{2} = \frac{\Gamma(r_2+1)\Gamma(r_1+1)\sum_{k=0}^{n-r}\binom{n-r}{k}}{A_{1k}} \left[\begin{array}{l} \dfrac{B(\vartheta_1+1,\vartheta_2)}{\left\{\xi_{2j}(x_{2j})\right\}^{r_2+1}} \left\{ \dfrac{1}{\left\{\xi_{1j}(x_{1j})-\ln(1-U^{\alpha})\right\}^{r_1+1}} \right\} \\ +\dfrac{B(\vartheta_1,\vartheta_2+1)}{\left\{\xi_{1j}(x_{1j})\right\}^{r_1+1}} \left\{ \dfrac{1}{\left\{\xi_{2j}(x_{2j})-\ln(1-U^{\alpha})\right\}^{r_2+1}} \right\} \end{array} \right] \tag{8.1.1b}$$

The Posterior Predictive Distributions and intervals under Jeffrey's and Jeffrey's Gamma Prior can be obtained in a similar manner. As the limits of the Posterior Predictive Intervals cannot be derived in the closed form, we have used iterative methods (such as Newton Raphson Method) on Mathematica software to obtain the numerical results of the limits. The Posterior Predictive Distributions and intervals under Jeffrey's and Jeffrey's Gamma Prior can be obtained in a similar manner.

9. Credible Interval

According to Eberly and Casella (2003), the Credible Interval can be defined as:

$$\int_0^L g(\beta\,|\,\mathbf{x})d\beta = \frac{k}{2}, \int_U^{\infty} g(\beta\,|\,\mathbf{x})d\beta = \frac{k}{2}$$

where

L and U are the lower and upper limits of the Credible Interval, respectively. And k is the level of significance.

9.1 Credible Interval for β_1, β_2 ***and w using Uniform Prior:*** The $(1-k)100\%$ Credible Interval for β_1 on the basis of Uniform Prior can be obtained by the simultaneous solution of the following equations for *L* and *U*, respectively. For convenience following notations have been used.

$$v_{K(r_1\pm i),r_2\pm i}\left(\vartheta_1 \pm i, \vartheta_2\right) = \sum_{k=0}^{n-r}\binom{n-r}{k}\frac{B(\vartheta_1 \pm i, \vartheta_2)\Gamma(r_1 \pm i, K\xi_{1j}(x_{1j}))\Gamma(r_2 \pm i)}{\left\{\xi_{1j}(x_{1j})\right\}^{r_1\pm i}\left\{\xi_{2j}(x_{2j})\right\}^{r_2\pm i}},$$

$$v_{r_1\pm i,K(r_2\pm i)}\left(\vartheta_1 \pm i, \vartheta_2\right) = \sum_{k=0}^{n-r}\binom{n-r}{k}\frac{B(\vartheta_1 \pm i, \vartheta_2)\Gamma(r_1 \pm i)\Gamma(r_2 \pm i, K\xi_{1j}(x_{1j}))}{\left\{\xi_{1j}(x_{1j})\right\}^{r_1\pm i}\left\{\xi_{2j}(x_{2j})\right\}^{r_2\pm i}},$$

$$v_{r_1\pm i,r_2\pm i}\left(K, \vartheta_1 \pm i, \vartheta_2\right) = \sum_{k=0}^{n-r}\binom{n-r}{k}\frac{B(K, \vartheta_1 \pm i, \vartheta_2)\Gamma(r_1 \pm i)\Gamma(r_2 \pm i)}{\left\{\xi_{1j}(x_{1j})\right\}^{r_1\pm i}\left\{\xi_{2j}(x_{2j})\right\}^{r_2\pm i}} \quad ,$$

where

$i = 0,1,2$

$$\frac{v_{L(r_1+1),r_2+1}\left(\vartheta_1, \vartheta_2\right)}{v_{r_1+1,r_2+1}\left(\vartheta_1, \vartheta_2\right)} = 1-\frac{k}{2} \text{ and } \frac{v_{U(r_1+1),r_2+1}\left(\vartheta_1, \vartheta_2\right)}{v_{r_1+1,r_2+1}\left(\vartheta_1, \vartheta_2\right)} = \frac{k}{2} \qquad (9.1.1)$$

where

$\Gamma\left(p,q\right) = \int_0^q x^{p-1}e^{-x}dx$ is an Incomplete Gamma function.

The $(1-k)100\%$ Credible Interval for β_2 on the basis of Uniform Prior can be obtained by the simultaneous solution of the following equations for *L* and *U*, respectively.

$$\frac{v_{r_1+1,L(r_2+1)}\left(\vartheta_1, \vartheta_2\right)}{v_{r_1+1,r_2+1}\left(\vartheta_1, \vartheta_2\right)} = 1-\frac{k}{2} \text{ and } \frac{v_{r_1+1,U(r_2+1)}\left(\vartheta_1, \vartheta_2\right)}{v_{r_1+1,r_2+1}\left(\vartheta_1, \vartheta_2\right)} = \frac{k}{2} \qquad (9.1.2)$$

The $(1-k)100\%$ Credible Interval for w on the basis of Uniform Prior can be obtained by the simultaneous solution of the following equations for *L* and *U*, respectively.

$$\frac{v_{r_1+1,r_2+1}\left(L, \vartheta_1, \vartheta_2\right)}{v_{r_1+1,r_2+1}\left(\vartheta_1, \vartheta_2\right)} = 1-\frac{k}{2} \text{ and } \frac{v_{r_1+1,r_2+1}\left(U, \vartheta_1, \vartheta_2\right)}{v_{r_1+1,r_2+1}\left(\vartheta_1, \vartheta_2\right)} = \frac{k}{2} \qquad (9.1.3)$$

The Credible Interval for the remaining Priors can be constructed by considering the mentioned method. As the limits of the Credible Intervals cannot be derived in the closed form, we have used iterative methods (such as Newton Raphson Method) on Mathematica software to obtain the numerical results of the limits. The Posterior Predictive Distributions and intervals under Jeffrey's and Jeffrey's Gamma Prior can be obtained in a similar manner.

10. Simulation study

A simulation study is carried out to investigate the performance of Bayes Estimators and the impact of sample size and mixing proportion. We take random samples of sizes *n* = 50, 100 and 500 from the two component mixture of Kumaraswamy Distribution with $(\beta_1, \beta_2) = (2,4)$, $w = 0.4, 0.6$. To generate a mixture data, we make use of probabilistic mixing with probability; w and $(1-w)$. A uniform number u is generated n times and if $u < w$ the observation is taken randomly from F_1 (the Kumaraswamy Distribution with parameter β_1) otherwise from F_2 (from the Kumaraswamy Distribution with parameter β_2). Hence, the parameters to be estimated are known to be β_1, β_2 and w. To implement censored sampling, all the observations greater than *T* are declared as censored ones. The tests termination time is considered to be so that the censoring rate in each sample is 20%. To avoid an extreme sample, we simulate 1000 data sets each of size *n*. The Bayes Estimates and Posterior risks are computed using Mathematica 7.0. The average of these Estimates and corresponding risks are reported in Tables 1-9.Predictive Intervals and Credible Intervals are presented in Table 10-13. The comparison observed has been summarized in last Section. For convenience, we have assumed $\alpha_1 = \alpha_2 = \alpha = 1$.

11. Conclusion

In order to form a new informative Prior, which perform better than the informative and non-informative Prior, a new methodology has been introduced to mingle the Prior information to update the information about the unknown parameters of mixture model under study.

Tables 1-13, appended above give Bayes estimates, corresponding risks, 95% Predictive and Credible Intervals for the parameters of the mixture model. Which are empirically evaluated based on a Monte Carlo simulation study of samples.

This simulation study has displayed some interesting properties of the Bayes Estimates. The interesting remark concerning the Posterior risks of the Estimators of β_1, β_2 and w is that increasing (decreasing) the proportion of a component in the mixture reduces (increases) the Posterior risk of the corresponding β_1 parameter's Estimate. Further, the increase in the sample size reduces the Posterior risks of the Estimators of β_1, β_2 and w. The Bayes estimates are underestimated with few exceptions, but the extent of this under-estimation is inversely proportional to the sample size. The Posterior risk of the mixing parameter w is same under WLF assuming uniform and Jeffrey's Prior. The Bayes Estimates with new informative (Jeffrey's-Gamma) Prior are more precise than existing non-informative counterparts. Tables 10-13, give the results of Interval Estimation which are in accordance with the Point Estimation. The credible and Predictive Intervals work quite well under the new informative Prior. The width of Predictive and Credible Interval is inversely proportional to sample size. The findings of the present study suggest that in order to estimate β_1, β_2 and w, the use of Quadratic Loss Function under the new proposed Prior can be preferred.

Table 1: Bayes Estimates (Uniform) of Kumaraswamy mixture parameters and their risks (in parenthesis) with $\beta_1 = 2, \beta_2 = 4, w = 0.4, 0.6$ under WLF.

n	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.4$	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.6$
50	2.10697 (0.105348)	4.14820 (0.138273)	0.39216 (0.011689)	2.06893 (0.068964)	4.19734 (0.209867)	0.58824 (0.007919)
100	2.05933 (0.051483)	4.08328 (0.068055)	0.39604 (0.005212)	2.02702 (0.033784)	4.10490 (0.102622)	0.59406 (0.003980)
500	2.01206 (0.010060)	4.01600 (0.013387)	0.39920 (0.001197)	2.00730 (0.006691)	4.02030 (0.020102)	0.59880 (0.000799)

Table 2: Bayes Estimates (Jeffrey's) of Kumaraswamy mixture parameters and their risks (in parenthesis) with $\beta_1 = 2, \beta_2 = 4, w = 0.4, 0.6$ under WLF.

n	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.4$	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.6$
50	2.10462 (0.105063)	3.98993 (0.137584)	0.39216 (0.011689)	1.99803 (0.068899)	3.98411 (0.20969)	0.58824 (0.007919)
100	2.00318 (0.051338)	3.99141 (0.067651)	0.39604 (0.005212)	1.99807 (0.033765)	3.98436 (0.102163)	0.59406 (0.003980)
500	1.99904 (0.010045)	3.99869 (0.013368)	0.39920 (0.001197)	2.00017 (0.006684)	3.99596 (0.020082)	0.59880 (0.000799)

Table 3: Bayes Estimates (Jeffrey's-Gamma) of Kumaraswamy mixture parameters and their risks (in parenthesis) with $\beta_1 = 2, \beta_2 = 4, w = 0.4, 0.6$ under WLF.

n	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.4$	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.6$
50	1.83500 (0.083405)	3.28273 (0.117234)	0.39266 (0.011579)	1.88170 (0.058801)	2.98415 (0.165771)	0.58837 (0.007910)
100	1.91087 (0.045496)	3.63228 (0.062624)	0.39614 (0.005210)	1.94152 (0.031314)	3.43705 (0.090445)	0.59416 (0.003978)
500	1.98205 (0.009812)	3.92604 (0.013175)	0.39930 (0.001191)	1.98589 (0.006576)	3.88560 (0.019624)	0.59884 (0.000791)

Table 4: Bayes Estimates (Uniform) of Kumaraswamy mixture parameters and their risks (in parenthesis) with $\beta_1 = 2, \beta_2 = 4, w = 0.4, 0.6$ under QLF.

n	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.4$	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.6$
50	1.99642 (0.050000)	3.99760 (0.035712)	0.38000 (0.031000)	1.99762 (0.03333)	4.99875 (0.055556)	0.58000 (0.014000)
100	1.99798 (0.025000)	3.99127 (0.017241)	0.39125 (0.015250)	1.99836 (0.016667)	4.99572 (0.026315)	0.59000 (0.006833)
500	2.00176 (0.005000)	4.00131 (0.003356)	0.39800 (0.003010)	1.99964 (0.003333)	3.99419 (0.005050)	0.59800 (0.001340)

Table 5: Bayes Estimates (Jeffrey's) of Kumaraswamy mixture parameters and their risks (in parenthesis) with $\beta_1 = 2, \beta_2 = 4, w = 0.4, 0.6$ under QLF.

n	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.4$	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.6$
50	1.89940 (0.052632)	3.85521 (0.034483)	0.381421) (0.030100)	1.92727 (0.034483)	3.79060 (0.052632)	0.58315 (0.01390)
100	1.95103 (0.025641)	3.94053 (0.016949)	0.39315 (0.151000)	1.96743 (0.016949)	3.88466 (0.025641)	0.59210 (0.006812)
500	1.98874 (0.005025)	3.98976 (0.003344)	0.39835 (0.003000)	1.99339 (0.003344)	3.97902 (0.005025)	0.59873 (0.001330)

Table 6: Bayes Estimates (Jeffrey's-Gamma) of Kumaraswamy mixture parameters and their risks (in parenthesis) with $\beta_1 = 2, \beta_2 = 4, w = 0.4, 0.6$ under QLF.

n	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.4$	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.6$
50	1.74589 (0.045452)	3.91531 (0.033333)	0.38412 (0.030000)	1.83078 (0.031249)	2.80724 (0.050000)	0.58425 (0.013800)
100	1.86735 (0.023809)	3.56320 (0.008333)	0.39510 (0.150000)	1.90543 (0.016129)	3.36077 (0.025000)	0.59420 (0.006801)

n	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.4$	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.6$
500	1.97199 (0.004950)	3.90531 (0.003333	0.39853 (0.002900)	1.98127 (0.003311)	3.86458 (0.005000)	0.598913 (0.001310)

Table 7: Bayes Estimates (Uniform) of Kumaraswamy mixture parameters and their risks (in parenthesis) with $\beta_1 = 2, \beta_2 = 4, w = 0.4, 0.6$ under SELF.

n	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.4$	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.6$
50	2.21144 (0.245958)	4.28778 (0.614637)	0.403846 (0.004543)	2.13209 (0.151891)	4.42284 (0.982007)	0.59615 (0.004520)
100	2.10026 (0.110418)	4.13689 (0.285314)	0.401961 (0.002334)	2.06055 (0.070759)	4.20677 (0.44288)	0.59804 (0.002333)
500	2.01695 (0.020340)	4.02025 (0.053876)	0.400040 (0.000477)	2.01475 (0.013530)	4.04145 (0.081668)	0.59972 (0.000475)

Table 8: Bayes Estimates (Jeffrey's) of Kumaraswamy mixture parameters and their risks (in parenthesis) with $\beta_1 = 2, \beta_2 = 4, w = 0.4, 0.6$ under SELF.

n	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.4$	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.6$
50	2.11222 (0.235108)	4.12871 (0.588059)	0.40375 (0.004542)	2.06245 (0.146674)	4.23410 (0.947475)	0.59625 (0.004510)
100	2.04154 (0.106905)	4.06376 (0.280039)	0.401619 (0.002333)	2.03417 (0.070168)	4.10804 (0.4331160)	0.59903 (0.002330)
500	2.01358 (0.020372)	4.01375 (0.053823)	0.40038 (0.000472)	2.00612 (0.013461)	4.01965 (0.081201)	0.59990 (0.000470)

Table 9: Bayes Estimates (Jeffrey's-Gamma) of Kumaraswamy mixture parameters and their risks (in parenthesis) with $\beta_1 = 2, \beta_2 = 4, w = 0.4, 0.6$ under SELF.

n	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.4$	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.6$
50	1.91686 (0.165449)	3.41129 (0.411474)	0.40360 (0.004540)	1.94120 (0.117184)	3.14402 (0.537526)	0.59572 (0.004500)
100	1.96132 ().091401)	3.68192 (0.233202)	0.401423 (0.002301)	1.97270 (0.062673)	3.53184 (0.32622)	0.59924 (0.002310)
n	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.4$	$\beta_1 = 2$	$\beta_2 = 4$	$w = 0.6$
500	1.99155 (0.019633)	3.93020 (0.051824)	0.40024 (0.00069)	1.99437 (0.013169)	3.90911 (0.077159)	0.59995 (0.000390)

Table 10: The Lower Limit (LL), the Upper Limit (UL) of the 95% Prediction Intervals for, $\beta_2 = 4$ w=0.4

n	Uniform Prior		Jeffreys Prior		Jeffreys-Gamma	
	LL	UL	LL	UL	LL	UL
50	0.089008	0.887352	0.087394	0.876921	0.096307	0.658140
100	0.088953	0.879832	0.088071	0.874445	0.092474	0.630017
200	0.088863	0.875641	0.088481	0.873502	0.090659	0.616520
300	0.088822	0.874085	0.088515	0.872458	0.090049	0.613753
500	0.088837	0.873411	0.088688	0.872423	0.089569	0.610101

Table 11: The Lower Limit (LL), the Upper Limit (UL) of the 95% Credible Intervals under Uniform Prior

n	$\beta_1 = 2$		$\beta_2 = 4$		$w = 0.4$	
	LL	UL	LL	UL	LL	UL
50	1.806320	1.312590	1.303840	5.722060	0.27584	0.53886
100	1.251330	1.469960	1.467730	5.144510	0.30931	0.49826
500	0.556780	1.745730	1.742270	4.480240	0.35797	0.44358

Table 12: The Lower Limit (LL), the Upper Limit (UL) of the 95% Credible Intervals under Jeffrey's Prior

n	$\beta_1 = 2$		$\beta_2 = 4$		$w = 0.4$	
	LL	UL	LL	UL	LL	UL
50	1.22222	2.96846	1.22324	5.58370	0.27584	0.53885
100	1.42851	2.66512	1.42754	5.06772	0.30931	0.49826
500	1.73310	2.28744	1.73632	4.46147	0.35797	0.44358

Table 13: The Lower Limit (LL), the Upper Limit (UL) of the 95% Credible Intervals under Jeffrey's-Gamma

n	$\beta_1 = 2$		$\beta_2 = 4$		$w = 0.4$	
	LL	UL	LL	UL	LL	UL
50	1.18050	2.69680	1.17187	4.67150	0.27584	0.53886
100	1.38818	2.53248	1.38924	4.60789	0.30931	0.49826
500	1.71816	2.26305	1.71886	4.388247	0.35797	0.44358

References

1. Ahmad, K. E., Moustafa, H. M. and Abd-El-Rahman, A. M. (1997). Approximate Bayes Estimation for mixtures of two Weibull Distributions under Type II censoring. *Journal of Statistical Computation and Simulation*, **58**, 269-285.
2. Aslam, M. (2003). An application of Prior Predictive Distribution to elicit the Prior density. *Journal of Statistical Theory and Applications*, **2(1)**, 70-83.
3. Bansal, A. K. (2007). *Bayesian Parametric Inference*. Narosa Publishing House (Pvt.) Ltd. New Delhi.
4. Bolstad, W. M. (2004). *Introduction to Bayesian Statistics*. John Wiley and Sons, Inc., New Jersey.
5. Demidenko, E. (2004). *Mixed Models: Theory and applications*. John Wiley and Sons, Inc., New Jersey.
6. Eberly, L. E. and Casella, G. (2003). Estimating Bayesian Credible Intervals. *Journal of Statistical Planning and Inference*, **112**, 115-132.
7. Feroze, N. and Aslam, M. (2012). On Posterior Analysis of mixture of two components of Gumbel Type II Distribution. *International Journal of Probability and Statistics*, **1(4)**, 119-132.
8. Feroze, N. and Aslam, M. (2013). Statistical properties of two component mixture of Topp Leone Distribution under a Bayesian approach. *International Journal of Intelligent Technology and Applied Statistics*, **6(1)**, 65-99.
9. Garg, M. (2009). On Generalized Order Statistics from Kumaraswamy Distribution. *Tamsui Oxford Journal of Mathematical Sciences*, **25(2)**, 153-166.
10. Gholizadeh, R., Khalilpor, M. and Hsadian, M. (2011). Bayesian estimation in the Kumaraswamy Distribution under Progressively Type II censoring data. *International Journal of Engineering, Science and Technology*, **3(9)**, 47- 65.
11. Ismail, S. A. and El-Khodary, I. H. (2001). Characterization of mixtures of Exponential Family Distributions through conditional expectation. *Annual Conference on Statistics and Computer Modeling in Human and Social Sciences*, **13**, 64-73.
12. Jones, M. C. (2009). Kumaraswamy’s Distribution: A Beta Type Distribution with some tractability advantages. *Statistical Methodology*, **6**, 70-81.
13. Kumaraswamy, P. (1980). A generalized probability density function for double-bounded random processes. *Journal of Hydrology*, **46**, 79-88.
14. Landsay, B. G. (1995). *Mixture models: Theory, Geometry and Applications*. The Institute of Mathematical Statistics, Hayward, CA.

15. Maclachlan, G. J. and Peel, D. (2000). *Finite Mixture Models*. Wiley, New York.
16. Majeed, M. Y. and Aslam, M. (2012). Bayesian Analysis of the two component mixture of Inverted Exponential Distribution under Quadratic Loss Functions. *International Journal of Physical Sciences*, **7(9)**, 1424-1434.
17. Mcculloch, C. E. and Searle, S. R. (2001). *Generalized, Linear and Mixed Models*. Wiley, New York.
18. Mendenhall, W. and Hadar, R. J. (1958). *Estimation of Parameters of Mixed Exponential Distributions from Censored Life Test Data*. Wiley, New York.
19. Nadarajah, S. (2008). On the Distribution of Kumaraswamy. *Journal of Hydrol*ogy, **348**, 568-569.
20. Nassar, M. M. (1988). Two properties of mixtures of Exponential Distributions. *IEEE Transactions on Reliability*, **37(4)**, 383-385.
21. Nassar, M. M. and Mahmoud, M. R. (1985). On characterizations of a mixture of Exponential Distributions. *IEEE Transactions on Reliability*, **34(5)**, 484-488.
22. Ponnambalam, K., Seifi, A. and Vlach, J. (2001). Probabilistic design of systems with general Distributions of parameters. *International Journal of Circuit Theory and Application*, **29**, 527-536.
23. Saleem, M. and Aslam, M. (2008a). Bayesian Analysis of the two component mixture of the Rayleigh Distribution: With the Uniform and the Jeffrey's Priors. *Journal of Applied Statistical Science*, **16(4)**, 105-113.
24. Saleem, M. and Aslam, M. (2008b). On Prior selection for the mixture of Rayleigh Distribution using Predictive Intervals. *Pakistan Journal of Statistics*, **24(1)**, 21-35.
25. Saleem, M., Aslam, M. and Economou, P. (2010). On the Bayesian Analysis of the mixture of Power Function Distribution using the complete and the censored sample. *Journal of Applied Statistics*, **37(1)**, 25-40.
26. Saleem, M. and Irfan, M. (2010). On properties of the Bayes Estimates of the Rayleigh mixture parameters: A simulation study. *Pakistan Journal of Statistics*, **26(3)**, 547-555.
27. Sindhu, T. N., Feroze, N. and Aslam, M. (2013). Bayesian Analysis of the Kumaraswamy Distribution under failure censoring sampling scheme. *International Journal of Advanced Science and Technology*, **51**, 39-58.
28. Sultan, K. S., Ismail, A. M. and Al-Moisheer, A. S. (2007). Mixture of two inverse Weibull Distributions: Properties and estimation. *Computational Statistics and Data Analysis*, **51**, 5377-5387.
29. Titterington, D. M., Smith A. F. M. and Makov, U. E. (1985). *Statistical Analysis of Finite Mixture Distributions*. Wiley, London.