

Comparison of Bayesian and Maximum Likelihood Estimation for Analysis of Burr Type V Distribution under Left Censored Samples

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Abstract

The paper deals with the Bayesian Maximum Likelihood Estimation of the parameters of the Burr Type V Distribution based on left censored samples. The Bayesian and Maximum Likelihood Estimators (MLE) of the parameters have been derived and the Fisher information matrix for the parameters of the said Distribution has been obtained explicitly. The confidence intervals and Credible Intervals for the parameters have also been discussed. A simulation study has been conducted to investigate the performance of the Point and Interval Estimates. The comparison among the performance of Bayesian and ML Estimation have been made.

Keywords

Fisher information matrix, Confidence intervals, Censoring

1. Introduction

Burr family of Distributions consists of a dozen of Distributions which can be used to fit almost any given set of Uni-modal data. Burr (1942) proposed these Distributions. From these twelve Distributions Burr Type X and XII have received the maximum attention during last ten years. The authors considering the Analysis of Burr Type X and XII include: Aludaat et al. (2008), Amjad and Ayman (2006), Dasgupta (2011), Feroze and Aslam (2012a), Makhdoom and Jafari (2011), Mousa and Jaheen (2002), Panahi and Asadi (2011), Shao (2004), Shao et al. (2004), Silva et al. (2008), Soliman (2002), Soliman (2005), Surles and Padgett (2001), Wahed (2006), Wu et al. (2007), Wu and Yu (2005) and Yarmohammadi and Pazira (2010).

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The remaining types of the Burr family of Distributions have not received a considerable interest of the analysts; same is the case with Burr Type V Distributions. The Burr Type V Distribution can be used to model the lifetime data. The probability density function of the Distribution is,

$$f(y) = \theta \lambda e^{-\tan y} \sec^2 y (1 + \lambda e^{-\tan y})^{-\theta-1}, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}, \quad \lambda, \theta > 0 \quad (1.1)$$

The cumulative distribution function of this Distribution is,

$$F(y) = (1 + \lambda e^{-\tan y})^{-\theta} \quad (1.2)$$

where

λ and θ are the shape parameters of the Distribution.

This Distribution is still waiting for the attention of the researchers may be due to its complex probability density function. Many properties of the parameters of the Distribution under different Estimation procedures are still to be revealed. To deal with characteristics of such deprived Distributions is always important for the researchers. The investigation of properties of such Distributions can be beneficial to the professionals looking to use those Distributions as models. The rare consideration of the Burr Type V Distribution in the literature is a motivation for the recent study.

Censoring is useful procedure when the value of a measurement or observation is only partially known. That is, all information regarding a portion of the sample/population is omitted or do not exist. In practice, it occurs when an observed value is outside the range of a measuring instrument or the measure outside a range is not desired. Censoring has many types; however, we will concentrate on the left censored samples for the Estimation of the said parameters. The left censored data is very likely to occur in Survivor Analysis. It can happen where and event of interest has already occurred at the observation time, but it is not known exactly when. For example, the situations such as: the infection with a sexually-transmitted disease such as HIV/AIDS, onset of a pre-symptomatic illness such as cancer and time at which teenagers begin to drink alcohol can lead to left censored data. In case of left censored samples, we can only observe those individuals whose event time is greater than some truncation point. This truncation point may or may not be the same for all individuals. For example, in case of actuarial life studies, those individuals who died in the womb are often ignored. Another example: suppose you wish to study how long patients who have been hospitalized for a heart attack survive taking some treatment at home. In such situations, the starting time is often considered to be the time of the heart

attack. Only those patients who survive their stay in hospital are able to be included in the study. The more illustrations on left censoring can be found in Antweller and Taylor (2008), Asselineau et al. (2007), Feroze and Aslam (2012b), Jerald and Lawless (2003), Sinha (2006) and Thompson et al. (2011). The other authors considering the Analysis of Distributions under MLE and Bayesian framework are: Feroze and Aslam (2012c), Feroze and Aslam (2012d), Feroze and Aslam (2012e), Feroze and Aslam (2014), Sindhu et al. (2013a) and Sindhu et al. (2013b),

We have considered the Bayesian and Maximum Likelihood Estimation (MLE) of the Burr Type V Distribution under left censored samples. As the explicit expressions for the Maximum Likelihood Estimators of the parameters cannot be obtained, a fixed Point Iteration technique has been used to obtain the MLE of shape parameter λ . Once the MLE of λ has been obtained the MLE of the second shape parameter θ becomes possible to be solved explicitly. In addition, the Fisher information matrix has been derived explicitly and the variance covariance matrix has been obtained by inverting the information matrix. The approximate confidence intervals for both the parameters have also been constructed. A comprehensive simulation study has been carried out to assess the behavior and performance of the Estimates under different sample sizes, parametric space and various degrees of censoring rate.

2. Maximum Likelihood Estimation

Based on the left censored sample the Maximum Likelihood function along with Maximum Likelihood Estimators of the parameters of the Burr Type V Distribution are discussed as follows. Let $Y_{(r+1)} \dots Y_{(n)}$ be the last $n-r$ ordered statistics from the Burr Type V Distribution. Then, the Likelihood function for the sample of $n-r$ left censored sample is,

$$L(\theta, \lambda) \propto \left\{ F(y_{(r+1)}) \right\}^r f(y_{(r+1)}) \dots f(y_{(n)}) \quad (2.1)$$

$$L(\theta, \lambda) \propto \left(1 + \lambda e^{-\tan y_{(r+1)}}\right)^{-\theta r} (\theta \lambda)^{n-r} e^{-\sum_{i=r+1}^n \tan y_{(i)}} \left\{ \prod_{i=r+1}^n \sec^2 y_{(i)} \right\} \prod_{i=r+1}^n \left(1 + \lambda e^{-\tan y_{(i)}}\right)^{-\theta-1} \quad (2.2)$$

The logarithmic of the Likelihood function can be written as:

$$l(\theta, \lambda) \propto -\theta r \ln\left(1 + \lambda e^{-\tan y_{(r+1)}}\right) + (n-r) \ln(\theta \lambda) - (\theta+1) \sum_{i=r+1}^n \ln\left(1 + \lambda e^{-\tan y_{(i)}}\right) \quad (2.3)$$

The normal equations for the derivation of the MLE of the λ and θ parameters are

$$\frac{\partial l}{\partial \theta} = -r \ln\left(1 + \lambda e^{-\tan y_{(r+1)}}\right) + \frac{n-r}{\theta} - \sum_{i=r+1}^n \ln\left(1 + \lambda e^{-\tan y_{(i)}}\right) = 0 \quad (2.4)$$

$$\frac{\partial l}{\partial \lambda} = \frac{-\theta r e^{-\tan y_{(r+1)}}}{1 + \lambda e^{-\tan y_{(r+1)}}} + \frac{n-r}{\lambda} - (\theta+1) \sum_{i=r+1}^n \frac{e^{-\tan y_{(i)}}}{1 + \lambda e^{-\tan y_{(i)}}} = 0 \quad (2.5)$$

From equation (2.4), the MLE of θ can be derived as a function of λ that can be denoted as:

$$\hat{\theta}(\lambda) = \frac{n-r}{r \ln\left(1 + \lambda e^{-\tan y_{(r+1)}}\right) + \sum_{i=r+1}^n \ln\left(1 + \lambda e^{-\tan y_{(i)}}\right)} \quad (2.6)$$

It is immediate from equation (2.5) that the MLE of λ cannot be obtained in an explicit form. So, we have to play some mathematical/numerical manipulations to find out the approximate MLE of λ . Firstly, the parameter θ in Log-likelihood equation (2.3) has been replaced by its MLE given in equation (2.6) the resultant Log-likelihood becomes

$$l(\lambda) \propto \frac{-r(n-r) \ln\left(1 + \lambda e^{-\tan y_{(r+1)}}\right)}{r \ln\left(1 + \lambda e^{-\tan y_{(r+1)}}\right) + \sum_{i=r+1}^n \ln\left(1 + \lambda e^{-\tan y_{(i)}}\right)} + (n-r) \ln(\lambda) + (n-r) \ln \left\{ \frac{(n-r)}{r \ln\left(1 + \lambda e^{-\tan y_{(r+1)}}\right) + \sum_{i=r+1}^n \ln\left(1 + \lambda e^{-\tan y_{(i)}}\right)} \right\} \\ - \left\{ \frac{(n-r)}{r \ln\left(1 + \lambda e^{-\tan y_{(r+1)}}\right) + \sum_{i=r+1}^n \ln\left(1 + \lambda e^{-\tan y_{(i)}}\right)} + 1 \right\} \sum_{i=r+1}^n \ln\left(1 + \lambda e^{-\tan y_{(i)}}\right)$$

After some simplifications it can be presented as:

$$l(\lambda) \propto (n-r) \ln(\lambda) - \sum_{i=r+1}^n \ln\left(1 + \lambda e^{-\tan y_{(i)}}\right) - (n-r) \ln \left\{ r \ln\left(1 + \lambda e^{-\tan y_{(r+1)}}\right) + \sum_{i=r+1}^n \ln\left(1 + \lambda e^{-\tan y_{(i)}}\right) \right\} \quad (2.7)$$

For MLE of λ , the normal equation can be given as:

$$\frac{\partial l(\lambda)}{\partial \lambda} = \frac{n-r}{\lambda} - \sum_{i=r+1}^n \frac{e^{-\tan y_{(i)}}}{1 + \lambda e^{-\tan y_{(i)}}} - (n-r) \left\{ \frac{\frac{r e^{-\tan y_{(r+1)}}}{1 + \lambda e^{-\tan y_{(r+1)}}} + \sum_{i=r+1}^n \frac{e^{-\tan y_{(i)}}}{1 + \lambda e^{-\tan y_{(i)}}}}{r \ln(1 + \lambda e^{-\tan y_{(r+1)}}) + \sum_{i=r+1}^n \ln(1 + \lambda e^{-\tan y_{(i)}})} \right\} = 0 \quad (2.8)$$

Again the explicit solution for MLE of λ is not possible. We have used the fixed Point Iteration method to have the approximate solution MLE of λ by considering the following function.

$$\lambda = g(\lambda) = \left\{ \frac{1}{n-r} \sum_{i=r+1}^n \frac{e^{-\tan y_{(i)}}}{1 + \lambda e^{-\tan y_{(i)}}} + \frac{\frac{r e^{-\tan y_{(r+1)}}}{1 + \lambda e^{-\tan y_{(r+1)}}} + \sum_{i=r+1}^n \frac{e^{-\tan y_{(i)}}}{1 + \lambda e^{-\tan y_{(i)}}}}{r \ln(1 + \lambda e^{-\tan y_{(r+1)}}) + \sum_{i=r+1}^n \ln(1 + \lambda e^{-\tan y_{(i)}})} \right\}^{-1} \quad (2.9)$$

The final result of the above function has been considered as an MLE of λ and denoted by $\hat{\lambda}$. Now, the value of $\hat{\lambda}$ facilitated to find out the solution for MLE of θ given in equation (2.6), that can be denoted by $\hat{\theta}(\hat{\lambda})$.

3. Approximate Fisher information matrix

In this section, the elements of the Fisher information matrix for the parameters of the Burr Type V Distribution based on left censored samples have been derived explicitly. The variance covariance matrix for the parameters of the Burr Type V Distribution can be obtained by inverting the Fisher information matrix which has been used construct the confidence intervals for the said parameters. The Fisher information matrix can be defined as:

$$I(\theta, \lambda) = -E \begin{bmatrix} \frac{\partial^2 l}{\partial \theta^2} & \frac{\partial^2 l}{\partial \theta \partial \lambda} \\ \frac{\partial^2 l}{\partial \lambda \partial \theta} & \frac{\partial^2 l}{\partial \lambda^2} \end{bmatrix} \quad (3.1)$$

The equations for the elements of the Fisher information matrix can be written as:

$$\frac{\partial^2 l}{\partial \theta^2} = -\frac{n-r}{\theta^2} \quad (3.2)$$

$$\frac{\partial^2 l}{\partial \lambda^2} = \frac{r\theta e^{-2 \tan y_{(r+1)}}}{\{1 + \lambda e^{-\tan y_{(r+1)}}\}^2} - \frac{n-r}{\lambda^2} + (\theta+1) \sum_{i=r+1}^n \frac{e^{-2 \tan y_{(i)}}}{\{1 + \lambda e^{-\tan y_{(i)}}\}^2} \quad (3.3)$$

$$\frac{\partial^2 l}{\partial \theta \partial \lambda} = \frac{-re^{-\tan y_{(r+1)}}}{1 + \lambda e^{-\tan y_{(r+1)}}} - \sum_{i=r+1}^n \frac{e^{-\tan y_{(i)}}}{1 + \lambda e^{-\tan y_{(i)}}} \quad (3.4)$$

Now, the expected values of the equation (3.3) and (3.4) require the Distribution of the i^{th} order statistics from the Burr Type V Distribution which can be written as:

$$g(y_{(i)}) = C_{n,i} \theta \lambda e^{-\tan y_{(i)}} \sec^2 y_{(i)} \left(1 + \lambda e^{-\tan y_{(i)}}\right)^{-\theta i - 1} \left\{1 - \left(1 + \lambda e^{-\tan y_{(i)}}\right)^{-\theta}\right\}^{n-i}, \quad -\frac{\pi}{2} < y_{(i)} < \frac{\pi}{2}$$

where

$$C_{n,i} = \frac{n!}{(i-1)!(n-i)!}$$

Here, the expectations necessary to derive the elements of the Fisher information matrix are

$$E\left[\frac{e^{-\tan y_{(i)}}}{1 + \lambda e^{-\tan y_{(i)}}}\right] = C_{n,i} \theta \lambda \int_{-\pi/2}^{\pi/2} e^{-\tan y_{(i)}} \sec^2 y_{(i)} \left(1 + \lambda e^{-\tan y_{(i)}}\right)^{-\theta i - 1} \left\{1 - \left(1 + \lambda e^{-\tan y_{(i)}}\right)^{-\theta}\right\}^{n-i} dy_{(i)}$$

After simplifications it becomes

$$E\left[\frac{e^{-\tan y_{(i)}}}{1 + \lambda e^{-\tan y_{(i)}}}\right] = \frac{2\theta C_{n,i}}{\lambda} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} B\left(\frac{5}{2}, \theta(i+j) - \frac{1}{2}\right) \quad (3.5)$$

where

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \text{ is a Beta function.}$$

Similarly

$$E\left[\frac{e^{-2\tan y_{(i)}}}{\left\{1 + \lambda e^{-\tan y_{(i)}}\right\}^2}\right] = \frac{2\theta C_{n,i}}{\lambda^2} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} B(3, \theta(i+j) - 1) \quad (3.6)$$

Hence, the elements of the Fisher information matrix becomes

$$-E\left(\frac{\partial^2 l}{\partial \lambda^2}\right) = \frac{n-r}{\lambda^2} - \frac{2\theta(\theta+1)C_{n,i}}{\lambda^2} \sum_{i=r+1}^n \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} B(3, \theta(i+j) - 1) - \frac{2r\theta^2 C_{n,i}}{\lambda^2} \sum_{j=0}^{n-r-1} (-1)^j \binom{n-r-1}{j} B(3, \theta(r+1+j) - 1)$$

$$\begin{aligned} -E\left(\frac{\partial^2 l}{\partial \theta \partial \lambda}\right) &= \frac{2\theta C_{n,i}}{\lambda} \sum_{i=r+1}^n \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} B\left(\frac{5}{2}, \theta(i+j) - \frac{1}{2}\right) \\ &\quad + \frac{2r\theta C_{n,i}}{\lambda} \sum_{j=0}^{n-r-1} (-1)^j \binom{n-r-1}{j} B\left(\frac{5}{2}, \theta(r+1+j) - \frac{1}{2}\right) \end{aligned}$$

The variance covariance can be obtained by inverting the Fisher information matrix as:

$$I^{-1}(\theta, \lambda) = \begin{bmatrix} V(\hat{\theta}) & Cov(\hat{\lambda}, \hat{\theta}) \\ Cov(\hat{\lambda}, \hat{\theta}) & V(\hat{\lambda}) \end{bmatrix}$$

where

the diagonal elements of the matrix are the variances of the MLEs of θ and λ respectively. The approximate confidence intervals for θ and λ as discussed by Wu and Kus (2009) are

$$\hat{\theta} \pm Z_{\alpha/2} \sqrt{V(\hat{\theta})} \quad \text{and} \quad \hat{\lambda} \pm Z_{\alpha/2} \sqrt{V(\hat{\lambda})}$$

4. Limiting Fisher information matrix

This section discusses the asymptotic efficiencies and limiting information matrix when r/n converges to, say, p which lies in $(0,1)$. According to Gupta et al. (2004), for the left censored observations at the time point T , the limiting Fisher information matrix can be written as:

$$I(\theta, \lambda) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

where

$$b_{ij} = \int_T^{\infty} \left(\frac{\partial}{\partial \psi_i} \ln r(y, \psi) \right) \left(\frac{\partial}{\partial \psi_j} \ln r(y, \psi) \right) f(y; \psi) dy$$

and

$$\psi = (\theta, \lambda), \quad r(y, \psi) = \frac{f(y; \psi)}{F(y; \psi)} \quad \text{the Reversed Hazard function.}$$

Zheng and Gastwirth (2000) have shown that for location and scale family, the Fisher information matrix for Type-I and Type-II (both for left and right censored data) are asymptotically equivalent. They further described that for general case (not for location and scale family) the results for Type-II censored data (both for left and right) of the asymptotic Fisher information matrices are very difficult to obtain. We cannot obtain the explicit expression for the limiting Fisher information matrix for Burr Type V Distribution under left censored samples as it does not belong to the location and scale family. Numerically, we have studied the limiting behavior of the Fisher information matrix by taking $n = 5000$

(assuming it is very large) and compare them with the different small samples and different ‘p’ values. The numerical results have been presented in Section 5.

5. Bayesian Estimation

Numerous practical situations suggest that the behavior of the parameters appearing in the various model characteristics cannot be treated as pre-determined constant throughout the life testing period. Therefore, it would be logical to assume the parameters included in the life time model as random variables. Keeping this fact in mind, we have also carried out a Bayesian study of the parameters. However, it is difficult to specify a particular Prior for a particular Sampling Distribution. In situations, where we do not have sufficient prior information about the parameters of the Distribution, we can use a non-informative Prior for the Analysis of the concerned parameters. So, here we have assumed non-informative Uniform Prior for the Bayesian Analysis of the parameters of the Burr Type V Distribution. As we are dealing with two parameters of the Burr Type V Distribution a Prior should be specified for each parameter based on the assumption that both Priors are independent. The expressions of Uniform Prior for both parameters can be given as:

$$g_1(\theta) \propto 1 \text{ and } g_2(\lambda) \propto 1.$$

Now the joint prior for the parameters can be written as:

$$g(\theta, \lambda) \propto 1; \theta, \lambda > 0 \tag{5.1}$$

The Posterior Distribution by combining equation (2.2) and (5.1) can be derived as:

$$p(\theta, \lambda) = \frac{\left(1 + \lambda e^{-\tan y_{(r+1)}}\right)^{-\theta r} (\theta \lambda)^{n-r} e^{-\sum_{i=r+1}^n \tan y_{(i)}} \left\{ \prod_{i=r+1}^n \sec^2 y_{(i)} \right\} \prod_{i=r+1}^n \left(1 + \lambda e^{-\tan y_{(i)}}\right)^{-\theta-1}}{\int_0^\infty \int_0^\infty \left(1 + \lambda e^{-\tan y_{(r+1)}}\right)^{-\theta r} (\theta \lambda)^{n-r} e^{-\sum_{i=r+1}^n \tan y_{(i)}} \left\{ \prod_{i=r+1}^n \sec^2 y_{(i)} \right\} \prod_{i=r+1}^n \left(1 + \lambda e^{-\tan y_{(i)}}\right)^{-\theta-1} d\theta d\lambda} \tag{5.2}$$

5.1. Loss functions and Bayes Estimators: The decision theory suggests that in order to select the best Estimator a Loss function must be specified and used to Estimate the risk associated with each of the possible Estimates. Since, there is no definite analytical process that allows us to identify the proper Loss function to be used, most of the analysts use the Squared Error Loss Function which is symmetrical and associates equal importance to the losses due to over-estimation and under-estimation of equal magnitude and obtain the Posterior mean as

Bayesian Estimate. But in situations where the loss is asymmetric; we have to employ some asymmetric Loss functions. Therefore, we have included both symmetric and asymmetric Loss functions for Posterior Estimation.

5.2. Squared Error Loss Function (SELF): The Squared Error Loss Function is defined as:

$$L(\theta, \theta_{SELF}) = (\theta - \theta_{SELF})^2.$$

The Bayes Estimator under this Loss function is,

$$\theta_{SELF} = E(\theta).$$

The risk associated with this Loss function can be viewed as variance of the Estimator and is related to Classical Least Square theory.

The Bayes Estimator of parameter θ along with its Posterior risk under Squared Error Loss Function (SELF) based on Uniform Prior can be presented as:

$$\theta_{SELF} = \frac{\int_0^\infty \int_0^\infty \theta (1 + \lambda e^{-\tan y_{(r+1)}})^{-\theta r} (\theta \lambda)^{n-r} e^{-\sum_{i=r+1}^n \tan y_{(i)}} \left\{ \prod_{i=r+1}^n \sec^2 y_{(i)} \right\} \prod_{i=r+1}^n (1 + \lambda e^{-\tan y_{(i)}})^{-\theta-1} d\theta d\lambda}{\int_0^\infty \int_0^\infty (1 + \lambda e^{-\tan y_{(r+1)}})^{-\theta r} (\theta \lambda)^{n-r} e^{-\sum_{i=r+1}^n \tan y_{(i)}} \left\{ \prod_{i=r+1}^n \sec^2 y_{(i)} \right\} \prod_{i=r+1}^n (1 + \lambda e^{-\tan y_{(i)}})^{-\theta-1} d\theta d\lambda}$$

$$\rho(\theta_{SELF}) = \frac{\int_0^\infty \int_0^\infty \theta^2 (1 + \lambda e^{-\tan y_{(r+1)}})^{-\theta r} (\theta \lambda)^{n-r} e^{-\sum_{i=r+1}^n \tan y_{(i)}} \left\{ \prod_{i=r+1}^n \sec^2 y_{(i)} \right\} \prod_{i=r+1}^n (1 + \lambda e^{-\tan y_{(i)}})^{-\theta-1} d\theta d\lambda}{\int_0^\infty \int_0^\infty (1 + \lambda e^{-\tan y_{(r+1)}})^{-\theta r} (\theta \lambda)^{n-r} e^{-\sum_{i=r+1}^n \tan y_{(i)}} \left\{ \prod_{i=r+1}^n \sec^2 y_{(i)} \right\} \prod_{i=r+1}^n (1 + \lambda e^{-\tan y_{(i)}})^{-\theta-1} d\theta d\lambda}$$

$$- \left[\frac{\int_0^\infty \int_0^\infty \theta (1 + \lambda e^{-\tan y_{(r+1)}})^{-\theta r} (\theta \lambda)^{n-r} e^{-\sum_{i=r+1}^n \tan y_{(i)}} \left\{ \prod_{i=r+1}^n \sec^2 y_{(i)} \right\} \prod_{i=r+1}^n (1 + \lambda e^{-\tan y_{(i)}})^{-\theta-1} d\theta d\lambda}{\int_0^\infty \int_0^\infty (1 + \lambda e^{-\tan y_{(r+1)}})^{-\theta r} (\theta \lambda)^{n-r} e^{-\sum_{i=r+1}^n \tan y_{(i)}} \left\{ \prod_{i=r+1}^n \sec^2 y_{(i)} \right\} \prod_{i=r+1}^n (1 + \lambda e^{-\tan y_{(i)}})^{-\theta-1} d\theta d\lambda} \right]$$

Similarly, the expressions for the Bayes Estimator of parameter λ and associated Posterior risk under Squared Error Loss Function (SELF) based on Uniform Prior are given below.

$$\lambda_{SELF} = \frac{\int_0^\infty \int_0^\infty \lambda (1 + \lambda e^{-\tan y_{(r+1)}})^{-\theta r} (\theta \lambda)^{n-r} e^{-\sum_{i=r+1}^n \tan y_{(i)}} \left\{ \prod_{i=r+1}^n \sec^2 y_{(i)} \right\} \prod_{i=r+1}^n (1 + \lambda e^{-\tan y_{(i)}})^{-\theta-1} d\theta d\lambda}{\int_0^\infty \int_0^\infty (1 + \lambda e^{-\tan y_{(r+1)}})^{-\theta r} (\theta \lambda)^{n-r} e^{-\sum_{i=r+1}^n \tan y_{(i)}} \left\{ \prod_{i=r+1}^n \sec^2 y_{(i)} \right\} \prod_{i=r+1}^n (1 + \lambda e^{-\tan y_{(i)}})^{-\theta-1} d\theta d\lambda}$$

$$\rho(\lambda_{SELF}) = \frac{\int_0^\infty \int_0^\infty \lambda^2 (1 + \lambda e^{-\tan y_{(r+1)}})^{-\theta r} (\theta \lambda)^{n-r} e^{-\sum_{i=r+1}^n \tan y_{(i)}} \left\{ \prod_{i=r+1}^n \sec^2 y_{(i)} \right\} \prod_{i=r+1}^n (1 + \lambda e^{-\tan y_{(i)}})^{-\theta-1} d\theta d\lambda}{\int_0^\infty \int_0^\infty (1 + \lambda e^{-\tan y_{(r+1)}})^{-\theta r} (\theta \lambda)^{n-r} e^{-\sum_{i=r+1}^n \tan y_{(i)}} \left\{ \prod_{i=r+1}^n \sec^2 y_{(i)} \right\} \prod_{i=r+1}^n (1 + \lambda e^{-\tan y_{(i)}})^{-\theta-1} d\theta d\lambda} \left\{ \int_0^\infty \int_0^\infty \lambda (1 + \lambda e^{-\tan y_{(r+1)}})^{-\theta r} (\theta \lambda)^{n-r} e^{-\sum_{i=r+1}^n \tan y_{(i)}} \left\{ \prod_{i=r+1}^n \sec^2 y_{(i)} \right\} \prod_{i=r+1}^n (1 + \lambda e^{-\tan y_{(i)}})^{-\theta-1} d\theta d\lambda \right\} \left\{ \int_0^\infty \int_0^\infty (1 + \lambda e^{-\tan y_{(r+1)}})^{-\theta r} (\theta \lambda)^{n-r} e^{-\sum_{i=r+1}^n \tan y_{(i)}} \left\{ \prod_{i=r+1}^n \sec^2 y_{(i)} \right\} \prod_{i=r+1}^n (1 + \lambda e^{-\tan y_{(i)}})^{-\theta-1} d\theta d\lambda \right\}$$

These integrals cannot further be solved analytically. Same is the case with Bayes Estimators and Posterior risks under other Loss functions and Priors. We have used the computer software Mathematica to find out the approximate solutions of these integrals.

6. Results and discussions

This section covers the discussions regarding the results of the simulation study for $n = 30, 50, 70, 100$ and 150 using parametric space $(\theta, \lambda) = \{(0.5, 1.5, 2, 2.5, 3), (0.5, 1.5, 2, 2.5, 3)\}$ under 10% and 20% left censored samples. The purpose of the simulation study is to assess the behavior of the MLEs and confidence intervals for the parameters of the Burr Type V Distribution. As the MLE of parameter λ cannot be obtained in the explicit form, a fixed Point Iteration scheme has been proposed to have the approximate MLE of the parameter λ . The performance of the MLEs and Bayesian Estimates have been evaluated in terms of their Mean Square Errors (MSEs); while, the performance of the confidence/credible intervals have been discussed on the basis of the widths of the intervals along with corresponding coverage probabilities. The inverse transformation method has been used to generate the random samples from the Distribution. The function used for the generation of the random numbers is,

$$Y = \tan^{-1} \left[-\ln \left\{ \lambda^{-1} (U^{-1/\theta} - 1) \right\} \right]$$

where

U is the random variable following the Uniform Distribution.

For the whole parametric space of the θ we have assumed $\lambda = 2$ and for the entire parametric space of λ we assumed $\theta = 2$. The entries in the Tables are the average of the results under 1000 replications. The Average Relative Estimate (ARE) defined as the ratio of MLE/(Bayesian Estimates) to the true parametric value, MSE, Lower Confidence/Credible Limits (LCL), Upper Confidence/Credible Limits (UCL), width of the confidence/credible limits and associated Coverage Probabilities (C.P) calculated by the proportion of the intervals containing the true parametric values to the total (1000) intervals, have been presented in the Tables. The performance of the Bayesian and MLE Estimates has been compared using the simulated samples.

It is immediate from the above Analysis that the shape parameter θ has been over Estimated; while the parameter λ has been under Estimated for all sample sizes and under each censoring rate. The degree of over/under Estimation is inversely proportional to sample size, while it is directly proportional to the true parametric value and the censoring rate. It has also been assessed that the Estimates of parameter λ are comparatively closer to the actual values. However, the magnitudes of the Mean Square Error (MSE) associated with the Estimates of both the parameters tend to decrease by increasing the sample size. Again, greater sample sizes lead to the smaller widths of the confidence intervals and larger Coverage Probabilities. This simply indicates that the Estimators of the parameters are consistent. It is interesting to note that for higher true parametric values, the Coverage Probabilities are relatively bigger due to the larger widths of the concerned confidence intervals. On the other hand, larger levels of the censoring rate result in the slower convergence (towards the true parametric values) of the Estimates with inflated amounts of MSEs, hence providing wider confidence intervals. So, the performance of the Estimators has been negatively affected by the increased censoring rates. It is a natural consequence of the censoring. However, it has been observed that the effects of the left censored observations are not that much severe in case of bigger sample sizes. Further, for fixed sample size and censoring rate, the higher actual values of the parameters impose a negative impact on the performance (in terms of MSEs, convergence rate and widths of confidence intervals) of the Estimates. It leads to the conclusion that the Estimation of extremely large values of the parameters of the Burr Type V Distribution may become difficult and the Fisher information matrix may be the decreasing function of the parameters. But the moderate to huge sample sizes can face off this problem. However, it can be assessed that the performance of the Bayesian Estimates is better than those MLEs as the

corresponding MSEs under Bayesian Estimates are smaller than those under MLE. The widths of the Credible Intervals are smaller than corresponding confidence intervals.

In the Tables 21-24, we have discussed the limiting behavior of the variance covariance matrix obtained by inverting the Fisher information matrix given in equation (3.1). As the analytical results of the Fisher information matrix for $n \rightarrow \infty$ cannot be obtained, we have calculated the entries of the Fisher information/variance covariance matrix by taking $n = 5000$ (extremely large). Different levels of the censoring rate have been employed for the analysis. Each Table contains the variance of the concerned Estimator along with their covariance. The covariance terms have been presented in the parenthesis.

7. Conclusion

It is evident from the Tables that even the small samples sizes with higher censoring rates are closely related to the limiting Figures of the variance covariance matrix. It implies that the approximate variance covariance matrix can be used for the Analysis of the unknown parameters of the Burr Type V Distribution. It can further be assessed that the both MLEs and Bayesian Estimates are consistent. However, the performance of Bayesian Estimates is better than MLEs as the corresponding MSEs under Bayesian Estimates are smaller than those under MLEs. The widths of the Credible Intervals are also smaller than corresponding confidence intervals. It simply indicates that the proposed Point and Interval Estimates can effectively be applied to the real life situations using moderate sample sizes. The findings are useful for researchers dealing with left censored data especially in the field of medical sciences.

Table 1: Average Relative Estimates of MLEs, MSEs, confidence limits and coverage probability of θ when $n = 30$

θ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	1.1987	0.0232	0.3733	0.8255	0.4522	0.9520
1.00	1.2587	0.1256	0.7839	1.7335	0.9496	0.9540
1.50	1.2713	0.3002	1.1876	2.6262	1.4386	0.9580
2.00	1.2738	0.5403	1.5867	3.5086	1.9219	0.9560
2.50	1.2776	0.8596	1.9893	4.3989	2.4096	0.9620
3.00	1.2789	1.2453	2.3895	5.2840	2.8945	0.9610

	20% Censored Samples					
0.50	1.2280	0.0287	0.3684	0.8597	0.4913	0.9430
1.00	1.2894	0.1531	0.7736	1.8053	1.0318	0.9470
1.50	1.3023	0.3647	1.1719	2.7351	1.5631	0.9500
2.00	1.3049	0.6558	1.5657	3.6540	2.0883	0.9460
2.50	1.3089	1.0423	1.9630	4.5813	2.6183	0.9540
3.00	1.3102	1.5095	2.3580	5.5030	3.1450	0.9550

Table 2: Average Relative Estimates of MLEs, MSEs, confidence limits and coverage probability of θ when $n = 50$

θ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	1.1250	0.0109	0.3981	0.7268	0.3287	0.9570
1.00	1.1812	0.0638	0.8361	1.5263	0.6903	0.9590
1.50	1.1930	0.1550	1.2667	2.3124	1.0457	0.9630
2.00	1.1954	0.2798	1.6923	3.0894	1.3971	0.9610
2.50	1.1990	0.4472	2.1217	3.8733	1.7516	0.9670
3.00	1.2002	0.6488	2.5486	4.6526	2.1040	0.9680
	20% Censored Samples					
0.50	1.1525	0.0141	0.3977	0.7548	0.3571	0.9490
1.00	1.2101	0.0807	0.8351	1.5851	0.7500	0.9510
1.50	1.2222	0.1951	1.2651	2.4014	1.1363	0.9520
2.00	1.2246	0.3518	1.6902	3.2083	1.5181	0.9550
2.50	1.2283	0.5615	2.1191	4.0224	1.9033	0.9580
3.00	1.2295	0.8143	2.5455	4.8317	2.2862	0.9600

Table 3: Average Relative Estimates of MLEs, MSEs, confidence limits and coverage probability of θ when $n = 70$

θ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	1.0736	0.0059	0.4042	0.6693	0.2651	0.9470
1.00	1.1272	0.0364	0.8489	1.4056	0.5567	0.9500
1.50	1.1385	0.0895	1.2861	2.1295	0.8434	0.9540
2.00	1.1408	0.1619	1.7182	2.8450	1.1268	0.9600
2.50	1.1442	0.2599	2.1542	3.5669	1.4127	0.9660
3.00	1.1454	0.3776	2.5876	4.2846	1.6970	0.9750
	20% Censored Samples					
0.50	1.0998	0.0079	0.4059	0.6939	0.2881	0.9380
1.00	1.1548	0.0478	0.8523	1.4572	0.6049	0.9410
1.50	1.1663	0.1169	1.2913	2.2077	0.9164	0.9420

θ	ARE	MSE	LCL	UCL	Width	C.P
2.00	1.1687	0.2113	1.7251	2.9495	1.2244	0.9500
2.50	1.1722	0.3386	2.1629	3.6979	1.5351	0.9580
3.00	1.1733	0.4917	2.5981	4.4420	1.8439	0.9670

Table 4: Average Relative Estimates of MLEs, MSEs, confidence limits and coverage probability of θ when $n = 100$

θ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	1.0286	0.0031	0.4081	0.6206	0.2125	0.9570
1.00	1.0801	0.0194	0.8569	1.3032	0.4463	0.9600
1.50	1.0909	0.0483	1.2983	1.9744	0.6761	0.9720
2.00	1.0931	0.0877	1.7345	2.6378	0.9033	0.9690
2.50	1.0963	0.1415	2.1746	3.3071	1.1325	0.9760
3.00	1.0974	0.2059	2.6121	3.9725	1.3604	0.9850
	20% Censored Samples					
0.50	1.0538	0.0042	0.4114	0.6424	0.2309	0.9490
1.00	1.1065	0.0266	0.8640	1.3489	0.4849	0.9520
1.50	1.1175	0.0662	1.3090	2.0436	0.7347	0.9620
2.00	1.1198	0.1201	1.7488	2.7303	0.9815	0.9610
2.50	1.1231	0.1933	2.1925	3.4231	1.2306	0.9650
3.00	1.1243	0.2811	2.6337	4.1119	1.4782	0.9770

Table 5: Average Relative Estimates of MLEs, MSEs, confidence limits and coverage probability of θ when $n = 150$

θ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	1.0119	0.0019	0.4206	0.5913	0.1707	0.9590
1.00	1.0624	0.0123	0.8832	1.2417	0.3584	0.9620
1.50	1.0731	0.0312	1.3381	1.8811	0.5430	0.9650
2.00	1.0752	0.0569	1.7877	2.5132	0.7255	0.9730
2.50	1.0784	0.0923	2.2413	3.1509	0.9096	0.9780
3.00	1.0795	0.1346	2.6922	3.7849	1.0926	0.9870
	20% Censored Samples					
0.50	1.0366	0.0026	0.4256	0.6110	0.1855	0.9480
1.00	1.0884	0.0177	0.8937	1.2831	0.3895	0.9510

θ	ARE	MSE	LCL	UCL	Width	C.P
1.50	1.0993	0.0448	1.3539	1.9440	0.5901	0.9540
2.00	1.1015	0.0816	1.8088	2.5971	0.7883	0.9600
2.50	1.1048	0.1322	2.2678	3.2562	0.9884	0.9670
3.00	1.1059	0.1927	2.7241	3.9113	1.1872	0.9750

Table 6: Average Relative Estimates of MLEs, MSEs, confidence limits and coverage probability of λ when $n = 30$

λ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	0.9527	0.0090	0.2967	0.6561	0.3594	0.9470
1.00	0.9481	0.0360	0.5905	1.3057	0.7153	0.9570
1.50	0.9472	0.0810	0.8848	1.9567	1.0718	0.9550
2.00	0.9491	0.1438	1.1821	2.6141	1.4320	0.9590
2.50	0.9519	0.2242	1.4821	3.2774	1.7953	0.9660
3.00	0.9529	0.3226	1.7803	3.9368	2.1565	0.9750
	20% Censored Samples					
0.50	0.9385	0.0101	0.2815	0.6570	0.3755	0.9370
1.00	0.9339	0.0407	0.5603	1.3075	0.7473	0.9480
1.50	0.9330	0.0917	0.8395	1.9593	1.1198	0.9430
2.00	0.9348	0.1626	1.1216	2.6177	1.4960	0.9470
2.50	0.9376	0.2533	1.4062	3.2819	1.8756	0.9590
3.00	0.9386	0.3643	1.6892	3.9422	2.2530	0.9680

Table 7: Average Relative Estimates of MLEs, MSEs, confidence limits and coverage probability of λ when $n = 50$

λ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	0.9625	0.0055	0.3406	0.6219	0.2812	0.9510
1.00	0.9578	0.0222	0.6780	1.2377	0.5597	0.9610
1.50	0.9569	0.0500	1.0159	1.8547	0.8387	0.9590
2.00	0.9588	0.0885	1.3573	2.4778	1.1205	0.9630
2.50	0.9616	0.1376	1.7017	3.1066	1.4049	0.9700
3.00	0.9626	0.1979	2.0441	3.7316	1.6875	0.9780
	20% Censored Samples					
0.50	0.9481	0.0063	0.3271	0.6209	0.2938	0.9430
1.00	0.9435	0.0255	0.6511	1.2358	0.5848	0.9540

λ	ARE	MSE	LCL	UCL	Width	C.P
1.50	0.9425	0.0574	0.9756	1.8519	0.8763	0.9480
2.00	0.9444	0.1016	1.3034	2.4741	1.1707	0.9550
2.50	0.9472	0.1576	1.6342	3.1019	1.4677	0.9600
3.00	0.9482	0.2265	1.9630	3.7260	1.7630	0.9690

Table 8: Average Relative Estimates of MLEs, MSEs, confidence limits and coverage probability of λ when $n = 70$

λ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	0.9694	0.0040	0.3650	0.6044	0.2394	0.9480
1.00	0.9646	0.0160	0.7264	1.2028	0.4764	0.9510
1.50	0.9637	0.0361	1.0886	1.8025	0.7139	0.9550
2.00	0.9656	0.0639	1.4543	2.4081	0.9538	0.9600
2.50	0.9685	0.0993	1.8234	3.0192	1.1958	0.9670
3.00	0.9695	0.1427	2.1902	3.6266	1.4364	0.9760
	20% Censored Samples					
0.50	0.9548	0.0046	0.3524	0.6024	0.2501	0.9360
1.00	0.9502	0.0186	0.7013	1.1990	0.4977	0.9430
1.50	0.9492	0.0420	1.0509	1.7968	0.7459	0.9460
2.00	0.9511	0.0742	1.4040	2.4005	0.9965	0.9510
2.50	0.9540	0.1148	1.7603	3.0096	1.2493	0.9550
3.00	0.9549	0.1648	2.1145	3.6151	1.5007	0.9660

Table 9: Average Relative Estimates of MLEs, MSEs, confidence limits and coverage probability of λ when $n = 100$

λ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	0.9829	0.0028	0.3899	0.5930	0.2031	0.9530
1.00	0.9781	0.0111	0.7760	1.1802	0.4041	0.9560
1.50	0.9771	0.0250	1.1629	1.7685	0.6056	0.9600
2.00	0.9791	0.0444	1.5536	2.3627	0.8091	0.9660
2.50	0.9820	0.0690	1.9478	2.9622	1.0144	0.9720
3.00	0.9830	0.0992	2.3397	3.5582	1.2185	0.9810
	20% Censored Samples					
0.50	0.9681	0.0032	0.3780	0.5901	0.2121	0.9440
1.00	0.9634	0.0129	0.7523	1.1745	0.4222	0.9470
1.50	0.9625	0.0292	1.1273	1.7600	0.6327	0.9520

λ	ARE	MSE	LCL	UCL	Width	C.P
2.00	0.9644	0.0516	1.5061	2.3514	0.8453	0.9590
2.50	0.9673	0.0798	1.8883	2.9481	1.0598	0.9630
3.00	0.9682	0.1145	2.2682	3.5412	1.2730	0.9740

Table 10: Average Relative Estimates of MLEs, MSEs, credible limits and coverage probability of λ when $n = 150$

λ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	0.9912	0.0018	0.4120	0.5792	0.1672	0.9610
1.00	0.9864	0.0074	0.8200	1.1528	0.3328	0.9640
1.50	0.9854	0.0167	1.2287	1.7274	0.4987	0.9680
2.00	0.9874	0.0295	1.6416	2.3078	0.6662	0.9740
2.50	0.9903	0.0460	2.0581	2.8934	0.8353	0.9810
3.00	0.9913	0.0662	2.4722	3.4756	1.0033	0.9900
	20% Censored Samples					
0.50	0.9763	0.0021	0.4008	0.5755	0.1747	0.9520
1.00	0.9716	0.0087	0.7977	1.1454	0.3477	0.9550
1.50	0.9706	0.0196	1.1954	1.7164	0.5210	0.9590
2.00	0.9725	0.0345	1.5971	2.2931	0.6960	0.9660
2.50	0.9755	0.0533	2.0023	2.8750	0.8727	0.9730
3.00	0.9764	0.0765	2.4052	3.4534	1.0482	0.9860

Table 11: Average Relative Estimates, MSEs, credible limits and coverage probability of θ when $n = 30$

θ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	1.1811	0.0198	0.3827	0.8080	0.4253	0.957
1.00	1.2442	0.1107	0.8065	1.7129	0.9064	0.960
1.50	1.2435	0.2691	1.2132	2.5591	1.3459	0.965
2.00	1.2569	0.4814	1.6112	3.4463	1.8351	0.965
2.50	1.2423	0.7535	2.0406	4.2999	2.-	0.972
3.00	1.2579	1.0818	2.4161	5.2170	2.8009	0.969
	20% Censored Samples					
0.50	1.1971	0.0252	0.3769	0.8500	0.4732	0.950
1.00	1.2639	0.1352	0.7921	1.7692	0.9771	0.953
1.50	1.2875	0.3201	1.1909	2.6627	1.4718	0.956

θ	ARE	MSE	LCL	UCL	Width	C.P
2.00	1.2771	0.5588	1.5864	3.6129	2.0264	0.955
2.50	1.2775	0.8942	2.0184	4.5334	2.5149	0.959
3.00	1.2739	1.2860	2.4095	5.3930	2.9835	0.963

Table 12: Average Relative of Bayes Estimates, MSEs, credible limits and coverage probability of θ when $n = 50$

θ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	1.1085	0.0093	0.4081	0.7114	0.3032	0.962
1.00	1.1676	0.0562	0.8602	1.5082	0.6480	0.965
1.50	1.1669	0.1389	1.2940	2.2533	0.9593	0.970
2.00	1.1795	0.2493	1.7185	3.0345	1.3161	0.970
2.50	1.1659	0.3920	2.1764	3.7862	1.6097	0.977
3.00	1.1805	0.5636	2.5770	4.5936	2.0167	0.976
	20% Censored Samples					
0.50	1.1235	0.0124	0.4068	0.7463	0.3395	0.956
1.00	1.1862	0.0713	0.8551	1.5534	0.6983	0.957
1.50	1.2083	0.1712	1.2856	2.3378	1.0523	0.958
2.00	1.1985	0.2998	1.7125	3.1722	1.4596	0.964
2.50	1.1988	0.4817	2.1789	3.9803	1.8014	0.963
3.00	1.1954	0.6937	2.6011	4.7351	2.1340	0.968

Table 13: Average Relative of Bayes Estimates, MSEs, credible limits and coverage probability of θ when $n = 70$

θ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	1.0578	0.0050	0.4144	0.6551	0.2407	0.952
1.00	1.1142	0.0321	0.8734	1.3889	0.5155	0.956
1.50	1.1136	0.0802	1.3138	2.0751	0.7613	0.961
2.00	1.1256	0.1443	1.7448	2.7945	1.0497	0.969
2.50	1.1126	0.2278	2.2098	3.4867	1.2769	0.976
3.00	1.1266	0.3280	2.6164	4.2303	1.6139	0.983
	20% Censored Samples					
0.50	1.0722	0.0069	0.4152	0.6861	0.2709	0.945
1.00	1.1320	0.0422	0.8727	1.4280	0.5553	0.947

θ	ARE	MSE	LCL	UCL	Width	C.P
1.50	1.1530	0.1026	1.3122	2.1492	0.8371	0.948
2.00	1.1438	0.1801	1.7479	2.9163	1.1684	0.959
2.50	1.1441	0.2905	2.2240	3.6592	1.4352	0.963
3.00	1.1408	0.4189	2.6549	4.3532	1.6983	0.975

Table 14: Average Relative of Bayes Estimates, MSEs, credible limits and coverage probability of θ when $n = 100$

θ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	1.0135	0.0026	0.4184	0.6074	0.1890	0.962
1.00	1.0677	0.0171	0.8816	1.2877	0.4061	0.966
1.50	1.0671	0.0433	1.3263	1.9240	0.5977	0.979
2.00	1.0786	0.0781	1.7613	2.5910	0.8296	0.978
2.50	1.0660	0.1240	2.2307	3.2327	1.0020	0.986
3.00	1.0794	0.1789	2.6412	3.9222	1.2810	0.993
	20% Censored Samples					
0.50	1.0273	0.0037	0.4208	0.6352	0.2143	0.956
1.00	1.0847	0.0235	0.8847	1.3219	0.4372	0.958
1.50	1.1048	0.0581	1.3302	1.9895	0.6593	0.968
2.00	1.0960	0.1023	1.7719	2.6996	0.9276	0.970
2.50	1.0962	0.1658	2.2544	3.3873	1.1329	0.970
3.00	1.0931	0.2395	2.6913	4.0297	1.3385	0.985

Table 15: Average Relative of Bayes Estimates, MSEs, credible limits and coverage probability of θ when $n = 150$

θ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	0.9970	0.0016	0.4312	0.5787	0.1475	0.964
1.00	1.0502	0.0108	0.9086	1.2269	0.3183	0.968
1.50	1.0497	0.0280	1.3670	1.8330	0.4661	0.972
2.00	1.0609	0.0507	1.8153	2.4686	0.6532	0.982
2.50	1.0486	0.0809	2.2991	3.0800	0.7809	0.988
3.00	1.0618	0.1169	2.7222	3.7369	1.0148	0.995
	20% Censored Samples					
0.50	1.0106	0.0023	0.4354	0.6041	0.1688	0.955
1.00	1.0669	0.0156	0.9151	1.2574	0.3423	0.957
1.50	1.0868	0.0393	1.3758	1.8925	0.5167	0.960

θ	ARE	MSE	LCL	UCL	Width	C.P
2.00	1.0780	0.0695	1.8327	2.5679	0.7351	0.969
2.50	1.0783	0.1134	2.3318	3.2221	0.8903	0.972
3.00	1.0752	0.1642	2.7836	3.8331	1.0495	0.983

Table 16: Average Relative of Bayes Estimates, MSEs, credible limits and coverage probability of λ when $n = 30$

λ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	0.9776	0.0077	0.3042	0.6422	0.3380	0.952
1.00	0.9684	0.0317	0.6075	1.2902	0.6827	0.963
1.50	0.9734	0.0726	0.9039	1.9067	1.0028	0.962
2.00	0.9803	0.1281	1.2004	2.5677	1.3673	0.968
2.50	0.9696	0.1965	1.5203	3.2037	1.6833	0.976
3.00	0.9820	0.2802	1.8001	3.8869	2.0868	0.983
	20% Censored Samples					
0.50	0.9625	0.0089	0.2880	0.6496	0.3616	0.944
1.00	0.9465	0.0360	0.5737	1.2813	0.7076	0.954
1.50	0.9492	0.0805	0.8531	1.9074	1.0543	0.949
2.00	0.9658	0.1386	1.1364	2.5882	1.4518	0.956
2.50	0.9533	0.2173	1.4459	3.2476	1.8016	0.964
3.00	0.9630	0.3104	1.7261	3.8634	2.1373	0.976

Table 17: Average Relative of Bayes Estimates, MSEs, credible limits and coverage probability of λ when $n = 50$

λ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	0.9877	0.0047	0.3492	0.6087	0.2595	0.956
1.00	0.9783	0.0196	0.6975	1.2230	0.5255	0.967
1.50	0.9833	0.0448	1.0378	1.8073	0.7695	0.966
2.00	0.9903	0.0789	1.3783	2.4338	1.0555	0.972
2.50	0.9795	0.1206	1.7456	3.0367	1.2911	0.980
3.00	0.9920	0.1719	2.0669	3.6843	1.6175	0.986
	20% Censored Samples					
0.50	0.9723	0.0055	0.3346	0.6139	0.2793	0.950
1.00	0.9562	0.0225	0.6667	1.2111	0.5444	0.961

λ	ARE	MSE	LCL	UCL	Width	C.P
1.50	0.9589	0.0504	0.9914	1.8029	0.8115	0.954
2.00	0.9758	0.0866	1.3206	2.4462	1.1256	0.964
2.50	0.9630	0.1352	1.6803	3.0694	1.3891	0.965
3.00	0.9729	0.1930	2.0059	3.6515	1.6456	0.977

Table 18: Average Relative of Bayes Estimates, MSEs, credible limits and coverage probability of λ when $n = 70$

λ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	0.9947	0.0034	0.3742	0.5916	0.2174	0.953
1.00	0.9853	0.0141	0.7473	1.1885	0.4412	0.957
1.50	0.9903	0.0324	1.1121	1.7565	0.6444	0.962
2.00	0.9973	0.0569	1.4768	2.3653	0.8885	0.969
2.50	0.9865	0.0870	1.8704	2.9513	1.0808	0.977
3.00	0.9991	0.1240	2.2146	3.5806	1.3661	0.984
	20% Censored Samples					
0.50	0.9792	0.0040	0.3605	0.5956	0.2351	0.943
1.00	0.9630	0.0164	0.7181	1.1750	0.4569	0.949
1.50	0.9657	0.0369	1.0679	1.7492	0.6813	0.952
2.00	0.9827	0.0632	1.4226	2.3735	0.9509	0.960
2.50	0.9699	0.0985	1.8100	2.9781	1.1681	0.960
3.00	0.9797	0.1404	2.1607	3.5428	1.3821	0.974

Table 19: Average Relative of Bayes Estimates, MSEs, credible limits and coverage probability of λ when $n = 100$

λ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	1.0086	0.0024	0.3997	0.5804	0.1807	0.958
1.00	0.9990	0.0098	0.7984	1.1662	0.3678	0.962
1.50	1.0041	0.0224	1.1880	1.7233	0.5353	0.967
2.00	1.0113	0.0396	1.5776	2.3207	0.7431	0.975
2.50	1.0003	0.0605	1.9981	2.8956	0.8975	0.982
3.00	1.0130	0.0862	2.3657	3.5131	1.1474	0.989
	20% Censored Samples					
0.50	0.9929	0.0028	0.3867	0.5835	0.1968	0.951
1.00	0.9764	0.0114	0.7703	1.1510	0.3807	0.953

1.50	0.9792	0.0256	1.1455	1.7134	0.5679	0.958
2.00	0.9964	0.0440	1.5260	2.3249	0.7989	0.968
2.50	0.9835	0.0685	1.9416	2.9172	0.9756	0.968
3.00	0.9934	0.0975	2.3178	3.4704	1.1526	0.982

Table 20: Average Relative of Bayes Estimates, MSEs, credible limits and coverage probability of λ when $n = 150$

λ	ARE	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	1.0171	0.0015	0.4224	0.5669	0.1445	0.966
1.00	1.0075	0.0065	0.8436	1.1391	0.2955	0.970
1.50	1.0126	0.0150	1.2552	1.6833	0.4281	0.975
2.00	1.0198	0.0263	1.6670	2.2668	0.5998	0.983
2.50	1.0087	0.0403	2.1112	2.8283	0.7171	0.991
3.00	1.0216	0.0575	2.4997	3.4316	0.9318	0.998
	20% Censored Samples					
0.50	1.0013	0.0018	0.4100	0.5690	0.1590	0.959
1.00	0.9847	0.0077	0.8168	1.1225	0.3057	0.962
1.50	0.9874	0.0172	1.2147	1.6710	0.4562	0.965
2.00	1.0048	0.0294	1.6182	2.2673	0.6491	0.975
2.50	0.9918	0.0457	2.0588	2.8449	0.7861	0.978
3.00	1.0018	0.0652	2.4578	3.3844	0.9266	0.994

Table 21: Elements of variance-covariance matrix including $v(\hat{\theta})$ and $Cov(\hat{\lambda}, \hat{\theta})$ for 10% censored data ($\lambda = 2$)

θ	Sample Size					
	30	50	70	100	150	5000
0.50	0.0133	0.0070	0.0045	0.0029	0.0018	0.0015
	(0.0157)	(0.0164)	(0.0107)	(0.0068)	(0.0044)	(0.0035)
1.00	0.0586	0.0310	0.0201	0.0129	0.0083	0.0066
	(0.0692)	(0.0780)	(0.0507)	(0.0326)	(0.0210)	(0.0168)
1.50	0.1346	0.0711	0.0462	0.0297	0.0191	0.0153
	(0.1590)	(0.1741)	(0.1132)	(0.0727)	(0.0469)	(0.0375)
2.00	0.2403	0.1270	0.0826	0.0531	0.0342	0.0274
	(0.2838)	(0.3488)	(0.2269)	(0.1458)	(0.0940)	(0.0752)
2.50	0.3778	0.1996	0.1298	0.0834	0.0538	0.0430
	(0.4461)	(0.5683)	(0.3697)	(0.2376)	(0.1532)	(0.1226)
3.00	0.5452	0.2880	0.1874	0.1204	0.0776	0.0621
	(0.6436)	(0.8488)	(0.5521)	(0.3548)	(0.2289)	(0.1831)

Table 22: Elements of variance-covariance matrix including $V(\hat{\theta})$ and $Cov(\hat{\lambda}, \hat{\theta})$ for 20% censored data ($\lambda = 2$)

θ	Sample Size					
	30	50	70	100	150	5000
0.50	0.0157	0.0083	0.0054	0.0034	0.0022	0.0017
	(0.0368)	(0.0194)	(0.0126)	(0.0081)	(0.0052)	(0.0042)
1.00	0.0692	0.0366	0.0238	0.0153	0.0098	0.0078
	(0.1743)	(0.0921)	(0.0599)	(0.0385)	(0.0248)	(0.0198)
1.50	0.1590	0.0840	0.0546	0.0351	0.0226	0.0181
	(0.3890)	(0.2055)	(0.1337)	(0.0859)	(0.0554)	(0.0443)
2.00	0.2838	0.1499	0.0975	0.0626	0.0404	0.0323
	(0.4000)	(0.2449)	(0.1774)	(0.1277)	(0.0865)	(0.0692)
2.50	0.4461	0.2357	0.1533	0.0985	0.0635	0.0508
	(1.2698)	(0.6710)	(0.4365)	(0.2805)	(0.1809)	(0.1447)
3.00	0.6436	0.3401	0.2212	0.1421	0.0917	0.0733
	(1.8966)	(1.0022)	(0.6519)	(0.4189)	(0.2702)	(0.2162)

Table 23: Elements of variance-covariance matrix including $V(\hat{\lambda})$ and $Cov(\hat{\lambda}, \hat{\theta})$ for 10% censored data ($\theta = 2$)

λ	Sample Size					
	30	50	70	100	150	5000
0.50	0.0084	0.0051	0.0037	0.0026	0.0018	0.0014
	(0.0197)	(0.0120)	(0.0087)	(0.0062)	(0.0042)	(0.0034)
1.00	0.0332	0.0203	0.0147	0.0106	0.0072	0.0057
	(0.0837)	(0.0513)	(0.0371)	(0.0267)	(0.0181)	(0.0145)
1.50	0.0747	0.0457	0.0331	0.0238	0.0161	0.0129
	(0.1829)	(0.1120)	(0.0811)	(0.0583)	(0.0395)	(0.0316)
2.00	0.1334	0.0817	0.0592	0.0426	0.0288	0.0231
	(0.2838)	(0.3488)	(0.2269)	(0.14584)	(0.0940)	(0.0752)
2.50	0.2097	0.1284	0.0930	0.0669	0.0454	0.0363
	(0.5970)	(0.3656)	(0.2648)	(0.1906)	(0.1292)	(0.1033)
3.00	0.3026	0.1853	0.1342	0.0966	0.0655	0.0524
	(0.8917)	(0.5460)	(0.3956)	(0.2847)	(0.1930)	(0.1544)

Table 24: Elements of variance-covariance matrix including $v(\hat{\lambda})$ and $Cov(\hat{\lambda}, \hat{\theta})$ for 20% censored data ($\theta = 2$)

λ	Sample Size					
	30	50	70	100	150	5000
0.50	0.0091	0.0056	0.0040	0.0029	0.0019	0.0015
	(0.0215)	(0.0131)	(0.0095)	(0.0068)	(0.0046)	(0.0037)
1.00	0.0363	0.0222	0.0161	0.0116	0.0078	0.0062
	(0.0914)	(0.0559)	(0.0405)	(0.0291)	(0.0197)	(0.0158)
1.50	0.0816	0.0499	0.0362	0.0260	0.0176	0.0141
	(0.1996)	(0.1222)	(0.0885)	(0.0637)	(0.0432)	(0.0345)
2.00	0.1456	0.0891	0.0646	0.0465	0.0315	0.0252
	(0.4000)	(0.2449)	(0.1774)	(0.1277)	(0.0865)	(0.0692)
2.50	0.2289	0.1401	0.1015	0.0730	0.0495	0.0396
	(0.6517)	(0.3990)	(0.2891)	(0.2080)	(0.1410)	(0.1128)
3.00	0.3303	0.2022	0.1465	0.1054	0.0715	0.0572
	(0.9733)	(0.5960)	(0.4318)	(0.3107)	(0.2106)	(0.1685)

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