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Bayesian Analysis of the Rao-Kupper Model for Paired Comparison with Order Effect

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Abstract

Because of the wide applicability of different paired comparison models in various fields of life, many Bayesian researchers have become interested in performing the Bayesian Analysis of these models. The present article is also a contribution in the theory of Bayesian statistics by performing the Bayesian Analysis of a paired comparison model, namely the Rao-Kupper model with Order Effect. For Bayesian Analysis using the Non-informative (Uniform and Jeffrey's) Priors for the parameters of the model, the Joint Posterior and Marginal Posterior distributions of the model are obtained. The Posterior Estimates of the parameters, the Posterior probabilities for testing of the hypotheses to compare and rank the treatment parameters, the preference and the Predictive probabilities are also computed in the present article.

Keywords

Bayesian analysis, Non-informative priors, Paired comparison models, Rao-Kupper model

1. Introduction

In the method of paired comparison, the treatments are presented in pairs to one or more judges who in the simplest situation, choose one from the pair or simply just have no preference. This method is being widely used in experimentation and research methodologies in situations where subjective judgment is involved.

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Because of its simple and realistic approach, this method has remained attractive for many of the Bayesian analytics; see for example, Bradley (1953), Davidson (1970) and Davidson and Solomon (1973). While, in the recent years, many statisticians have also performed Bayesian Analysis of the paired comparison models. These researchers include Aslam (2002, 2003, and 2005), Glickman (2008), Gilani and Abbas (2008),Kim (2005), and recently, Altaf et al. (2011), among many others who have provided a choice of dimensions to study the Bayesian Analysis of different paired comparison models.

The present article focuses on the Bayesian Analysis of the Rao-Kupper model with Order Effect, presented by Rao and Kupper (1967) and after the words by Davidson and Beaver (1977). It is a modified form of the Bradley and Terry (1952) model in which the amendment for the allowance of ties has been made. This model is assumed to be useful when the difference between two treatments or objects, being compared, becomes minute but meaningful. The article unfolds in the way that in Section 2, the Rao-Kupper model with Order Effect is stated along with the necessary notations and its Likelihood. The Bayesian Analysis for the model using Uniform Prior is presented in Section 3. The (reference) Jeffrey's Prior is also defined and Bayesian Analysis using the Jeffrey's Prior is presented in Section 4. Test for the Goodness of Fit is carried out in Section 5. Finally, Section 6 concludes the article.

2. The Rao-Kupper model with Order Effect for Paired Comparison

Rao and Kupper (1967) modify the Bradley and Terry (1952) model and incorporate the allowance for ties. They introduce a threshold parameter $\delta = \ln\lambda$, if X_i is the sensation experienced in response of the treatment T_i, X_j is the sensation experienced in response of the treatment T_j and suppose that if the observed difference $(X_i - X_j)$ is less than δ then the panelist may detect no difference between the treatments and will declare a tie. The probability $\mathrm{P}\{(X_i - X_j) > \delta \mid \theta_i, \theta_j\}$ that the treatment T_i is preferred to the treatment T_j when both the treatments are being compared is denoted by $\psi_{i.ij}$ and is defined as:

$$\psi_{i.ij} = \frac{1}{4}\int_{-(\ln\theta_i - \ln\theta_j)+\delta}^{\infty} \sec h^2(y/2)dy$$

$$= \frac{\theta_i}{\theta_i + \lambda\theta_j} \quad (2.1)$$

where

θ_i ($\theta_i > 0$) is the worth of the treatment T_i. The worth of the treatment is an indicator of relative merit. These worths are located on an underlying scale to describe the responses to the treatments and $\lambda \geq 1$ is the tie/threshold parameter and that the treatment T_j is preferred to T_i when both the treatments are being compared is denoted by $\psi_{j.ij}$ and is defined as:

$$\psi_{j.ij} = \frac{\theta_j}{\lambda\theta_i + \theta_j} \quad (2.2)$$

The probability that the preference of the treatments T_i and T_j ended up in a tie is denoted as $\psi_{o.ij}$ and is defined as:

$$\psi_{o.ij} = \frac{1}{4}\int_{-(\ln\theta_i - \ln\theta_j)-\delta}^{-(\ln\theta_i - \ln\theta_j)+\delta} \sec h^2(y/2)dy$$

$$= \frac{(\lambda^2 - 1)\theta_i\theta_j}{(\theta_i + \lambda\theta_j)(\lambda\theta_i + \theta_j)}. \quad (2.3)$$

The equation (2.1), (2.2) and (2.3), make the Rao-Kupper model.

To study the Effect of Order of presentation, Davidson and Beaver (1977) incorporate a multiplicative Order Effect in the Rao-Kupper model. Thus, assuming the multiplicative within pair Order Effect, the logarithms of the worths depict the additive Order Effect. A multiplicative Order Effect γ_{ij} is affecting the worths of the two objects being compared. It is assumed that $\gamma_{ij} = \gamma_{ji} = \gamma$ depends on the objects being compared. The range of the Order Effect parameter(s) does not depend on the worth parameters. When $\gamma > 1$, the worth of the treatment that is presented second increases and when $\gamma < 1$, the worth of the treatment presented first increases.

For a pair of treatments T_i and T_j, when the order of presentation is (i, j) the preference probabilities are:

$$\psi_{i.ij} = \frac{\theta_i}{(\theta_i + \gamma\lambda\theta_j)} \tag{2.4}$$

which is the preference probability for T_i over T_j. Now for the preference of T_j over T_i

$$\psi_{j.ij} = \frac{\gamma\theta_j}{(\lambda\theta_i + \gamma\theta_j)} \tag{2.5}$$

The probability of no preference

$$\psi_{o.ij} = \frac{(\lambda^2 - 1)\gamma\theta_j\theta_i}{(\theta_i + \gamma\lambda\theta_j)(\lambda\theta_i + \gamma\theta_j)} \tag{2.6}$$

When the Order of presentation is (j, i), the preference corresponding probabilities are:

$$\psi_{j.ji} = \frac{\theta_j}{(\theta_j + \gamma\lambda\theta_i)} \tag{2.7}$$

which is the preference probability for T_j over T_i. Now for the preference of T_i over T_j

$$\psi_{i.ji} = \frac{\gamma\theta_i}{(\lambda\theta_j + \gamma\theta_i)} \tag{2.8}$$

and the probability of no preference

$$\psi_{o.ij} = \frac{(\lambda^2 - 1)\gamma\theta_j\theta_i}{(\theta_j + \gamma\lambda\theta_i)(\lambda\theta_j + \gamma\theta_i)} \tag{2.9}$$

where

γ is the multiplicative Order Effect parameter. The Rao-Kupper model with Order Effect is given by equation (2.4) to (2.9).

If r_{ij} and r_{ji} are the total numbers of comparisons for the orders (i, j) and (j, i), respectively then the probability of the observed result in the kth repetition of the pair (T_i, T_j) for both the orders of presentation (i, j) and (j, i) is,

$$P_{ijk} = (\lambda^2 - 1)^T \left[\frac{\theta_i}{(\theta_i + \lambda\gamma\theta_j)}\right]^{r_{ij}(1)} \left[\frac{\gamma\theta_j}{(\lambda\theta_i + \gamma\theta_j)}\right]^{r_{ij}(2)} \left[\frac{\theta_j}{(\lambda\gamma\theta_i + \theta_j)}\right]^{r_{ji}(1)} \left[\frac{\gamma\theta_i}{(\gamma\theta_i + \lambda\theta_j)}\right]^{r_{ji}(2)} \tag{2.10}$$

where
$r_{ij}(1)=w_{ij}(1)+t_{ij}$, $r_{ij}(2)=w_{ij}(2)+t_{ij}$, similarly $r_{ji}(1)=w_{ji}(1)+t_{ji}$
$r_{ji}(2)=w_{ji}(2)+t_{ji}$.

We define the notations for the Rao-Kupper model with Order Effect as:

$w_{ijk}(1)$ =1 or 0, accordingly as the treatment T_i is preferred to the treatment T_j when the treatment T_i is presented first in the k^{th} repetition of the comparison. $w_{ijk}(2)$ =1 or 0, accordingly as the treatment T_j is preferred to the treatment T_i when the treatment T_i is presented first in the k^{th} repetition of the comparison. $w_{jik}(1)$ =1 or 0, accordingly as the treatment T_j is preferred to the treatment T_i when the treatment T_j is presented first in the k^{th} repetition of the comparison. $w_{jik}(2)$ =1 or 0, accordingly as the treatment T_i is preferred to treatment T_j when the treatment T_j is presented first in the k^{th} repetition of the comparison. t_{ijk} =1 or 0, accordingly as the treatment T_i is tied with the treatment T_j when T_i is presented first in the k^{th} repetition. t_{jik} =1 or 0, accordingly as the treatment T_i is tied with the treatment T_j when T_j is presented first in the k^{th} repetition.

3. Bayesian Analysis of the Rao-Kupper model using Uniform Prior

Bayes (1763) and Laplace (1812) proposed the Bayesian Analysis of unknown parameters using a Uniform (possibly Improper) Prior. This was the approach of "inverse probability". We call this Prior a Non-informative, because, it does not favor any possible value of the parameter over any other values; however, it is not invariant under re-parameterization.

According to Laplace (1812), it would be a good step if the Non-informative Prior for parameter θ is simply chosen to be the constant $\{p(\theta)=1\}$ on the parameter space Θ.

Here, we assume the Standard Uniform Distribution to be the Non-informative Prior for the parameters of the Rao-Kupper model.

Let,
$\theta = (\theta_1, \dots, \theta_m)$ be the vector of treatment parameters, λ be the threshold (tie) parameter and γ be the Order Effect parameter, the Uniform Prior Distribution after using the constraint $\sum_{i=1}^{m}\theta_i = 1$ for identification for the parameters $\theta_1,\ \theta_2,...,\theta_m,\ \lambda$ and γ of the Rao-Kupper model is assumed to be:

$$p(\theta_1,\theta_2,...,\theta_m,\lambda,\gamma) \propto 1, \qquad 0 < \theta_i < 1 \text{ for } i = 1,2,...,m, \sum_{i=1}^{m}\theta_i = 1,\ \lambda > 1,\ \gamma > 0$$

where
the parameters are independent. The Likelihood function after observing the data of the trial and putting the constraint $\sum_{i=1}^{m}\theta_i = 1$ for identification is given by:

$$l(\mathbf{x};\theta_1,...,\theta_m,\lambda,\gamma) = \prod_{i\neq j=1}^{m}\prod_{j=1}^{m} P_{ijk}$$

$$= \prod_{i\neq j=1}^{m}\prod_{j=1}^{m} \frac{r_{ij}\,!}{w_{ij}(1)!\,w_{ij}(2)!\,t_{ij}\,!} \times \frac{\theta_i^{r_i}(\lambda^2-1)^T\gamma^{\mathrm{K}}}{(\theta_i+\lambda\gamma\theta_j)^{r_{ij}(1)}(\gamma\theta_i+\lambda\theta_j)^{r_{ji}(2)}} \qquad (3.1)$$

$\theta_i > 0,\ i = 1,2,...,m,\ \sum_{i=1}^{m}\theta_i = 1, \lambda \geq 1,\ \gamma > 0$

where
$T = \sum_{i=1}^{m}\sum_{j\neq i}^{m} t_{ij}$ is the total number of times treatments T_i and T_j are tied.
The number of preferences for the treatment presented second is, $K = \sum_{i=1}^{m}\sum_{j\neq i=1}^{m} w_{ij}(2) + T$

$r_i = \sum_{i=1}^{m}(w_i + t_i)\,.\,\theta_i > 0, (i = 1,2,...,m),\ \sum_{i=1}^{m}\theta_i = 1,\quad w_i = \sum_{j}\{w_{ij}(1) + w_{ji}(2)\}$
which is the total number of times T_i is preferred.

$w_{ij}(1) =$ 1 or 0, accordingly as the treatment T_i is preferred to the treatment T_j when the treatment T_i is presented first. $w_{ij}(2) =$ 1 or 0, accordingly as the treatment T_j is preferred to the treatment T_i when the treatment T_i is presented first. t_{ij} =1 or 0, accordingly as the treatment T_i is tied up with the treatment T_j when T_i is presented first. The threshold (tie) parameter is $\lambda \geq 1$ and the Order

Effect parameter is $\gamma > 0$.The Joint Posterior Distribution for the parameters $\theta_1,...,\theta_m$, λ and γ using the above Uniform Prior, given the data *x* is,

$$p(\theta_1,...,\theta_m,\lambda,\gamma\,|\mathbf{x}) \propto \prod_{i\neq j=1}\prod_{j=1}\frac{\theta_i^{r_i}(\lambda^2-1)^T\gamma^{K}}{(\theta_i+\lambda\gamma\theta_j)^{r_{ij}(1)}(\lambda\theta_i+\gamma\theta_j)^{r_{ij}(2)}}, \qquad (3.2)$$

$\theta_i \geq 0,\ i=1,2,...,m,\ \sum_{i=1}^{m}\theta_i = 1, \lambda>1,\ \gamma>0$

The Marginal Posterior Distribution of θ_1 when we have the case of *m* treatments is,

$$p(\theta_1|\mathbf{x}) = \frac{\theta_1^{r_1}}{Q}\int_{\theta_2=0}^{1-\theta_1}...\int_{\theta_{m-1}=0}^{1-\theta_1-...-\theta_{m-2}}\int_{\lambda=1}^{\infty}\int_{\gamma=0}^{\infty}\prod_{i\neq j=1}^{m-1}\prod_{j}^{m-1}\frac{\theta_i^{r_i}(\lambda^2-1)^T\gamma^{K}}{(\theta_i+\lambda\gamma\theta_j)^{r_{ij}(1)}(\lambda\theta_i+\gamma\theta_j)^{r_{ij}(2)}}d\gamma d\lambda d\theta_{m-1}...d\theta_2 \qquad (3.3)$$

$0<\theta_1<1, \lambda>1,\ \gamma>0$

where

Q is the normalizing constant. From equation (3.2), the Posterior Distribution for *m* = 3 is obtained. The same pattern given in equation (3.3) is followed to derive the Marginal Posterior densities of the parameters $\theta_2,\ \theta_3,\ \lambda$ and γ.

For the complete Bayesian Analysis of the models using the Uniform and Jeffrey's Priors, we need data to carry out the effective procedures. The data given in Table 1 are obtained from Davidson and Beaver (1977) in which, a paired comparison experiment conducted is mentioned for the ordered pair (i, j) about the packaged food mixes where r_{ij} the independent responses for the pair are (i, j) and r_{ji} are the independent responses for the pair (j,i).

The graphs of the marginal Posterior densities of the parameters are given in Figure 1. All the curves depict Symmetric behavior.

The Posterior means for the parameters $\theta_1,\ \theta_2,\ \theta_3,\ \lambda$ and γ, taking Uniform Prior as the Non-informative Prior, are found out to be 0.3181, 0.4385, 0.2434, 1.3611 and 0.4441, respectively with the help of a Quardrature method. The Posterior modes are obtained by maximizing the Posterior density. The values of the modes of the parameters $\theta_1,\ \theta_2,\ \theta_3,\ \lambda$ and γ are obtained by solving the following system of equations (3.4) simultaneously, by the use of calculus rule in the SAS package. The modes of the parameters are found to be 0.3179, 0.4401, 0.2420, 1.3463 and 0.4380, respectively. The Posterior Estimates (means and modes) illustrate that the treatment T_2 shows its dominance upon other treatments as the most preferred one. The treatments T_1 and T_3 come at second and third position, respectively according to the preference pattern.

$$r_i/\theta_i - \sum_{j\neq i=1}^{m} r_{ij}(1)/(\theta_i+\lambda\gamma\theta_j) - \sum_{j\neq i=1}^{m} \lambda r_{ij}(2)/(\lambda\theta_i+\gamma\theta_j) - \sum_{j\neq i=1}^{m} \lambda\gamma r_{ji}(1)/(\lambda\gamma\theta_i+\theta_j) - \sum_{j\neq i=1}^{m} \lambda\gamma r_{ji}(2)/(\gamma\theta_i+\lambda\theta_j) = 0, \quad i=1,2,...,m.$$

$$2\lambda T/(\lambda^2-1) - \sum_{j\neq i=1}^{m} \gamma\theta_j r_{ij}(1)/(\theta_i+\lambda\gamma\theta_j) - \sum_{j\neq i=1}^{m} \theta_i r_{ij}(2)/(\lambda\theta_i+\gamma\theta_j) - \sum_{j\neq i=1}^{m} \theta_i\gamma r_{ji}(1)/(\lambda\gamma\theta_i+\theta_j) - \sum_{j\neq i=1}^{m} \theta_j r_{ji}(2)/(\gamma\theta_i+\lambda\theta_j) = 0,$$

$$K/\gamma - \sum_{j\neq i=1}^{m} \lambda\theta_j r_{ij}(1)/(\theta_i+\lambda\gamma\theta_j) - \sum_{j\neq i=1}^{m} \theta_j r_{ij}(2)/(\lambda\theta_i+\gamma\theta_j) - \sum_{j\neq i=1}^{m} \theta_i\lambda r_{ji}(1)/(\lambda\gamma\theta_i+\theta_j) - \sum_{j\neq i=1}^{m} \theta_i r_{ji}(2)/(\gamma\theta_i+\lambda\theta_j) = 0, \sum_{i=1}^{m}\theta_i - 1 = 0. \quad (3.4)$$

For the Bayesian testing of hypotheses, we have to obtain the Posterior probabilities of the treatment parameters. The hypotheses compared in this case of three treatments are:

$$H_{ij}: \theta_i > \theta_j \text{ and } H_{ij}^{c}: \theta_j \geq \theta_i, \quad i \neq j = 1,2,3. \quad (3.5)$$

The Posterior probability for H_{ij} is $p_{ij} = p(\theta_i > \theta_j)$ and that for H_{ij}^{c} is given by: $q_{ij} = 1 - p_{ij}$.

The Posterior probability p_{ij} is obtained as:

$$p_{ij} = p(\varphi > 0 \mid \mathrm{x}) = \int_{\varphi=0}^{1} \int_{\xi=\varphi}^{(1+\varphi)/2} \int_{\lambda=1}^{\infty} \int_{\gamma=0}^{\infty} p(\varphi,\xi,\lambda,\gamma \mid \mathrm{x})\, d\gamma\, d\lambda\, d\xi\, d\varphi, \quad i \neq j = 1,2,3 \quad (3.6)$$

where

$\varphi = \theta_i - \theta_j$ and $\xi = \theta_i$.

For example, the Posterior probability p_{12} for H_{12} is obtained as:

$$p_{12} = p(\theta_1 > \theta_2) = p(\theta_1 - \theta_2 > 0) = p(\varphi > 0)$$

$$\begin{aligned} p_{12} = \int_{\varphi=0}^{1} \int_{\xi=\varphi}^{(1+\varphi)/2} \int_{\gamma=0}^{\infty} \int_{\lambda=1}^{\infty} & \xi^{r_1} (\xi-\varphi)^{r_2} (1-2\xi+\varphi)^{r_3} \gamma^{K} (\lambda^2-1)^{T} / C((\xi+\gamma\lambda(\xi-\varphi))^{r_{12}(1)} \{\lambda(\xi-\varphi) \\ & +\gamma\xi\}^{r_{12}(2)} (\gamma\lambda\xi+(\xi-\varphi))^{r_{21}(1)} \{\gamma(\xi-\varphi)+\lambda\xi\}^{r_{21}(2)} \{\xi+\gamma\lambda(1-2\xi+\varphi)\}^{r_{13}(1)} \{\gamma\xi+\lambda(1- \\ & 2\xi+\varphi)\}^{r_{13}(2)} \{\gamma\lambda\xi+(1-2\xi+\varphi)\}^{r_{31}(1)} \{\lambda\xi+\gamma(1-2\xi+\varphi)\}^{r_{31}(2)} \{(\xi-\varphi)+\gamma\lambda(1-2\xi+ \\ & \varphi)\}^{r_{23}(1)} \{\gamma(\xi-\varphi)+\lambda(1-2\xi+\varphi)\}^{r_{23}(2)} \{\gamma\lambda(\xi-\varphi)+(1-2\xi+\varphi)\}^{r_{32}(1)} \{\lambda(\xi-\varphi)+\gamma(1 \\ & -2\xi+\varphi)\}^{r_{32}(2)} d\lambda \gamma \lambda d\xi d\varphi \end{aligned} \quad (3.7)$$

A small transformation has been made i.e., $\varphi = \theta_1 - \theta_2$, $\xi = \theta_1$, to acquire the Posterior probabilities which are given in Table 2.

To test the above mentioned hypotheses, we apply the rule given by Aslam (1996), which states to let $s = \min(p_{ij}, q_{ij})$. If p_{ij} is small then H_{ij}^c is accepted with high probability. If q_{ij} is small then H_{ij} is accepted with high probability. If s is small, we can reject one hypothesis otherwise, if $s > 0.1$ then the evidence is inconclusive. Thus, using the same criteria, we test the hypotheses (3.5). The Posterior probabilities are given in the following Table 2. The hypotheses H_{12} and H_{23}^c are rejected, H_{12}^c and H_{23} are accepted and for H_{13}, the decision is inconclusive.

The Predictive probabilities for the three treatments are obtained by a SAS program and are given in Table 3. The Effect of Order of presentation is quite noticeable in the pair-wise comparison of the treatments. The Predictive probabilities for no preference in a future single comparison between the treatments for all the pairs are less than 0.15.

4. Choice of the Jeffrey's Prior

According to Datta and Ghosh (1995), a Uniform Prior, though most widely used as Non-informative Prior, does not typically lead to another Uniform Prior under an alternate one-to-one re-parameterization, as a remedy, Jeffreys (1961) proposes the Non-informative Prior which remains invariant under any one-to-one parameterization.

Bernardo (1979a and 1979b) shows that the reference Prior for θ, providing there are no nuisance parameters, is the Jeffrey's (1961) Prior $\pi(\theta) = (|I(\theta)|)^{1/2}$; where $|I(\theta)|$ is the expected Fisher Information Matrix. According to Johnson and Ladalla (1979), the Jeffrey's Prior is appropriate as the Non-informative Prior if the Likelihood function belongs to the Exponential Family.

4.1 Bayesian Analysis of the Rao-Kupper model using the Jeffrey's Prior: The Likelihood function of the Davidson model with Order Effect given as: $l(\mathbf{x};\theta_1,...,\theta_m,\nu,) = \prod_{i\neq j}\prod_{j} \frac{\gamma_{ij}{}^{K}\nu^{T}\theta_i^{s_i}}{(\theta_i + \gamma\theta_j + \nu\sqrt{\theta_i\theta_j})^{r_{ij}}}$ can be represented as:

$$\exp\left\{\sum_{i=1}^{m} S_i \ln\theta_i + K\ln\gamma + T\ln\nu - \sum_{i(<j)=1}^{m} r_{ij}\ln(\theta_i + \gamma\theta_j + \nu\sqrt{\theta_i\theta_j})\right\}$$

and this Likelihood function belongs to the Exponential family. In addition to it, if there is no nuisance parameter in the function, then according to Johnson and Ladalla (1979) and Sareen (2001), the Jeffrey's Prior is chosen as the appropriate Non-informative Prior. The Bayesian Analysis for the Davidson model using the Jeffrey's Prior is performed on the same data set given in Table 1 with r_{ij} and r_{ji} numbers of comparisons for each pair.

If we take $p_J(\theta_1,...,\theta_m,\lambda,\gamma)$ as the Jeffrey's Prior then the Posterior Distribution of the parameters $\theta_1,...,\theta_m$, λ and γ for m treatments is,

$$p(\theta_1,...,\theta_m,\lambda,\gamma \mid \mathbf{x}) \propto p_J(\theta_1,...,\theta_m,\lambda,\gamma)\prod_{i\neq j=1}^{m}\prod_{j=1}^{m}\frac{\theta_i^{r_i}(\lambda^2-1)^T\gamma^{K}}{(\theta_i+\lambda\gamma\theta_j)^{r_{ij}(1)}(\lambda\theta_i+\gamma\theta_j)^{r_{ij}(2)}}, \qquad (4.1.1)$$

$$\theta_i \geq 0,\ i=1,2,..,m,\ \sum_{i=1}^{m}\theta_i = 1,\ \lambda > 1, \gamma > 0$$

and the Marginal Posterior density of θ_1 is,

$$p(\theta_1|\mathbf{x}) = \frac{\theta_1^{r_1}}{Q}\int_{\theta_2=0}^{1-\theta_1}\cdots\int_{\theta_{m-1}=0}^{1-\theta_1-...-\theta_{m-2}}\int_{\lambda=1}^{\infty}\int_{\gamma=0}^{\infty}\prod_{i\neq j=1}^{m-1}\prod_{j}^{m-1} p_J(\theta_1,...,\theta_m,\lambda,\gamma)\times$$
$$\frac{\theta_i^{r_i}(\lambda^2-1)^T\gamma^{K}}{(\theta_i+\lambda\gamma\theta_j)^{r_{ij}(1)}(\lambda\theta_i+\gamma\theta_j)^{r_{ij}(2)}}\,d\gamma\, d\lambda\, d\theta_{m-1}...d\theta_2,$$
$$0 < \theta_1 \leq 1,\ \lambda > 1,\ \gamma > 0$$

where
Q is the normalizing constant. The same method can be followed to find the Marginal Posterior Distributions for the other parameters.

The Joint Posterior and Marginal Posterior Distributions for the parameters using the Jeffrey's Prior are derived from equation (3.6) for m = 3. The same data set given in Table 1 is used for the Analysis of the model.

The graphs of the Marginal Posterior densities of the parameters are shown in Figure 2. It is apparent that the graphs of the Marginal Distributions using both the Non-informative Priors (Uniform, Jeffrey's) look alike.

The Posterior means for the parameters θ_1, θ_2, θ_3, λ and γ are obtained with the help of a Quardrature method and are found to be 0.2989, 0.4687, 0.2324, 1.3733 and 0.4170, respectively. The Posterior modes are obtained to be 0.2956, 0.4456, 0.2265, 1.3763 and 0.4154, respectively via a SAS program. It is obvious that the Posterior Estimates (means and modes) obtained using the Uniform and the Jeffrey's Priors are almost alike and hence the ranking of the treatments is same.

Same hypotheses as given in equation (3.5) are observed for the Jeffrey's Prior and thus, the Posterior probability for H_{ij}, $p_{ij} = p(\theta_i > \theta_j)$ is obtained. For testing of the following hypotheses, the Posterior probabilities using the Jeffrey's Prior are acquired and shown in Table 4.

The hypotheses H_{12}^c and H_{23} are accepted and the decision is inconclusive for H_{23}. The same decisions are drawn as are shown using Uniform Prior in Figure 2.

The Predictive probabilities for the three treatments obtained using the Jeffrey's Prior are shown in Table 5.

The Order Effect factor is quite noticeable here because none of the treatment can prove its apparent preference over any other one. The values of the probabilities are almost identical to those obtained using Uniform Prior so they are interpreted in the same way.

5. Appropriateness of the model

To test the appropriateness of the model in case of three treatments, observed number of preferences is compared with the expected number of preferences. The χ^2 Statistic is used to test the Goodness of Fit of the model for paired comparison. The hypotheses to be tested are:

H_o: The model is considered to be true for any value of $\theta = \theta_o$.

H^c: The model is considered not to be true for any value of θ.

The χ^2 Statistic is,

$$\chi^2 = \sum_{i \neq j}^{m} \{ \frac{(w_{ij}(1) - \hat{w}_{ij}(1))^2}{\hat{w}_{ij}(1)} + \frac{(w_{ij}(2) - \hat{w}_{ij}(2))^2}{\hat{w}_{ij}(2)} + \frac{(t_{ij} - \hat{t}_{ij})^2}{\hat{t}_{ij}} + \frac{(w_{ji}(1) - \hat{w}_{ji}(1))^2}{\hat{w}_{ji}(1)} + \frac{(w_{ji}(2) - \hat{w}_{ji}(2))^2}{\hat{w}_{ji}(2)} + \frac{(t_{ji} - \hat{t}_{ji})^2}{\hat{t}_{ji}} \}$$

with $2m(m-1)-(m+1)$ degrees of freedom as given by Davidson and Beaver (1977), where $\hat{w}_{ij}(1)$ and $\hat{w}_{ij}(2)$ are the expected number of times T_i and T_j are preferred respectively and $\hat{t}_{ij}$ is the expected number of times T_i and T_j end up in a tie when T_i is presented first. $\hat{w}_{ji}(1)$, $\hat{w}_{ji}(2)$ and $\hat{t}_{ji}$ are described in the same way when T_j is presented first. The value of χ^2 statistic is obtained to be 9.534 with the p-value 0.2993. We have no evidence to reject the null hypothesis which means that the model is suitable to Fit.

6. Conclusion

The test of Goodness of Fit construes that the Rao-Kupper model is appropriate for the data and is enormously useful in paired comparison experiments especially where Effect of Order of presentation is involved. In this study, the Bayesian Analysis of the model using the Non-informative (Uniform and Jeffrey's) Priors is performed and nearly same conclusions are drawn for the results obtained. The ranking of the treatments obtained via Posterior Estimates is also the same using both the Priors, which is $T_2 \rightarrow T_1 \rightarrow T_3$. It may further, be concluded that as both the Non-informative Priors show quite similar results, for simplicity, Uniform Prior may be used as the Non-informative Prior.

Table 1: Responses for the preference testing experiment for $m = 3$

Pairs (i, j)	r_{ij}	$w_{ij}(1)$	$w_{ij}(2)$	t_{ij}
(1,2)	42	23	11	8
(2,1)	43	29	6	8
(1,3)	43	27	11	5
(3,1)	42	22	14	6
(2,3)	41	34	6	1
(3,2)	42	23	16	3

Table 2: Posterior probabilities using Uniform Prior

Hypotheses	p_{ij}	Hypotheses	q_{ij}
$H_{12}: \theta_1 > \theta_2$	0.0156	$H^c{}_{12}: \theta_2 \geq \theta_1$	0.9844
$H_{13}: \theta_1 > \theta_3$	0.8608	$H^c{}_{13}: \theta_3 \geq \theta_1$	0.1667
$H_{23}: \theta_2 > \theta_3$	0.9981	$H^c{}_{23}: \theta_3 \geq \theta_2$	0.0019

Table 3: Predictive probabilities using Uniform Prior

Pairs (i, j)	(1,2)	(2,1)	(1,3)	(3,1)	(2,3)	(3,2)
$P_{ij}(1)$	0.5470	0.6843	0.6955	0.5598	0.7487	0.4804
$P_{ij}(2)$	0.3118	0.2011	0.1928	0.3001	0.1548	0.3709
$P_{ij}(0)$	0.1412	0.1146	0.1117	0.1401	0.0965	0.1487

Table 4: Posterior probabilities using the Jeffrey's Prior

Hypotheses	p_{ij}	Hypotheses	q_{ij}
$H_{12}: \theta_1 > \theta_2$	0.0320	$H^c{}_{12}: \theta_2 \geq \theta_1$	0.9844
$H_{13}: \theta_1 > \theta_3$	0.8608	$H^c{}_{13}: \theta_3 \geq \theta_1$	0.1392
$H_{23}: \theta_2 > \theta_3$	0.9981	$H^c{}_{23}: \theta_3 \geq \theta_2$	0.0019

Table 5: Predictive probabilities using the Jeffrey's Prior

Pairs (i, j)	(1,2)	(2,1)	(1,3)	(3,1)	(2,3)	(3,2)
$P_{ij}(1)$	0.5557	0.6767	0.7210	0.5251	0.7298	0.4922
$P_{ij}(2)$	0.3050	0.2077	0.2086	0.3048	0.1690	0.3616
$P_{ij}(0)$	0.1361	0.1156	0.0704	0.1701	0.1013	0.1462

Table 6: Observed and expected number of preferences

Pairs (i, j)	$w_{ij}(1)$	$\hat{w}_{ij}(1)$	$w_{ij}(2)$	$\hat{w}_{ij}(2)$	t_{ij}	$\hat{t}_{ij}$
(1,2)	23	22.91	11	13.04	8	6.05
(2,1)	29	29.90	6	8.23	8	4.86
(1,3)	27	29.40	11	8.59	5	5.01
(3,1)	22	23.47	14	12.56	6	6.11
(2,3)	34	30.70	6	6.29	1	4.00
(3,2)	23	20.12	16	15.56	3	6.33

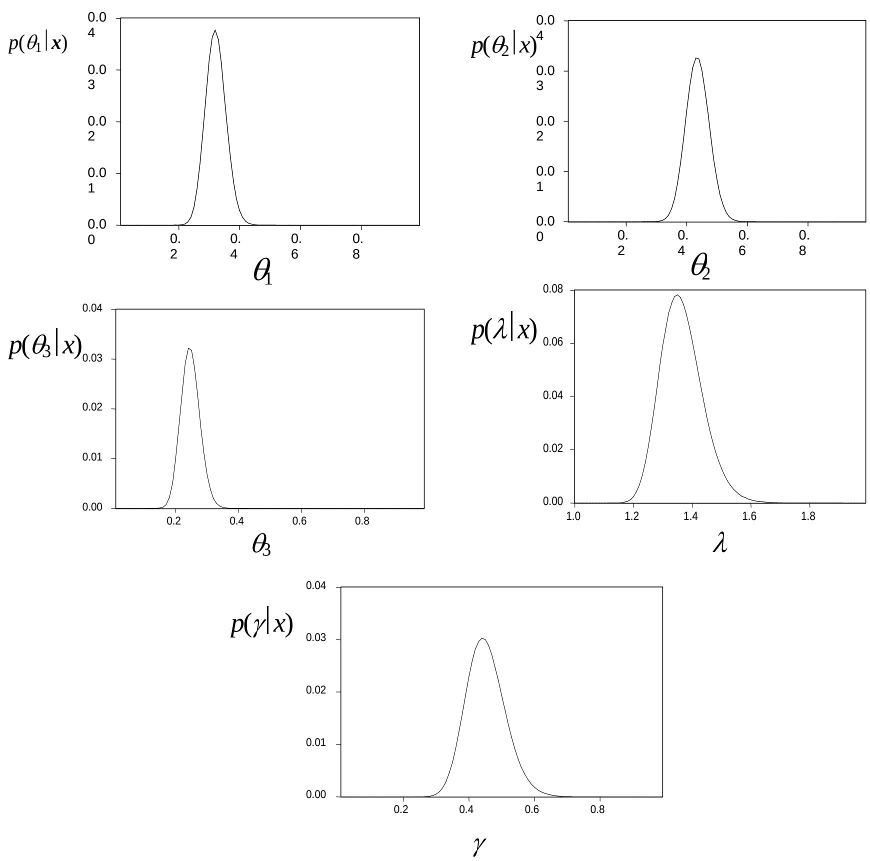

Figure 1: The Marginal Posterior Distributions of the parameters

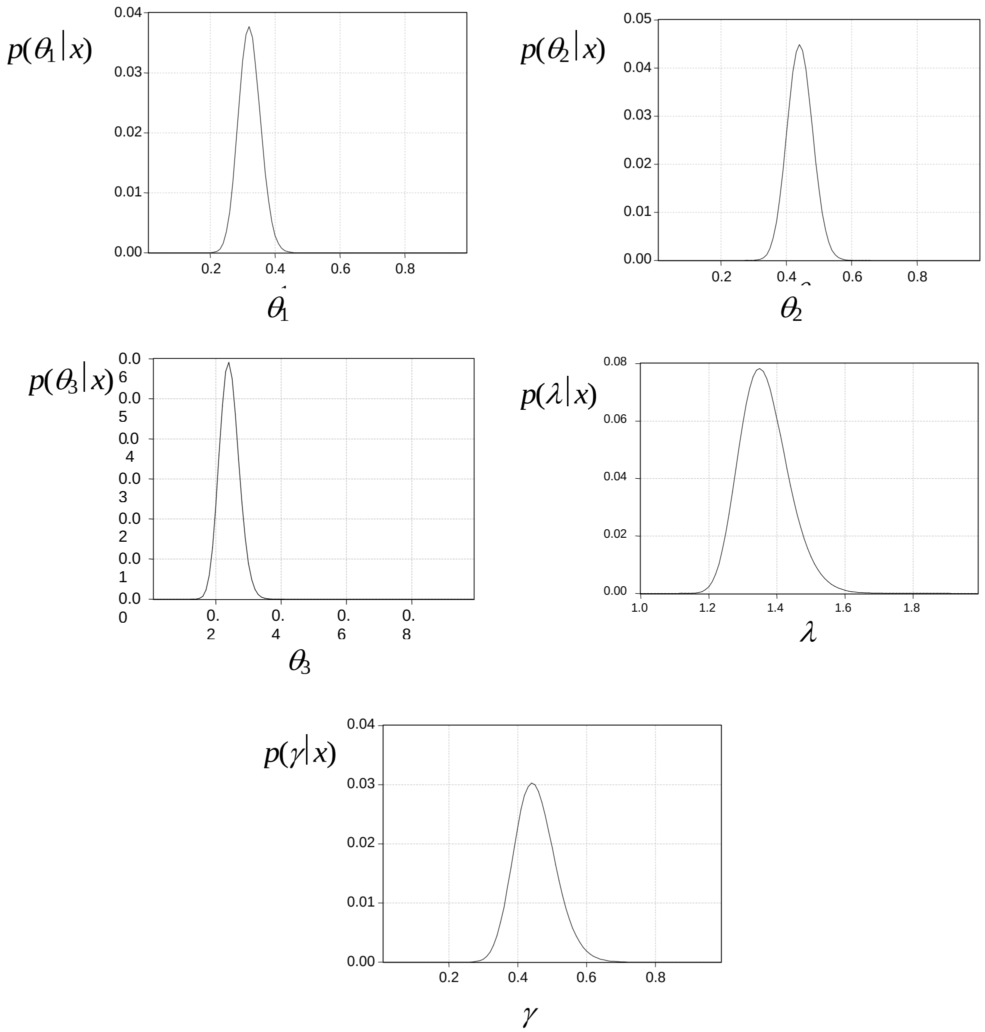

Figure 2: The Marginal Posterior Distributions of the parameters

References

1. Altaf, S., Aslam, M. and Aslam, M. (2011). Bayesian Analysis of the Davidson model for Paired comparison with order effect using Non-informative Priors. *Pakistan of Journal of Statistics*, **27(2)**, 171-185.
2. Aslam, M. (1996). Bayesian Analysis for paired comparisons data. Ph.D. Thesis, Department of Mathematics, University of Wales, Aberystwyth.

3. Aslam, M. (2002). Bayesian Analysis for paired comparisons models allowing ties and not allowing ties. *Pakistan of Journal of Statistics*, **18(1)**, 55-69.
4. Aslam, M. (2003). An application of Prior Predictive distribution to elicit the Prior Density. *Journal of Statistical Theory and Applications*, **2(1)**, 70-83.
5. Aslam, M. (2005). Bayesian comparison of the paired comparison models allowing ties. *Journal of Statistical Theory and Applications*, **4(2)**, 161-171.
6. Bayes, T. (1763). An Essay towards Solving a Problem in the Doctrine of Chances. *Philosophical Transactions of the Royal Society of London*, **53**, 370-418. Reprinted in Barnard (1958). *Biometrika*, **45**, 293-315.
7. Bernardo, J. M. (1979a). Expected information as expected utility. *The Annals of Statistics*, **7(3)**, 686–690.
8. Bernardo, J. M. (1979b). Reference Posterior distributions for Bayesian Inference. *Journal of the Royal Statistical Society: Series (B)*, **41**, 113-147.
9. Bradley, R. A. (1953). Some statistical methods in taste testing and quality evaluation. *Biometrics*, **9**, 22-28.
10. Bradley, R. A. and Terry. M. E. (1952). Rank Analysis of Incomplete Block Design: The method of paired comparisons. *Biometrika*, **39**, 324-345.
11. Datta, G. S. and Ghosh, M. (1995). Some Remarks on Non-informative Priors. *Journal of the American Statistical Association*, **90(432)**, 1357-1363.
12. Davidson, R. R. (1970). On extending the Bradley-Terry model to accommodate ties in paired comparison experiments. *Journal of American Statistical* Association, **65(329)**, 317-328.
13. Davidson, R. R. and Beaver, R.J. (1977). On extending the Bradley-Terry model to incorporate within pair Order Effects. *Biometrics*, **33(4)**, 693-702.
14. Davidson, R. R. and Solomon, D.L. (1973). A Bayesian approach to paired comparison experimentation. *Biometrika*, **60(3)**, 477-487.
15. Gilani, G. M. and Abbas, N. (2008). Bayesian Analysis for the paired comparison model with Order Effects (Using Non-informative Priors). *Pakistan Journal of Statistics and Operational Research*, **4(2)**, 85-93.
16. Glickman, M. E. (2008). Bayesian locally Optimal Design of knockout tournaments. *Journal of Statistical Planning and Inference*, **138**, 2117 – 2127.
17. Jeffreys, H. (1961). *Probability Theory*. Oxford University Press, New York.
18. Johnson, R. A. and Ladalla, J. N. (1979). The large sample behavior of Posterior distributions which sample from Multi-parameter Exponential Family models and allied results. *Sankhya: Series (B)*, **41**, 196-215.
19. Kim, H. J. (2005).A Bayesian approach to paired comparison. Rankings based on a graphical model. *Computational Statistics and Data Analysis*, **48(2)**, 269-290.
20. Laplace, P. S. (1812). *Thkorie Analytique des Probabilitks*. Courcier, Paris.

21. Rao, P. V. and Kupper, L. L. (1967). Ties in paired comparison experiments: A generalization of the Bradley-Terry model. *Journal of the American Statistical Association*, **62(317)**, 194-204.
22. Sareen, S. (2001). Relating Jeffrey's Prior and reference Prior in non-regular problems. In Edward, G.I. (Ed), *Bayesian Methods with Applications to Science: Policy and Official Statistics*. Luxembourg.