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A New Class of Bivariate Distributions with Lindley Conditional Hazard Functions

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Abstract

In this paper, we introduce a new class of Bivariate Distributions such that both the Conditional Hazard Functions are Lindley. Some properties of the new class are studied. Estimation of the parameters of the new class is discussed using the Maximum Likelihood and Pseudo-likelihood methods. Further, a set of real data is used to compare the results obtained by these two methods of Estimation. Finally, a comparison is given between the fitting of a real set of data to the new class and to the Bivariate class of Exponential Conditionals (BEC).

Keywords

Bivariate distribution, Conditional Hazard Function, Functional equation, Lindley distribution, Bivariate exponential conditionals

1. Introduction

The problem of constructing new classes of Bivariate Distributions with fixed marginals or conditionals has received an obvious attention during the last few years (Arnold et al., 1999, 2001; Balakrishnan and Lai, 2009 and Kotz et al., 2000). This problem arises specially in modeling, because once one moves away from Bivariate (Multivariate) Normality, there few Bivariate (Multivariate) Distributions flexible enough to fit correlated data. In this sense, the study of Bivariate Distributions when both conditionals belong to some specified parametric families has received special attention. Balakrishnan et al. (2010) introduced a new technique for constructing Bivariate Continuous Distributions with specified Conditional Hazard Functions.

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Generally, the use of Bivariate Continuous Distributions with specified Hazard Conditionals, in modeling, makes the possibility of studying and specifying several reliability properties including Survival and Hazard Bivariate functions, the Clayton-Oakes measure, conditional densities and Hazard Rate functions of the marginal and conditional Distributions. Added to these the possibility of studying the reliability properties for the series and parallel systems with component lifetimes having these particular dependence models (Navarro and Sarabia, 2011).

In the present paper, a new class of Bivariate Distributions with Lindley Conditional Hazard Functions is given. An absolutely continuous random variable is said to have Lindley Distribution if its density function (Lindley 1958, 1965) is,

$$f(x) = \frac{\theta^2}{\theta+1}(1+x)\exp(-\theta x), x > 0, \theta > 0 \tag{1.1}$$

The corresponding cumulative distribution function is,

$$F(x) = 1 - \left(\frac{1+\theta(1+x)}{\theta+1}\right)\exp(-\theta x), x > 0, \theta > 0 \tag{1.2}$$

and the Hazard Function is,

$$\lambda(x) = \frac{\theta^2(1+x)}{1+\theta(1+x)}, x > 0, \theta > 0 \tag{1.3}$$

In some practical situations, Lindley Mixed models provide better fit compared with other models (Ghitanyet al., (2008); Hossein and Noriszura, (2010); Mahmoudi and Zakerzadeh, (2010) and Selen and Gamze (2014)). For the properties of the Lindley Distribution, see Ghitanyet al., (2007). In the sequel, we will need the following two relations between Survival $\overline{F}(x)$, density $f(x)$ and Hazard $\lambda(x)$ functions:

$$\overline{F}(x) = \exp\left(-\int_{-\infty}^{x}\lambda(u)du\right) \tag{1.4}$$

and

$$f(x) = \lambda(x)\exp\left(-\int_{-\infty}^{x}\lambda(u)du\right) \tag{1.5}$$

2. The Bivariate Lindley Conditional Hazard new class

Suppose, $X > 0, Y > 0$ be two random variables and that the corresponding Conditional Hazard Functions be Lindley Distributed and specified as:

$$\lambda_1(x \mid y) = \frac{\theta_1^2(y)(1+x)}{1+\theta_1(y)(1+x)} \tag{2.1}$$

and

$$\lambda_2(y \mid x) = \frac{\theta_2^2(x)(1+y)}{1+\theta_2(x)(1+y)} \tag{2.2}$$

where

$\theta^1(y)$ and $\theta_2(x)$ are functions of y and x, respectively.

A Bivariate Distribution satisfying equation (2.1) and (2.2) will be called a Bivariate Lindley Conditional Hazard (BLCH) Distribution.

From equation (2.1) and (2.2), using equation (1.5), the conditional density functions are,

$$f_{X|Y}(x \mid y) = \frac{\theta_1^2(y)(1+x)}{1+\theta_1(y)(1+x)} \exp\left(-\int_0^x \frac{\theta_1^2(y)(1+u)}{1+\theta_1(y)(1+u)} du\right)$$

and

$$f_{Y|X}(y \mid x) = \frac{\theta_2^2(x)(1+y)}{1+\theta_2(x)(1+y)} \exp\left(-\int_0^y \frac{\theta_2^2(x)(1+u)}{1+\theta_2(x)(1+u)} du\right)$$

With $f_X(x)$ and $f_Y(y)$ denoting the marginal densities, the identity

$f_{X|Y}(x \mid y) f_Y(y) = f_{Y|X}(y \mid x) f_X(x)$

yields the following relation

$$f_Y(y) \frac{\theta_1^2(y)(1+x)}{1+\theta_1(y)(1+x)} \exp\left(-\int_0^x \frac{\theta_1^2(y)(1+u)}{1+\theta_1(y)(1+u)} du\right)$$

$$= f_X(x) \frac{\theta_2^2(x)(1+y)}{1+\theta_2(x)(1+y)} \exp\left(-\int_0^y \frac{\theta_2^2(x)(1+u)}{1+\theta_2(x)(1+u)} du\right)$$

then, after achieving the existing integrals,

$$f_Y(y) \frac{\theta_1^2(y)}{\theta_1(y)+1} (1+x)\exp(-\theta_1(y)x) = f_X(x) \frac{\theta_2^2(x)}{\theta_2(x)+1} (1+y)\exp(-\theta_2(x)y) \tag{2.3}$$

Denoting

$$\begin{cases} n(y) = \log\left(f_Y(y) \frac{\theta_1^2(y)}{(\theta_1(y)+1)(1+y)}\right) \\ m(x) = \log\left(f_X(x) \frac{\theta_2^2(x)}{(\theta_2(x)+1)(1+x)}\right) \end{cases} \tag{2.4}$$

Then, upon taking logarithms in equation (2.3) and using equation (2.4), we get:

$n(y) - \theta_1(y)x = m(x) - \theta_2(x)y$

which is a functional equation of the form

$$\sum_{k=1}^{n} f_k(x)g_k(y) = 0$$

It's solution is given by Aczel (1966), as:

$$\theta_1(y) = \beta + \gamma y, \quad \theta_2(x) = -\alpha + \gamma x \tag{2.5}$$

Substitute equation (2.5) in (2.3), we get:

$$f_Y(y)\frac{(\beta+\gamma y)^2}{(\beta+\gamma y)+1}(1+x)\exp(-(\beta+\gamma y)x) = f_X(x)\frac{(-\alpha+\gamma x)^2}{(-\alpha+\gamma x)+1}(1+y)\exp(-(-\alpha+\gamma x)y)$$

Equating joint probability distribution function and rearrange x and y terms to get for all x and y

$$f_X(x)\frac{\exp(\beta x)(\alpha-\gamma x)^2}{(1-\alpha+\gamma x)(1+x)} = f_Y(y)\frac{\exp(-\alpha x)(\beta+\gamma y)^2}{(1+\beta+\gamma y)(1+y)}$$

Hence, each side is a constant $N(\alpha,\beta,\gamma)$ (provided that $\beta = -\alpha$) and hence the marginal is,

$$f_X(x) = \frac{\{N(\alpha,\beta,\gamma)\}^{-1}(1-\alpha+\gamma x)(1+x)}{(\alpha-\gamma x)^2}\exp(-\beta x) \qquad x > 0$$

Put the integral of this probability distribution function over its space to get the constant

$$N(\alpha,\beta,\gamma) = \frac{\exp\left(-\frac{\alpha\beta}{\gamma}\right)}{\alpha\beta\gamma^3}\left\{\exp\left(\frac{\alpha\beta}{\gamma}\right)\gamma\big(-\beta\gamma+\alpha(-\beta+\gamma)\big) + \alpha\beta(\alpha(\beta-\gamma)+(-1+\beta-\gamma)\gamma)\mathrm{Ei}\left[\frac{\alpha\beta}{\gamma}\right]\right\}$$

where

$\mathrm{Ei}(z)$is the Exponential Integral function,

$$\mathrm{Ei}(z) = -\int_{-z}^{\infty}\frac{\exp(-t)}{t}dt$$

then

$$f_{X,Y}(x,y) = [N(\alpha,\beta,\gamma)]^{-1}(1+x)(1+y)\exp(\alpha y-(\beta+\gamma y)x) \tag{2.6}$$

$x > 0, y > 0, \alpha < 0, \beta > 0, \gamma \geq 0$

Equation (2.6) describes the complete class of BLCH Distribution that has the three parameters $\alpha < 0$ (intensity parameter for Y), $\beta > 0$ (intensity parameter forX) and $\gamma \geq 0$ (interaction or dependence parameter).

Figure 1 and 2 shows the three dimensional curve of the BLCH Distribution given by equation (2.6) for the special cases for α, β, and γ.

The specific forms of the Conditional Hazard Functions for the equation (2.6) are

$$\lambda_1(x \mid y) = \frac{(\beta+\gamma y)^2(1+x)}{1+(\beta+\gamma y)(1+x)}, \quad x > 0, y > 0, \beta > 0, \gamma \geq 0 \tag{2.7}$$

$$\lambda_2(y \mid x) = \frac{(-\alpha+\gamma x)^2(1+y)}{1+(-\alpha+\gamma x)(1+y)}, \quad x > 0,\ y > 0,\ \alpha < 0,\ \gamma \geq 0 \tag{2.8}$$

The compatibility of equation (2.7) and (2.8) (Balakrishnanet al., 2010) guarantees the existence of the equation (2.6).

3. Main properties of BLCH

First of all, direct calculations using equation (2.6) gives,

$$E(XY) = \frac{[N(\alpha,\beta,\gamma)]^{-1}}{\alpha\beta\gamma^5}\exp\left(-\frac{\alpha\beta}{\gamma}\right)\left\{\exp\left(\frac{\alpha\beta}{\gamma}\right)\gamma\left[\alpha^2\beta(\beta-\gamma) - \beta\gamma^2 + \alpha\gamma\left(\beta^2 + \gamma -\right.\right.\right.$$
β3+γ−αβα2ββ−γ+γ22−2β+γ+αγβ2+2γ−β4+γEiαβγ

(3.1)

For $\beta = \gamma$, equation (3.1) reduces to

$$E(XY) = \frac{[N(\alpha,\gamma,\gamma)]^{-1}}{\alpha\gamma^6}[(\gamma+2\alpha)(\alpha Ei(\alpha)\exp(-\alpha)-1) - 2\alpha Ei(\alpha)\exp(-\alpha)]$$

which for $\alpha = -\frac{\gamma}{2}$ again reduces to

$$E(XY) = -\frac{2\left[N\left(-\left(\frac{\gamma}{2}\right),\gamma,\gamma\right)\right]^{-1}}{\gamma^3}\exp\left(\frac{\gamma}{2}\right)Ei\left(-\frac{\gamma}{2}\right)$$

and hence E(XY) is a decreasing function in γ.

3.1. Conditional densities, marginals and moments**:** The conditional densities of the equation (2.6) are

$$f_{X|Y}(x \mid y) = \frac{(\beta+\gamma y)^2}{(\beta+\gamma y+1)}(1+x)\exp(-(\beta+\gamma y)x)$$
$$x > 0,\ y > 0,\ \alpha < 0,\ \beta > 0,\ \gamma \geq 0 \tag{3.3}$$

and

$$f_{Y|X}(\, y \mid x\,) = \frac{(-\alpha + \gamma x)^2}{(-\alpha + \gamma x + 1)}(1 + y)\exp(-(-\alpha + \gamma x)y)$$

$x > 0,\ y > 0,\ \alpha < 0,\ \beta > 0,\ \gamma \geq 0$ (3.4)

i.e.

$X \mid Y = y \sim \text{Lindley}(\beta + \gamma y)$

$Y \mid X = x \sim \text{Lindley}(-\alpha + \gamma x)$

We notice that for $\beta = -\alpha$ the Conditional Distributions are identical.

Now, using equation (3.3) the conditional moments are

$$E\big(X^k \mid Y = y\big) = \frac{(1 + k + \beta + \gamma y)\Gamma(1 + k)}{(\beta + \gamma y)^k (1 + \beta + \gamma y)} \qquad k \geq 0,$$

which are rational functions in the conditional variable.

For $k = 1$, we have,

$$E(\, X \mid Y = y\,) = \frac{\beta + \gamma y + 2}{(\beta + \gamma y)(1 + \beta + \gamma y)}$$

We notice that this Regression function is nonlinear and decreasing for all nonnegative values of the conditioned variable with curves both similar to that given in Figure 3.

Now, the marginal densities corresponding to equation (2.6) are

$$f_X(x) = \frac{\{N(\alpha, \beta, \gamma)\}^{-1}(1 + x)(1 - \alpha + \gamma x)}{(-\alpha + \gamma x)^2}\exp(-\beta x)$$

$$x > 0, \alpha < 0, \beta > 0, \gamma \geq 0,$$

$$f_Y(y) = \frac{\{N(\alpha, \beta, \gamma)\}^{-1}(1 + y)(1 + \beta + \gamma y)}{(\beta + \gamma y)^2}\exp(\alpha y)$$

$$y > 0, \alpha < 0, \beta > 0, \gamma \geq 0.$$

We notice that for $\beta = -\alpha$ the random variablesX and Y are identically distributed. Using these densities we can calculate directly the moments of X(Y).

For example:

$$E(X) = \frac{\{N(\alpha,\beta,\gamma)\}^{-1}\exp\left(-\frac{\alpha\beta}{\gamma}\right)}{\beta^2\gamma^4}\left\{\exp\left(\frac{\alpha\beta}{\gamma}\right)\gamma[\alpha\beta(-\beta+\gamma) + \gamma(\beta-\beta^2+\gamma+\beta\gamma)] + \beta^2[\alpha^2(\beta-\gamma)-\gamma^2-\alpha\gamma(2-\beta+\gamma)]\mathrm{Ei}\left(\frac{\alpha\beta}{\gamma}\right)\right\}$$

and

$$E(Y) = \frac{\{N(\alpha,\beta,\gamma)\}^{-1}\exp\left(-\frac{\alpha\beta}{\gamma}\right)}{\alpha^2\gamma^4}\left\{\exp\left(\frac{\alpha\beta}{\gamma}\right)\gamma[\alpha^2(\beta-\gamma)+\gamma^2-\alpha\gamma(1-\beta+\gamma)] - \alpha^2\left[\alpha\beta(\beta-\gamma)+\gamma\left(\beta^2+\gamma-\beta(2+\gamma)\right)\right]\mathrm{Ei}\left(\frac{\alpha\beta}{\gamma}\right)\right\}$$

For $\gamma = 0$ we notice that $E(XY) = E(X)E(Y)$.

3.2. Totally negative of order two: According to Holland and Wang (1987a, 1987b) a joint density function $f_{X,Y}(x,y)$ of two continuous random variables X and Y is said to be totally negative of order 2 (TN_2) iff $\gamma_f(x,y) < 0$
where

$$\gamma_f(x_1,x_2) = \frac{\partial^2}{\partial x_1\,\partial x_2}\ln f_{X_1,X_2}(x_1,x_2) \tag{3.5}$$

which is called the Local Dependence function.

Theorem:

The BLCH Distribution given by equation (2.6) is TN_2.

Proof:

From equation (2.6) and (3.5), we get:
$\gamma_f(x,y) = -\gamma$,
Thus, $f_{X,Y}(x,y)$ is TN_2 since $\gamma > 0$.

Shaked (1977) proved that if $f_{X,Y}(x,y)$ is TN_2, then the Conditional Hazard rate of (X|Y=y) is increasing iny. A similar property holds for $(Y|X = x)$. The property that the BLCH equation (2.6) is TN_2 explains the monotonicity of the Hazard Rate function of the conditional distribution of $X|Y = y$ $(Y|X = x)$ as a function of $y(x)$ as it is clear from equation (2.7) and (2.8).

Remarks:

- It follows from the previous theorem that the two *X* and *Y* jointly distributed by equation (2.6) are negatively correlated.
- It is easy to see that the conditional densities in equation (3.3) and (3.4) have the same local dependence functions as the joint density $f_{X,Y}(x,y)$. i.e., $\gamma_{f_{X|Y}}(x,y) = \gamma_{f_{Y|X}}(x,y) = \gamma_f(x,y)$.
- $\gamma_f(x,y) = 0$, iff $\gamma = 0$. Therefore X and Y are independent iff $\gamma = 0$ (Balakrishnan and Lai, 2009).

4. Estimation of the Parameters of BLCH

Assume that (x_i, y_i) $(i = 1, \dots, n)$ is a random sample from the BLCH given by equation (2.6).

4.1. Maximum Likelihood Estimation (MLE) of α, β and γ: The Log-likelihood function for the assumed sample is,

$$l(\alpha,\beta,\gamma) = -n\log\big(N(\alpha,\beta,\gamma)\big) + \sum_{i=1}^{n}\log\big((1+x_i)(1+y_i)\big) + \alpha\sum_{i=1}^{n} y_i - \beta\sum_{i=1}^{n} x_i - \gamma\sum_{i=1}^{n} x_i y_i$$

Differentiating and setting the partial derivatives equal to zero, we get the Likelihood equations as:

$$\frac{\frac{\partial N(\alpha,\beta,\gamma)}{\partial\alpha}}{N(\alpha,\beta,\gamma)} = \frac{1}{n}\sum_{i=1}^{n} y_i \qquad (4.1)$$

$$\frac{\frac{\partial N(\alpha,\beta,\gamma)}{\partial\beta}}{N(\alpha,\beta,\gamma)} = -\frac{1}{n}\sum_{i=1}^{n} x_i \qquad (4.2)$$

$$\frac{\frac{\partial N(\alpha,\beta,\gamma)}{\partial\gamma}}{N(\alpha,\beta,\gamma)} = -\frac{1}{n}\sum_{i=1}^{n} x_i y_i \qquad (4.3)$$

Now, choosing initial values for α and β then searching for a value for γ to make equation (4.3) holds. Then, using the value obtained for γ with the chosen initial value for β we search for a value for α that makes equation (4.1) holds. Finally, we use the obtained searched values for α and γ to search a value for β that makes equation (4.2) holds.

4.2. Maximum Pseudo-likelihood Estimation (MPLE): An alternative estimation technique is MPLE, which was introduced by Besag (1975, 1977) and Arnold and Strauss (1988b). The technique involves Pseudo-likelihood function that does not involve the normalizing constant equation (2.6). To estimate the parameters α, β and γ, according to this method, we have to maximize the function

$$\prod_{i=1}^{n} f_{X|Y}(x_i \mid y_i) f_{Y|X}(y_i \mid x_i)$$

When the sampling distribution is the BLCH equation (2.6), the required conditional densities are given by equation (3.3), (3.4) and therefore we have Log-Pseudo-likelihood function in the form

$$\begin{aligned}\log PL(\alpha,\beta,\gamma) &= \sum_{i=1}^{n}\log(1+x_i) + 2\sum_{i=1}^{n}\log(\beta+\gamma y_i) \\ &\quad -\sum_{i=1}^{n}\log(1+\beta+\gamma y_i) - \sum_{i=1}^{n}(\beta+\gamma y_i)x_i + \sum_{i=1}^{n}\log(1+y_i) \\ &\quad +2\sum_{i=1}^{n}\log(-\alpha+\gamma x_i) - \sum_{i=1}^{n}\log(-\alpha+\gamma x_i+1) - \sum_{i=1}^{n}(-\alpha+\gamma x_i)y_i.\end{aligned}$$

Differentiating with respect to α, β, and γ, respectively, we get the Pseudo-likelihood equations as:

$$\frac{\partial \log PL(\alpha,\beta,\gamma)}{\partial\alpha} = \sum_{i=1}^{n} y_i - 2\sum_{i=1}^{n}\frac{1}{(-\alpha+\gamma x_i)} + \sum_{i=1}^{n}\frac{1}{(-\alpha+\gamma x_i+1)} \tag{4.5}$$

$$\frac{\partial \log PL(\alpha,\beta,\gamma)}{\partial\beta} = -\sum_{i=1}^{n} x_i + 2\sum_{i=1}^{n}\frac{1}{(\beta+\gamma y_i)} - \sum_{i=1}^{n}\frac{1}{(\beta+\gamma y_i+1)} \tag{4.6}$$

$$\begin{aligned}\frac{\partial \log PL(\alpha,\beta,\gamma)}{\partial\gamma} &= 2\sum_{i=1}^{n}\frac{y_i}{(\beta+\gamma y_i)} - \sum_{i=1}^{n}\frac{y_i}{(\beta+\gamma y_i+1)} - 2\sum_{i=1}^{n} x_i y_i \\ &\quad +2\sum_{i=1}^{n}\frac{x_i}{(-\alpha+\gamma x_i)} - \sum_{i=1}^{n}\frac{x_i}{(-\alpha+\gamma x_i+1)}\end{aligned} \tag{4.7}$$

The Pseudo-likelihood estimates of α, β and γ can be obtained by solving

$$\frac{\partial \log PL(\alpha,\beta,\gamma)}{\partial\alpha} = 0, \quad \frac{\partial \log PL(\alpha,\beta,\gamma)}{\partial\alpha} = 0, \quad \frac{\partial \log PL(\alpha,\beta,\gamma)}{\partial\alpha} = 0,$$

in the same manner as it is explained in Section (4.1).

5. Application

The following data has been obtained from Andrews and Herzberg (1985). Using these data we shall estimate the parameters α, β and γ of the equation (2.6). The data includes the observations on patients having bladder tumors when they

entered the trial. These tumors were removed and patients were given a treatment called placebo pills. At subsequent follow-up visits, any tumors found were removed, and treatment was continued. The variables observed are *X*, time (in month) to first recurrence of a tumor, and Y, time (in month) to second recurrence of a tumor in Table 1.
For the given data, we have,

$$\frac{1}{30}\sum_{i=1}^{30} y_i = 418, \quad \frac{1}{30}\sum_{i=1}^{30} x_i = 217, \frac{1}{30}\sum_{i=1}^{30} x_i y_i = 4922.$$

These numerical results are used to obtain each of the MLE (using equation (4.1)-(4.3)), and the MPLE (using equation (4.5)-(4.7)).

Table 2 contains, the true (initial) values of α, β, γ, their MLE, MPLE and their Mean Squared Errors. The BEC Distribution discussed extensively by Arnold and Strauss (1988a) has the following joint density:

$$f_{X,Y}(x,y) = \exp(m_{11} - m_{12}y - m_{21}x + m_{22}xy)$$
$$x > 0,\ y > 0, m_{12} > 0, m_{21} > 0, m_{22} \leq 0,$$

where
m_{11} is the normalizing constant.
For the given data, the MLE of m_{12}, m_{21}, m_{22} are given by:

$$\widehat{m}_{12} = -0.063097, \qquad \widehat{m}_{21} = 0.081095, \qquad \widehat{m}_{22} = 0.000350.$$

The Likelihood Ratio Test (LRT) will be used to test the null hypothesis H_0: Distribution of the data is BEC. Under H_0, the Likelihood Ratio statistic:

$$X_L = 2\left(l(\hat{\alpha}, \hat{\beta}, \hat{\gamma}) - l(\widehat{m}_{12}, \widehat{m}_{21}, \widehat{m}_{22})\right)$$

which has an asymptotic Chi-square Distribution with 1 degree of freedom. H_0 is rejected at a significance level α if $X_L > \chi^2_{1.\alpha}$. In addition, for model selection, we use the AIC and BIC, defined as:

$$AIC = \log\text{likelihood} - 2p$$

$$BIC = \log\text{likelihood} - \frac{p}{2}\log(n)$$

where
p is the number of parameters in the model and *n* is the sample size.

The model with higher AIC (BIC) is the one that better fits the data. Table 3 gives a comparison between the Log-likelihood, AIC and BIC for BEC Distribution and BLCH Distribution for the given data in Table 3.

For the given data, the Likelihood Ratio statistic is,

$X_L = 74.59 > \chi^2_{1.0.05} = 3.84,$

then, we cannot accept the null hypothesis, i. e., the LRT rejects the assumption that the BEC model is suitable for the given data.

6. Conclusion

In this study, a Bivariate Lindley Distribution is introduced by specified Conditional Hazard Functions. In addition, the parameters of BLCH via the method of MLE and MPLE are shown. Based on the results, Table 2 the MLE are close to the MPLE. Therefore, the computationally simpler method of MPLE seems to be an attractive alternative, to the more difficult method of MLE since the conditional distributions and hence the MPLE are not based on the normalizing constant. Furthermore, Table 3 shows that the BLCH Distribution is better to fit the given data compared with the BEC Distribution.

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Table 1: Data on time of first recurrence (X_i), and second recurrence (Y_i) of bladder tumor for patients undergoing placebo pills treatment.

Patient	1	2	3	4	5	6	7	8	9	10
X_i	12	10	3	3	7	3	2	28	2	2
Y_i	16	15	16	9	10	15	26	30	17	6
Patient	11	12	13	14	15	16	17	18	19	20
X_i	12	9	16	3	9	3	2	5	2	3
Y_i	15	17	19	6	11	15	15	14	8	4
Patient	21	22	23	24	25	26	27	28	29	30
X_i	2	3	3	3	2	6	8	44	8	1
Y_i	3	10	9	7	6	20	15	47	14	3

Table 2: Estimation of parameters of the Bivariate equation (2.6)

True value	MLE	MSE	MPLE	MSE
α=-0.05	-0.1216	0.00513	-0.0854	0.00126
β=0.9	0.1665	0.53801	0.1312	0.59100
γ=0.01	0.0002	0.00009	0.0006	0.00008

Table 3: Selection criteria for BEC and BLCH to the given data

Class	Log-likelihood	AIC	BIC
BEC	-164.811	-170.81	-169.91
BLCH	-127.515	-133.52	-132.62

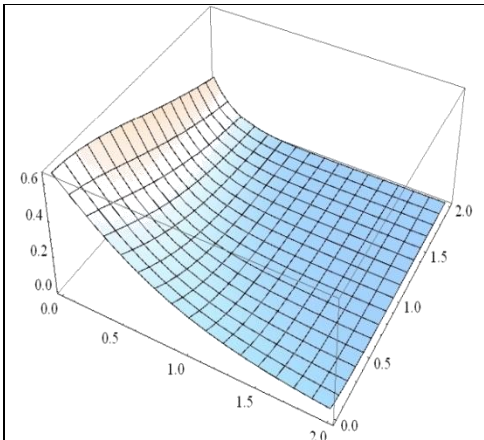

Figure 1: BLCH with $\alpha = -1, \beta, \gamma = 2$

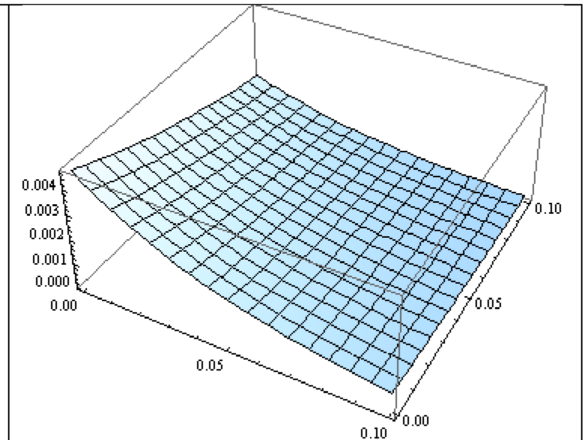

Figure 2: BLCH with $\alpha = -10, \beta = 20, \gamma = 50$

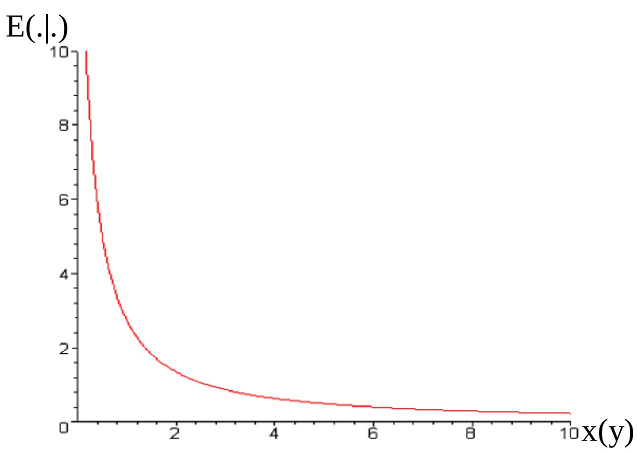

Figure 3: The Regression curve

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