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Bayesian Estimation and Prediction of Generalized Pareto Distribution Based on Type II Censored Samples

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Abstract

The study aims to estimate the parameter of the Generalized Pareto Distribution under Type II censored samples. The Bayes Estimators have been derived under a class of informative and non-informative Priors using different symmetric and asymmetric Loss functions. The Credible Intervals, Highest Posterior Density (HPD) intervals, Posterior Predictive Distributions and Posterior Predictive Intervals have been constructed under each Prior. The Bayesian hypothesis testing scheme has also been employed. The performance of the Point and Interval Estimators of the parameter has been evaluated under a simulation study. The findings of the study suggest that in order to have a Point Estimate of the parameter of the Generalized Pareto Distribution under a Bayesian framework, the use of Inverse Gamma Prior along with Entropy Loss Function can be preferred. The Bayesian Interval Estimates and the Posterior Predictive Intervals are also more precise under the assumption of Inverse Gamma Prior.

Keywords

Bayes estimators, Credible intervals, Posterior Predictive intervals, Censoring

1. Introduction

The Generalized Pareto Distribution (GPD), introduced by Pickands (1975), has been extensively used for reliability modeling and life testing.

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It has been used by many authors to model exceedances in several fields such as hydrology, finance, insurance and environmental science. In general, GPD can be applied to any situation in which the Exponential Distribution might be employed but in which some robustness is required against heavier tailed or lighter tailed alternatives. For more details see: Al-Zahrani (2012), Jockovic (2012) and the references cited there-in.

Nadarajah and Kotz (2005) derived the exact Distributions of $R = X + Y$, $P = XY$ and $W = X/(X + Y)$ and the corresponding moment properties when X and Y follow Muliere and Scarsini's Bivariate Pareto Distribution. Shawky and Abu-Zinadah (2008) derived the exact form of the probability density function and moments of single, double, triple and quadruple of lower record values from Exponentiated Pareto Distribution (EPD). The several recurrence relations between single, double, triple and quadruple moments of lower record values from EPD have also been established. Pandey and Rao (2009) discussed the Bayes Estimators of the shape parameter of the Generalized Pareto Distribution by taking Quasi, Inverted Gamma and Uniform Prior Distributions under the LINEX, precautionary and entropy Loss functions. Afify (2010) obtained Bayes and Classical Estimators for two parameters Exponentiated Pareto Distribution when sample is available from complete, Type I and Type II censoring scheme. Bayes Estimators have been developed under Squared Error Loss Function (SELF) as well as under LINEX Loss function using non-informative type of Priors for the parameters. Al-Zahrani (2012) dealt with the Estimation problem for the Generalized Pareto Distribution based on progressive Type-II censoring with random removals. Jockovic (2012) gave the review of the classical techniques for estimating GPD quantiles, and applied these methods in finance to estimate the Value at Risk (VaR) parameter, and discussed certain difficulties related to this subject. Lee (2012) focused on modeling and estimating tail parameters of Loss Distributions from Taiwanese commercial fire loss severity. Using extreme value theory, the Generalized Pareto Distribution (GPD) was employed and compared with standard parametric modeling based on Lognormal, Exponential, Gamma and Weibull Distributions.

Aban et al. (2006) derived Estimators for the Truncated Pareto Distribution, investigated their properties, and illustrated a way to check for fit. These methods have been illustrated with applications from finance, hydrology, and atmospheric science. Lee and Lim (2009) considered the characterizations of the Lomax, Exponential and Pareto Distributions by conditional expectations of record values.

Odat (2010) addressed the problem of estimation of $p(x > y)$ when x and y are two independent variables following Pareto Distribution. The Maximum Likelihood and its asymptotical Distribution were also obtained. Shanubhogue and Jain (2012) dealt with the problem of Uniformly Minimum Variance Unbiased Estimation of the parameter of Pareto Distribution of first kind based on Progressive Type II censored data with Binomial removals.

From the above discussion, it can be assessed that many authors have dealt with Generalized Pareto Distribution but none of them has considered the Interval Estimation of the parameter of the Distribution especially in Bayesian field. Further the Bayesian predictions for the future values from the Distribution haven't been seen. Pandey and Rao (2009) have considered the Bayesian Point Estimation of the Distribution but their article is silent towards the issue of Interval Estimation and predictions from the Distribution. We have addressed the problem of Point and Interval Estimation of the parameter of Generalized Pareto Distribution under a Bayesian framework. Moreover, the Posterior Predictive Intervals have been constructed and evaluated. The Bayesian hypothesis testing has also been considered. The paper provides precise arguments to explain the anomalous behavior of the Bayesian Estimation and prediction when sampling is done from the GPD. The other authors considering the Analysis of Distributions under MLE and Bayesian framework are: Feroze and Aslam (2012a), Feroze and Aslam (2012b), Feroze and Aslam (2012c), Feroze and Aslam (2012d), Feroze and Aslam (2012e), Feroze and Aslam (2014), Sindhu et al. (2013a) and Sindhu et al. (2013b),

The rest of the paper is organized as follow: In Section 2, the Model and Likelihood function for the Generalized Pareto Distribution has been presented. Section 3 contains the definition of Prior Distribution and derivation of Posterior Distributions. The derivation of Bayes Estimators and Posterior risks has been given in the Section 4. The construction of Credible Intervals and the highest Posterior density intervals has been discussed in the Section 5. Section 6 represents the Bayesian hypothesis testing scheme. The Posterior Predictive Distributions and Predictive Intervals have been constructed in the Section 7. Section 8 shows the simulation study for all the Estimators. Section 9 deals with real life application of the results. Finally, the conclusions regarding the study have been presented in the Section 10.

2. Model and Likelihood function

The probability density function of the Generalized Pareto Distribution as used by Pandey and Rao (2009) is given as:

$$f(x) = \frac{1}{\sigma\theta}\left(1-\frac{x}{\sigma}\right)^{\frac{1}{\theta}-1} \qquad , \; 0<x<\sigma\,, \;\; \theta>0$$

For $\sigma = 1$**,** the probability density function of the Generalized Pareto Distribution is,

$$f(x) = \frac{1}{\theta}(1-x)^{\frac{1}{\theta}-1} \qquad , \; 0<x<1, \;\; \theta>0 \tag{2.1}$$

And the cumulative distribution function of the Distribution is,

$$F(x) = 1-(1-x)^{\frac{1}{\theta}} \tag{2.2}$$

The problem of censoring can be considered in two ways. Firstly, the sample can be incomplete, that is, all information regarding a portion of the sample are omitted or do not exist. Secondly, the situation may arise when complete information regarding all the units in the sample cannot be obtained. These two kinds of censored samples will be denoted as Type I and Type II respectively. We have used Type II censoring for the Bayesian Estimation of the parameter of the Generalized Pareto Distribution. The Likelihood function for the Type II censored sample can be derived as:

Suppose *n* items are put on a life-testing experiment and only first *m* failure times have been observed, that is, $x_1 < x_2 ... < x_m$ and remaining *n* – *m* items are still working. Under the assumptions that the lifetimes of the items are independently and identically distributed (i.i.d) Generalized Pareto random variable, the Likelihood function of the observed data without the multiplicative constant can be written as:

$$L(\theta|\underline{x}) \propto \left[\prod_{i=1}^{m} f(x_i)\right]\left[1-F(x_m)\right]^{n-m} \tag{2.3}$$

The Likelihood function for the Generalized Pareto Distribution is,

$$L(\theta|\underline{x}) \propto \left[\prod_{i=1}^{m} \frac{1}{\theta}(1-x)^{\frac{1}{\theta}-1}\right]\left[1-\left\{1-(1-x)^{\frac{1}{\theta}}\right\}\right]^{n-m}$$

$$L(\theta|\underline{\text{x}}) \propto \left(\frac{1}{\theta}\right)^m \left[\prod_{i=1}^{m}(1-x)^{\frac{1}{\theta}}\right]\left[(1-x)^{\frac{1}{\theta}}\right]^{n-m}$$

$$L(\theta|\underline{\text{x}}) \propto \left(\frac{1}{\theta}\right)^m e^{-\frac{1}{\theta}\left\{\sum_{i=1}^{m}\ln(1-x_i)^{-1}\right\}} e^{-\frac{1}{\theta}\left\{(n-m)\ln(1-x_m)^{-1}\right\}} \text{, since } x^m = e^{-m\ln x^{-1}}$$

$$L(\theta|\underline{\text{x}}) \propto \left(\frac{1}{\theta}\right)^m e^{-\frac{1}{\theta}\left\{\sum_{i=1}^{m}\ln(1-x_i)^{-1}+(n-m)\ln(1-x_m)^{-1}\right\}} \tag{2.4}$$

3. Prior and Posterior Distributions

A probability distribution that expresses uncertainty about parameter θ or underlying variable before the data are taken into account is called the Prior Distribution of the parameter θ. The parameters of a Prior density are called Hyper-parameters, to discriminate them from the parameters of the model. The Prior Distribution is an important part of Bayesian inference and one of the main differences between classical and Bayesian inference. To choose the Prior Distribution/information is a vital issue in Bayesian inference, together with the sensitivity or robustness of the inferences to the choice of Prior, and the Likelihood of conflict between Prior and data. Detailed information regarding the derivation of Posterior Distribution can be seen from: Feroze and Aslam (2012a), Feroze and Aslam (2012b) and Pandey and Rao (2009).

The Priors can be categorized into uninformative and informative Priors. An uninformative Prior is a function which is used in place of a subjective Prior Distribution when little or no Prior information is available. These are also called diffuse, minimal, non-informative, objective, reference, uniform, or vague Priors. The Uniform Prior, introduced by Laplace (1812) and Jeffrey's Prior, proposed by Jeffrey's (1961), are the most commonly used uninformative Priors. So among uninformative Priors, the Uniform and Jeffrey's Priors have been used for the Posterior Analysis.

The Uniform Prior is assumed to be

$$p(\theta) \propto 1,\ \theta > 0 \tag{3.1}$$

The Jeffrey's Prior is defined as:

$p(\theta) \propto \sqrt{|I(\theta)|}$, where $|I(\theta)|$ is Fisher information matrix.

Here $$I(\theta)=-E\left[\frac{\partial^2 \ln f(x)}{\partial\theta^2}\right]=\frac{1}{\theta^2}$$

Therefore $$p(\theta)\propto\sqrt{|I(\theta)|}=\frac{1}{\theta} \tag{3.2}$$

This is Jeffrey's Prior for the Generalized Pareto Distribution.

If Prior information exists about parameter θ then, it should be utilized in the Prior Distribution of θ. For example, if the present model form is similar to a Prior model form, and the present model is proposed to be a rationalized version based on more existing data, then the Posterior Distribution of θ from the Prior model may be utilized as the Prior Distribution of θ for the current model. We have assumed Inverse Gamma, Inverse Chi-square and Inverse Exponential Priors as informative Priors for the Bayesian Analysis of the parameter of the Generalized Pareto Distribution.

The Inverse Gamma Prior is assumed to be

$$p(\theta)\propto\beta^{-(a+1)}e^{-\frac{b}{\theta}} \qquad ,\theta>0,\ a,b>0 \tag{3.3}$$

where

a and b are Hyper-parameters.

The Inverse Chi-square Prior is assumed to be

$$p(\theta)\propto\theta^{-\frac{v}{2}-1}e^{-\frac{1}{2\theta}} \qquad ,\ \theta>0,\ v>0 \tag{3.4}$$

where

v is the Hyper-parameter.

The Inverse Exponential Prior is assumed to be

$$p(\theta)\propto\frac{1}{\theta^2}\exp\left(-\frac{1}{h\theta}\right) \quad ,\theta>0,\ h>0 \tag{3.5}$$

where

h is a Hyper-parameter.

The Posterior Distribution summarizes the current state of knowledge about all the uncertain qualities in a Bayesian Analysis. Analytically, the Posterior

Distribution is the product of the Prior density and the Likelihood. It can be defined as:

$$p(\theta|\underline{x}) \propto p(\theta)L(\theta|\underline{x}) \qquad \text{or} \qquad p(\theta|\underline{x}) = \frac{p(\theta)L(\theta|\underline{x})}{\int_0^\infty p(\theta)L(\theta|\underline{x})\,d\theta} \tag{3.6}$$

where

$p(\theta|\underline{x})$, $p(\theta)$ and $L(\theta|\underline{x})$ are Posterior Distribution, Prior Distribution and Likelihood function, respectively.

The Posterior Distribution under the assumption of Uniform has been derived as: Using equation (3.6)

$$p(\theta|\underline{x}) = \frac{p(\theta)L(\theta|\underline{x})}{\int_0^\infty p(\theta)L(\theta|\underline{x})\,d\theta}$$

Here

$$\int_0^\infty p(\theta)L(\theta|\underline{x})\,d\theta = \int_0^\infty \left(\frac{1}{\theta}\right)^m e^{-\frac{1}{\theta}\left\{\sum_{i=1}^m \ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}} d\theta$$

$$= \frac{\Gamma(m-1)}{\left\{\sum_{i=1}^m \ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}^{m-1}}$$

Therefore, the Posterior Distribution under Uniform Prior becomes

$$p(\theta|\underline{x}) = \frac{\left\{\sum_{i=1}^m \ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}^{m-1}}{\Gamma(m-1)} \left(\frac{1}{\theta}\right)^m e^{-\frac{1}{\theta}\left\{\sum_{i=1}^m \ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}}, \quad \theta > 0 \tag{3.7}$$

The Posterior Distributions under the assumption of Jeffrey's, Inverse Gamma, Inverse Chi-square and Inverted Exponential Priors are derived respectively as:

$$p(\theta|\underline{x}) = \frac{\left\{\sum_{i=1}^m \ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}^{m}}{\Gamma(m)} \left(\frac{1}{\theta}\right)^{m+1} e^{-\frac{1}{\theta}\left\{\sum_{i=1}^m \ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}}, \quad \theta > 0 \tag{3.8}$$

$$p\left(\theta|\underline{\mathrm{x}}\right)=\frac{\left\{\sum_{i=1}^{m}\ln\left(1-x_i\right)^{-1}+\left(n-m\right)\ln\left(1-x_m\right)^{-1}+b\right\}^{m+a}}{\Gamma\left(m+a\right)}\left(\frac{1}{\theta}\right)^{m+a+1}e^{-\frac{1}{\theta}\left\{\sum_{i=1}^{m}\ln\left(1-x_i\right)^{-1}+\left(n-m\right)\ln\left(1-x_m\right)^{-1}+b\right\}},\theta>0 \quad (3.9)$$

$$p\left(\theta|\underline{\mathrm{x}}\right)=\frac{\left\{\sum_{i=1}^{m}\ln\left(1-x_i\right)^{-1}+\left(n-m\right)\ln\left(1-x_m\right)^{-1}+\frac{1}{2}\right\}^{m+\frac{v}{2}}}{\Gamma\left(m+\frac{v}{2}\right)}\left(\frac{1}{\theta}\right)^{m+\frac{v}{2}+1}e^{-\frac{1}{\theta}\left\{\sum_{i=1}^{m}\ln\left(1-x_i\right)^{-1}+\left(n-m\right)\ln\left(1-x_m\right)^{-1}+\frac{1}{2}\right\}},\quad\theta>0 \quad (3.10)$$

$$p\left(\theta|\underline{\mathrm{x}}\right)=\frac{\left\{\sum_{i=1}^{m}\ln\left(1-x_i\right)^{-1}+\left(n-m\right)\ln\left(1-x_m\right)^{-1}+\frac{1}{h}\right\}^{m+1}}{\Gamma\left(m+1\right)}\left(\frac{1}{\theta}\right)^{m+2}e^{-\frac{1}{\theta}\left\{\sum_{i=1}^{m}\ln\left(1-x_i\right)^{-1}+\left(n-m\right)\ln\left(1-x_m\right)^{-1}+\frac{1}{h}\right\}},\quad\theta>0 \quad (3.11)$$

4. Bayes Estimators and associated risks

The decision theory states that in order to select the best Estimator a Loss function must be specified and used to estimate the risk associated with each of the possible Estimates. Since, there is no definite analytical process that allows us to identify the proper Loss function to be used, most of the analysts use the Squared Error Loss Function which is symmetrical and associates equal importance to the losses due to over-estimation and under-estimation of equal magnitude and obtain the Posterior mean as Bayesian Estimate. It is mostly used in situations where the loss is symmetric. The progress of the Loss functions can be compared in terms of amount of risk associated with each Estimator. We have used Squared Error Loss Function (SELF), proposed by Legendre (1805) and Gauss (1810); Quadratic Loss Function (QLF), Precautionary Loss Function (PLF), introduced by Norstrom (1996); Entropy Loss Function (ELF), suggested by Calabria and Pulcini (1996); Squared Logarithmic Loss Function (SLLF) due to Brown (1968); and Weighted Loss Function (WLF), defined by DeGroot (1970), for Estimation of the parameter of the Generalized Pareto Distribution. The Bayes Estimators and corresponding risks have been discriminated by attaching the abbreviations of the concerned Loss functions as subscripts to the parameter θ. The results have been presented under the assumption of Uniform Prior. The formulae to derive the

Bayes Estimators and Posterior risks under these Loss functions have been presented in Table 1.

The formula for the derivation of risk under SELF is,

$$\theta_{SELF} = \mathrm{E}(\theta)$$

Here

$$\mathrm{E}(\theta) = \frac{\left\{\sum_{i=1}^{m}\ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}^{m-1}}{\Gamma(m-1)}\int_0^{\infty}\left(\frac{1}{\theta}\right)^{m-1} e^{-\frac{1}{\theta}\left\{\sum_{i=1}^{m}\ln(1-x_i)^{-1}+(n-m)\ln(1-x_m)^{-1}\right\}}$$

$$= \frac{\left\{\sum_{i=1}^{m}\ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}^{m-1}\Gamma(m-2)}{\Gamma(m-1)\left\{\sum_{i=1}^{m}\ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}^{m-2}}$$

Therefore

$$\theta_{SELF} = \frac{\left\{\sum_{i=1}^{m}\ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}}{m-2} \qquad (4.1)$$

The Bayes Estimators under Uniform Prior based on QLF, PLF, ELF, SLLF and WLF are

$$\theta_{QLF} = \frac{\left\{\sum_{i=1}^{m}\ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}}{m} \qquad (4.2)$$

$$\theta_{PLF} = \frac{\left\{\sum_{i=1}^{m}\ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}}{\sqrt{(m-2)(m-3)}} \qquad (4.3)$$

$$\theta_{ELF} = \frac{\left\{\sum_{i=1}^{m}\ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}}{m-1} \qquad (4.4)$$

$$\theta_{SLLF} = \exp\left[\sum_{g=1}^{m-1}\frac{1}{g} - \gamma - \ln\left\{\sum_{i=1}^{m}\ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}\right] \qquad (4.5)$$

$$\theta_{WLF} = \frac{\left\{\sum_{i=1}^{m} \ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}}{m-1} \tag{4.6}$$

The formula for the derivation of risk under SELF is,

$$\rho(\theta_{SELF}) = \mathrm{E}(\theta^2) - \{\mathrm{E}(\theta)\}^2$$

Here

$$\mathrm{E}(\theta) = \frac{\left\{\sum_{i=1}^{m} \ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}}{m-2}$$

and

$$\mathrm{E}(\theta^2) = \frac{\left\{\sum_{i=1}^{m} \ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}^{m-1} \Gamma(m-3)}{\Gamma(m-1)\left\{\sum_{i=1}^{m} \ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}^{m-3}}$$

$$= \frac{\left\{\sum_{i=1}^{m} \ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}^{2}}{(m-2)(m-3)}$$

Therefore

$$\rho(\theta_{SELF}) = \frac{\left\{\sum_{i=1}^{m} \ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}^{2}}{(m-2)(m-3)} - \frac{\left\{\sum_{i=1}^{m} \ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}^{2}}{(m-2)^2}$$

Hence, the Posterior risk under SELF using Uniform Prior becomes

$$\rho(\theta_{SELF}) = \frac{\left\{\sum_{i=1}^{m} \ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}^{2}}{(m-2)^2(m-3)} \tag{4.7}$$

The expressions for Posterior risks associated with the Bayes Estimators using Uniform Prior on the basis of QLF, PLF, ELF, SLLF and WLF are

$$\rho(\theta_{QLF}) = \frac{1}{m} \tag{4.8}$$

$$\rho\left(\theta_{PLF}\right)=2\left\{\sum_{i=1}^{m}\ln\left(1-x_i\right)^{-1}+\left(n-m\right)\ln\left(1-x_m\right)^{-1}\right\}\left\{\frac{1}{\sqrt{\left(m-2\right)\left(m-3\right)}}-\frac{1}{m-2}\right\} \qquad (4.9)$$

$$\rho\left(\theta_{ELF}\right)=\sum_{g=1}^{m-1}\frac{1}{g}-\gamma-\ln\left(m-1\right) \qquad (4.10)$$

where

$\gamma=0.57721$ is an Euler constant.

$$\rho\left(\theta_{SLLF}\right)=\mathrm{E}\left(\ln\theta^2\right)-\left\{\mathrm{E}\left(\ln\theta\right)\right\}^2 \qquad (4.11)$$

where

$$\mathrm{E}\left(\ln\theta\right)=\sum_{g=1}^{m-1}\frac{1}{g}-\gamma-\ln\left\{\sum_{i=1}^{m}\ln\left(1-x_i\right)^{-1}+\left(n-m\right)\ln\left(1-x_m\right)^{-1}\right\}$$

and

$\mathrm{E}\left(\ln\theta^2\right)$ has been evaluated numerically.

$$\rho\left(\theta_{WLF}\right)=\frac{\left\{\sum_{i=1}^{m}\ln\left(1-x_i\right)^{-1}+\left(n-m\right)\ln\left(1-x_m\right)^{-1}\right\}}{\left(m-1\right)\left(m-2\right)} \qquad (4.12)$$

The expressions for Bayes Estimators and corresponding Posterior risks under remaining Priors can be derived in a similar manner.

5. Credible and Highest Posterior Density (HPD) Intervals

The classical theory of confidence intervals for parameter estimates is not insightful; saying that 95% confidence interval means that if the repeated confidence intervals are constructed for different samples then 95% of them will contain the true value of the parameter. The particular confidence interval from any one sample may or may not contain the true parameter value. While, 95% Bayesian Credible Intervals contains the true parameter value with approximately 95% confidence. The Credible Intervals is defined as:

Let $\pi\left(\theta|\underline{\mathrm{x}}\right)$ *be the Posterior Distribution then a* $100\left(1-\alpha\right)\%$ *Credible Intervals in any set C* is *such that* $P_{\pi\left(\theta|\underline{\mathrm{x}}\right)}\left(C\right)=1-\alpha$. The $100\left(1-\alpha\right)\%$ Credible Intervals on the basis of Uniform Prior is given as:

$$\frac{2\left\{\sum_{i=1}^{m}\ln\left(1-x_i\right)^{-1}+\left(n-m\right)\ln\left(1-x_m\right)^{-1}\right\}}{\chi^2_{\left(1-\alpha/2\right)\left\{2\left(m-1\right)\right\}}}<\theta<\frac{2\left\{\sum_{i=1}^{m}\ln\left(1-x_i\right)^{-1}+\left(n-m\right)\ln\left(1-x_m\right)^{-1}\right\}}{\chi^2_{\left(\alpha/2\right)\left\{2\left(m-1\right)\right\}}} \qquad (5.1)$$

The interested readers may refer to Feroze and Aslam (2012a) and Feroze and Aslam (2012b). The Credible Intervals based on remaining Priors can be constructed accordingly.

The highest Posterior density intervals for θ can be obtained by solving the following two equations simultaneously.

$$p\left(\theta_1|\underline{x}\right)=p\left(\theta_2|\underline{x}\right),\ \int_{\theta_1}^{\theta_2}p\left(\theta|\underline{x}\right)d\theta=1-\alpha$$

By simplifying the above equations for the Posterior Distribution in equation (3.7) the resultant equations become

$$\left(m\right)\ln\left(\frac{\theta_2}{\theta_1}\right)-\left(\frac{1}{\theta_1}-\frac{1}{\theta_2}\right)\left\{\sum_{i=1}^{m}\ln\left(1-x_i\right)^{-1}+\left(n-m\right)\ln\left(1-x_m\right)^{-1}\right\}=0 \qquad (5.2)$$

$$\Gamma\left(m-1,\frac{\left\{\sum_{i=1}^{m}\ln\left(1-x_i\right)^{-1}+\left(n-m\right)\ln\left(1-x_m\right)^{-1}\right\}}{\theta_2}\right)-\Gamma\left(m-1,\frac{\left\{\sum_{i=1}^{m}\ln\left(1-x_i\right)^{-1}+\left(n-m\right)\ln\left(1-x_m\right)^{-1}\right\}}{\theta_1}\right)-\left(\alpha-1\right)\Gamma\left(m-1\right)=0 \qquad (5.3)$$

Now, the HPD Intervals can be obtained by solving the above equations simultaneously. Similarly, the HPD Intervals under Jeffrey's, Inverse Gamma, Inverse Chi-square and Inverse Exponential Priors can be constructed.

6. Bayesian Hypothesis Testing and Bayes Factor

This section covers the Bayesian hypothesis testing and corresponding Bayes factor for different values of the parameter of Generalized Pareto Distribution under Uniform, Jeffrey's, Inverse Gamma, Inverse Chi-square and Inverse Exponential Priors. The Posterior probabilities are defined for the hypotheses $H_0:\theta\le\theta_0$ and $H_1:\theta>\theta_0$ as:

$$P\left(H_0\right)=P\left(\theta\le\theta_0\right)=\int_0^{\theta_0}p\left(\theta|\underline{x}\right)d\theta\ ,$$

where

$p(\theta|\underline{x})$ is Posterior Distribution of θ

The Posterior probabilities are calculated under null and alternative hypothesis. The purpose is to show how well the Bayesian hypothesis for Generalized Pareto Distribution work under different Priors. Following scale, due to Jeffrey's (1961), can be used for the interpretation of Bayes factor which will facilitate the decision making process.

7. Posterior Predictive Distribution and Intervals

The Posterior Predictive Distribution can be obtained as:

$$p(y|\underline{x}) = \int_0^\infty p(\theta|\underline{x}) f(y;\theta) d\theta \qquad (7.1)$$

The Posterior Predictive Distribution under the assumption of Uniform Prior is derived to be

$$p(y|\underline{x}) = \frac{(m-1)\left\{\sum_{i=1}^{m} \ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1}\right\}^{m-1}}{(1-y)\left\{\sum_{i=1}^{m} \ln(1-x_i)^{-1} + (n-m)\ln(1-x_m)^{-1} - \ln(1-y)\right\}^{m}}, y > 0 \qquad (7.2)$$

The Posterior Predictive Distribution based on remaining Priors can be obtained by following the same lines.

The Posterior Predictive Interval can be obtained by solving the following two equations.

$$\int_0^L p(y|\underline{x})dy = \frac{\alpha}{2}, \qquad \int_U^\infty p(y|\underline{x})dy = \frac{\alpha}{2}$$

The Posterior Predictive Interval under the assumption of Uniform Prior is constructed as:

$$1-Anti\log\left[\left\{\sum_{i=1}^{m}\ln(1-x_i)^{-1}+(n-m)\ln(1-x_m)^{-1}\right\}\left\{1-\left(1-\frac{\alpha}{2}\right)^{\frac{1}{1-m}}\right\}\right] < \theta < 1-Anti\log\left[\left\{\sum_{i=1}^{m}\ln(1-x_i)^{-1}+(n-m)\ln(1-x_m)^{-1}\right\}\left\{1-\left(\frac{\alpha}{2}\right)^{\frac{1}{1-m}}\right\}\right] \qquad (7.3)$$

With little modifications, the Posterior Predictive intervals under Jeffrey's, Gamma, Chi-square and Exponential Priors can be constructed.

8. Simulation study

Simulation is a technique that can be used to examine the performance of different estimation procedures. In this technique, random samples are generated in such a way that Estimators under different estimation procedure can be compared and are in accordance with the real life problem. The major advantage of simulation is that its implementation is straightforward. In general, the simulation methods are easier to apply than analytical methods. Whereas, analytical methods may require user to employ many simplifying assumptions, simulation model has few such limitations, thus allowing much greater flexibility in representing the real system. Once a model is built, it can be used repeatedly to analyze different parameters or designs. Data simulations are often criticized because they are much cleaner than real data. However, simulating data remains an important component for most of the development projects. To this end, any developments to improve the complexity of the data simulations will permit investigators to better assess new analytical methods.

Here the simulation study has been carried out for estimates under Uniform, Jeffrey's, Inverse Gamma, Inverse Chi-square and Inverse Exponential Priors based on SELF, QLF, PLF, ELF, SLLF and WLF. The parametric space $\theta \in (3,6,9)$ has been assumed for Estimation. The amounts of Posterior risks associated with each Point Estimate have been presented in the parenthesis in the tables. All the calculations have been performed by assuming the samples to be 20% Type II censored. These samples have been drawn by following steps:

Step 1: Draw samples of size 'n' from the Generalized Pareto model using Inverse transformation technique, from the random no. generator $X = 1-(1-U)^{\theta}$, where, U is uniformly distributed random variable.

Step 2: Determine the test termination points on right, that is, determine the values of x_m

Step 3: The observations which are greater than x_m have been considered to be censored.

Step 4: Use the remaining observations for the analysis.

Step 5: Repeat step 1 to 5, 1,000 times.

The purpose of the simulation study is to evaluate and compare the performance of different estimates. In order to save the space in the tables the following abbreviations have been used:

Uniform Prior:	U.P	Lower Limit:	LL
Jeffrey's Prior:	J.P	Upper Limit:	UP
Inverse Gamma Prior:	I.G.P	Credible Intervals:	C.I
Inverse Chi Square Prior:	I.CS.P	Posterior Predictive Interval:	P.P.I
Inverse Exponential Prior:	I.E.P		

The simulation study explored some interesting properties of the Bayes Estimators. The rate of under estimation is 70% (i.e. 70% of the Estimates have been under estimated) with a significant proportion for the Estimates under Quadratic Loss Function for each Prior. This simply shows that the concerned Posterior Distributions are negatively skewed in majority of the cases. As suggested by corresponding formulae the estimates on the basis of Entropy and Weighted Loss Functions are same. As for as the Priors are concerned, the informative Priors (under each Loss function) have bigger contribution in under estimation, especially for larger choice of Hyper-parametric values. In case of non-informative Priors the added tendency of over estimation is assessed with the use of Uniform Prior. The degree of under estimation increases with increase in sample size. However, the rate of convergence of the estimates towards the true parametric values is positively affected by increasing the sample size. The greater true parametric values result in slower convergence of the estimates and higher level of under Estimation. The Estimates under Jeffrey's and Uniform Priors based on Squared Error, Entropy and Weighted Loss Functions (respectively) are same and provide the best convergence towards the true value of the parameter.

Similarly, the amounts of Posterior risks associated with the Bayes Estimates are observed to be decreasing with increasing the sample sizes. The minimum magnitudes of Posterior risks have been seen under Entropy Loss Function for each Prior. Two groups of Bayes risks (in terms of their magnitudes) can be identified one containing Estimates under Quadratic and Squared Logarithmic Loss Functions and the other containing Estimates based on Precautionary and Weighted Loss Functions. The performance of the Squared Error Loss Function is the worst as the corresponding Posterior Distributions are not symmetric. The larger choice of true parametric values tends to inflate the Posterior risks. On the other hand, the Estimates under Inverse Gamma Prior are having the least amounts of risks. In comparisons of the informative and non-informative Priors it can be observed that the informative Priors provide the better Estimates as the corresponding magnitudes of risks are lower. This simply indicates that the informative Priors are superior to non-informative Priors. A better choice of Hyper-parametric values or some Prior elicitation technique may further justify these findings. On the whole, it can be said that in order to have a Point Estimate of the parameter of the Generalized Pareto Distribution under a Bayesian framework, the Inverse Gamma Prior under Entropy Loss Function can affectively be employed.

The Bayesian Credible Intervals under different informative and non-informative Priors have been calculated. The widths of Credible Intervals tend to decrease by increasing the sample size. This property is similar under each Prior. In case of non-informative Priors, the minimum width of intervals is observed under Jeffrey's Prior. While for informative Priors, the Credible Intervals with least lengths are based on Inverse Gamma Prior. It is interesting to note that each Credible Intervals contains the true and estimated value of the parameter. The higher choices of Hyper-parametric values result in shorter intervals but at the cost of inflated Posterior risks (in some cases). On the other hand, the greater true parametric values lead to bigger widths of the Credible Intervals. Hence, the preference of the Prior Distribution is in accordance with that in the Point Estimation, that is, the performance of the Inverse Gamma Prior is found superior to that of other Priors.

The 95% Posterior Predictive Intervals follow the similar patterns as observed in case of Credible Intervals. The widths of the Posterior Predictive Intervals are inversely proportional to the sample size, while these are directly proportional to the true parametric values. The Posterior Predictions on the basis of Inverse Gamma Prior are seemed to be the most precise.

The Bayesian hypothesis testing indicates that the trend of information to support the null hypothesis starts (at least) from H_0: $\beta \leq 3.0$ and H_1: $\beta > 3.0$ under each Prior. The evidence in favor of H_0 increases as the hypothesized value of the parameter becomes closer to the true parametric range. However, the strength of the information to favor the H_0 under Inverse Gamma Prior is greater than that based on other Priors. The evidence in favor of H_0 increases with increase in the value of the parameter and vice versa. Again the Inverse Gamma Prior provides more reasonable Bayesian hypothesis testing for the Generalized Pareto Distribution.

9. Real life example

As Generalized Pareto Distribution is considered a suitable model in reliability studies, the application of a real life data regarding failure times of certain product(s) would be appropriate to illustrate the practical aspects of the study. So in this section, the data set of repair times (in days) of 30 air conditioning systems presented by Sinha and Prabha (2010) have been used to discuss the practical applications of the results obtained in previous sections.

The Table 12 shows the Bayes Estimators and posterior risks under different Priors and various Loss functions. From the results, it can be observed that the performance of the informative Priors is superior to the non-informative Priors. On the whole, the Inverse Gamma Prior provides the best estimates among all informative and non-informative Priors, as the amounts of Posterior risks associated with these Estimates are the minimum for each Loss function. In comparison of different Loss functions it can be assessed that the risks associated with Estimates under ELF are the least for every Prior. So, the patterns of results using the real life data are similar to those under simulation study. This simply indicates that the results derived in the previous sections are capable to be employed in the real situations.

Table 13 shows the details regarding 95% Credible Intervals (C.I) and Posterior Predictive Intervals (P.P.I) under different Priors. The results suggest that the widths of 95% Credible Intervals under informative Priors are smaller than those under non-informative Priors. The least amounts of the widths of the Credible Intervals are observed under the assumption of Inverse Gamma Prior. Similarly, the width of 95% Posterior Predictive Intervals, based on Inverse Gamma Prior, is the minimum. Therefore, the findings of Interval Estimations are again completely in accordance with the simulation study. This is another indication of the authenticity of the different Estimators derived above.

10. Conclusions

The study has been conducted in order to have suitable Bayesian Point and Interval Estimates of the parameter of the Generalized Pareto Distribution under Type II censored samples. Different Priors and Loss functions have been assumed for the estimation. The framework for the prediction of the future values from the Distribution has also been discussed. From the above Analysis it can be assessed that the performance of the Inverse Gamma Prior is the best for Bayesian Point and Interval Estimation. The Bayesian hypothesis testing scheme also indicated the preference of the Inverse Gamma Prior. Similarly, the Posterior Predictive Intervals are also more precise under the assumption of Inverse Gamma Prior. The Analysis under real life data further strengthened these beliefs. On the other hand, the performance of the Entropy Loss Function is supreme among all Loss functions. Therefore, for Bayesian Estimation and prediction of Generalized Pareto Distribution based on Type II censored samples, the use of Inverse Gamma Prior along with Entropy Loss Function can be preferred.

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Table 1: Formulae for Bayes Estimators and Posterior risks under different Loss functions

Loss Function	Bayes Estimator	Posterior Risk
SELF	$\theta_{SELF}=\mathrm{E}(\theta)$	$\rho(\theta_{SELF})=\mathrm{E}(\theta^2)-\{\mathrm{E}(\theta)\}^2$
QLF	$\theta_{QLF}=\mathrm{E}(\theta^{-1})\{\mathrm{E}(\theta^{-2})\}^{-1}$	$\rho(\theta_{QLF})=1-\{\mathrm{E}(\theta^{-1})\}^2\{\mathrm{E}(\theta^{-2})\}^{-1}$
PLF	$\theta_{PLF}=\{\mathrm{E}(\theta^2)\}^{\frac{1}{2}}$	$\rho(\theta_{PLF})=2(\theta_{PLF}-\mathrm{E}(\theta))$
ELF	$\theta_{ELF}=\{\mathrm{E}(\theta^{-1})\}^{-1}$	$\rho(\theta_{ELF})=\mathrm{E}(\ln\theta)-\ln(\theta_{ELF})$
SLLF	$\theta_{SLLF}=\exp\{\mathrm{E}(\ln\theta)\}$	$\rho(\theta_{SLLF})=\mathrm{E}(\ln\theta^2)-\{\mathrm{E}(\ln\theta)\}^2$
WLF	$\theta_{WLF}=\{\mathrm{E}(\theta^{-1})\}^{-1}$	$\rho(\theta_{WLF})=\mathrm{E}(\theta)-\theta_{WLF}$

Table 2: Jeffrey's scale of evidence for Bayes factor interpretation

Value of Bayes Factor (B)	Interpretation
$B \le 0.10$	Strong evidence for H_1
$0.10 < B \le 0.33$	Moderate evidence for H_1
$0.33 < B \le 1.00$	Weak evidence for H_1
$1.00 < B \le 3.00$	Weak evidence for H_0
$3.00 < B \le 10.00$	Moderate evidence for H_0
$B > 10.00$	Strong evidence for H_0

Table 3: Bayes estimates and risks under different Priors and Loss functions for θ = 3

n	Priors	Loss Functions					
		SELF	**QLF**	**PLF**	**ELF**	**SLLF**	**WLF**
50	**U.P**	3.1618	3.0037	3.2042	3.0807	3.1269	3.0807
		(0.2701)	(0.0250)	(0.0848)	(0.0119)	(0.0262)	(0.0810)
	J.P	3.0807	2.9304	3.1210	3.0037	3.0488	3.0037
		(0.2497)	(0.0250)	(0.0805)	(0.0117)	(0.0257)	(0.0770)
	I.G.P	2.9317	2.7953	2.9681	2.8619	2.9048	2.8619
		(0.2148)	(0.0238)	(0.0728)	(0.0112)	(0.0245)	(0.0698)
	I.CS.P	3.0162	2.8726	3.0546	2.9426	2.9868	2.9426
		(0.2332)	(0.0243)	(0.0768)	(0.0115)	(0.0251)	(0.0735)
	I.E.P	3.0087	2.8654	3.0470	2.9353	2.9793	2.9353
		(0.2321)	(0.0238)	(0.0766)	(0.0113)	(0.0250)	(0.0738)
70	**U.P**	3.0688	2.9592	3.0976	3.0130	3.0582	3.0130
		(0.1776)	(0.0178)	(0.0576)	(0.0085)	(0.0187)	(0.0557)
	J.P	3.0130	2.9073	3.0408	2.9592	3.0036	2.9592
		(0.1681)	(0.0178)	(0.0555)	(0.0084)	(0.0183)	(0.0538)
	I.G.P	2.9082	2.8096	2.9340	2.8580	2.9009	2.8580
		(0.1510)	(0.0172)	(0.0517)	(0.0081)	(0.0177)	(0.0501)
	I.CS.P	2.9681	2.8658	2.9950	2.9160	2.9598	2.9160
		(0.1601)	(0.0175)	(0.0537)	(0.0082)	(0.0180)	(0.0520)
	I.E.P	2.9628	2.8606	2.9896	2.9108	2.9544	2.9108
		(0.1596)	(0.0172)	(0.0536)	(0.0082)	(0.0181)	(0.0522)
100	**U.P**	3.0220	2.9464	3.0415	2.9837	3.0285	2.9837
		(0.1156)	(0.0125)	(0.0391)	(0.0059)	(0.0131)	(0.0382)
	J.P	2.9837	2.9100	3.0028	2.6190	2.6583	2.6190
		(0.1141)	(0.0125)	(0.0381)	(0.0058)	(0.0128)	(0.0372)
	I.G.P	2.9107	2.8405	2.9288	2.8752	2.9183	2.8752
		(0.1059)	(0.0121)	(0.0362)	(0.0057)	(0.0125)	(0.0354)
	I.CS.P	2.9527	2.8807	2.9713	2.9162	2.9600	2.9162
		(0.1103)	(0.0123)	(0.0372)	(0.0058)	(0.0127)	(0.0364)
	I.E.P	2.9489	2.8770	2.9675	2.9125	2.9562	2.9125
		(0.1100)	(0.0121)	(0.0372)	(0.0058)	(0.0128)	(0.0365)

Table 4: Bayes estimates and risks under different Priors and Loss functions for θ = 6

n	Priors	Loss Functions					
		SELF	**QLF**	**PLF**	**ELF**	**SLLF**	**WLF**
50	**U.P**	6.2671	5.9537	6.3512	6.1064	6.1980	6.1064
		(1.0615)	(0.0250)	(0.1682)	(0.0119)	(0.0262)	(0.1606)
	J.P	6.1064	5.8085	6.1862	5.9537	6.0430	5.9537
		(0.9812)	(0.0250)	(0.1596)	(0.0117)	(0.0257)	(0.1526)
	I.G.P	5.8097	5.5395	5.8819	5.6714	5.7565	5.6714
		(0.8438)	(0.0238)	(0.1443)	(0.0112)	(0.0245)	(0.1383)
	I.CS.P	5.9662	5.6821	6.0422	5.8207	5.9080	5.8207
		(0.9127)	(0.0243)	(0.1520)	(0.0115)	(0.0251)	(0.1455)
	I.E.P	5.9587	5.6750	6.0346	5.8134	5.9006	5.8134
		(0.9104)	(0.0238)	(0.1518)	(0.0113)	(0.0250)	(0.1458)
70	**U.P**	6.1429	5.9235	6.2005	6.0312	6.1216	6.0312
		(0.7119)	(0.0178)	(0.1153)	(0.0085)	(0.0187)	(0.1116)
	J.P	6.0312	5.8196	6.0868	5.9235	6.0123	5.9235
		(0.6736)	(0.0178)	(0.1111)	(0.0084)	(0.0183)	(0.1077)
	I.G.P	5.8204	5.6231	5.8722	5.7201	5.8059	5.7201
		(0.6049)	(0.0172)	(0.1034)	(0.0081)	(0.0177)	(0.1003)
	I.CS.P	5.9324	5.7278	5.9861	5.8283	5.9158	5.8283
		(0.6398)	(0.0175)	(0.1073)	(0.0082)	(0.0180)	(0.1040)
	I.E.P	5.9270	5.7227	5.9807	5.8231	5.9104	5.8231
		(0.6387)	(0.0172)	(0.1072)	(0.0082)	(0.0181)	(0.1042)
100	**U.P**	6.0861	5.9339	6.1255	6.0090	6.0992	6.0090
		(0.4688)	(0.0125)	(0.0787)	(0.0059)	(0.0131)	(0.0770)
	J.P	6.0090	5.8607	6.0474	5.2746	5.3537	5.2746
		(0.4629)	(0.0125)	(0.0767)	(0.0058)	(0.0128)	(0.0751)
	I.G.P	5.8613	5.7200	5.8978	5.7898	5.8766	5.7898
		(0.4294)	(0.0121)	(0.0730)	(0.0057)	(0.0125)	(0.0714)
	I.CS.P	5.9402	5.7953	5.9776	5.8668	5.9548	5.8668
		(0.4466)	(0.0123)	(0.0749)	(0.0058)	(0.0127)	(0.0733)
	I.E.P	5.9364	5.7916	5.9739	5.8631	5.9511	5.8631
		(0.4460)	(0.0121)	(0.0749)	(0.0058)	(0.0128)	(0.0734)

Table 5: Bayes estimates and risks under different Priors and Loss functions for θ = 9

n	Priors	Loss Functions					
		SELF	**QLF**	**PLF**	**ELF**	**SLLF**	**WLF**
50	**U.P**	9.4250	8.9537	9.5515	9.1833	9.3210	9.1833
		(2.4008)	(0.0250)	(0.2530)	(0.0119)	(0.0262)	(0.2416)
	J.P	9.1833	8.7353	9.3033	8.9537	9.0880	8.9537
		(2.2193)	(0.0250)	(0.2400)	(0.0117)	(0.0257)	(0.2295)
	I.G.P	8.7365	8.3302	8.8451	8.5285	8.6565	8.5285
		(1.9081)	(0.0238)	(0.2170)	(0.0112)	(0.0245)	(0.2080)
	I.CS.P	8.9662	8.5392	9.0804	8.7475	8.8787	8.7475
		(2.0613)	(0.0243)	(0.2284)	(0.0115)	(0.0251)	(0.2186)
	I.E.P	8.9587	8.5321	9.0728	8.7402	8.8713	8.7402
		(2.0579)	(0.0238)	(0.2282)	(0.0113)	(0.0250)	(0.2189)
70	**U.P**	9.2355	8.9056	9.3222	9.0675	9.2036	9.0675
		(1.6093)	(0.0178)	(0.1734)	(0.0085)	(0.0187)	(0.1679)
	J.P	9.0675	8.7494	9.1511	8.9056	9.0392	8.9056
		(1.5226)	(0.0178)	(0.1671)	(0.0084)	(0.0183)	(0.1619)
	I.G.P	8.7503	8.4536	8.8280	8.5994	8.7284	8.5994
		(1.3672)	(0.0172)	(0.1555)	(0.0081)	(0.0177)	(0.1508)
	I.CS.P	8.9145	8.6071	8.9952	8.7582	8.8895	8.7582
		(1.4449)	(0.0175)	(0.1613)	(0.0082)	(0.0180)	(0.1563)
	I.E.P	8.9092	8.6020	8.9898	8.7529	8.8842	8.7529
		(1.4431)	(0.0172)	(0.1612)	(0.0082)	(0.0181)	(0.1565)
100	**U.P**	9.15022	8.9214	9.2094	9.0343	9.1699	9.0343
		(1.0598)	(0.0125)	(0.1184)	(0.0059)	(0.0131)	(0.1158)
	J.P	9.0343	8.8113	9.0921	7.9301	8.0491	7.9301
		(1.0464)	(0.0125)	(0.1154)	(0.0058)	(0.0128)	(0.1129)
	I.G.P	8.8119	8.5996	8.8668	8.7044	8.8350	8.7044
		(0.9706)	(0.0121)	(0.1098)	(0.0057)	(0.0125)	(0.1074)
	I.CS.P	8.92771	8.7099	8.9840	8.8175	8.9497	8.8175
		(1.0089)	(0.0123)	(0.1126)	(0.0058)	(0.0127)	(0.1102)
	I.E.P	8.9239	8.7063	8.9802	8.8137	8.9460	8.8137
		(1.0080)	(0.0121)	(0.1126)	(0.0058)	(0.0128)	(0.1102)

Table 6: 95% Confidence Intervals (C.I) and Posterior Predictive Intervals (P.P.I) for θ = 3

n	Priors	95% C.I			95% P.P.I		
		LL	UL	UL – LL	LL	UL	UL – LL
50	U.P	2.3036	4.3324	2.0288	0.0656	1.0000	0.9344
	J.P	2.2536	4.2045	1.9509	0.0719	1.0000	0.9281
	I.G.P	2.2068	4.0853	1.8785	0.0728	1.0000	0.9272
	I.CS.P	2.2150	4.1006	1.8856	0.0729	1.0000	0.9271
	I.E.P	2.1920	4.1226	1.9306	0.0710	1.0000	0.9290
70	U.P	2.3520	3.9996	1.6476	0.0735	1.0000	0.9265
	J.P	2.3148	3.9175	1.6027	0.0722	1.0000	0.9278
	I.G.P	2.2795	3.8398	1.5603	0.0735	1.0000	0.9265
	I.CS.P	2.2857	3.8502	1.5645	0.0737	1.0000	0.9263
	I.E.P	2.2707	3.8614	1.5907	0.0711	1.0000	0.9289
100	U.P	2.4214	3.7688	1.3474	0.0751	1.0000	0.9249
	J.P	2.3941	3.7159	1.3218	0.0733	1.0000	0.9267
	I.G.P	2.3679	3.6652	1.2973	0.0751	1.0000	0.9249
	I.CS.P	2.3724	3.6722	1.2998	0.0754	1.0000	0.9246
	I.E.P	2.3629	3.6777	1.3148	0.0716	1.0000	0.9284

Table 7: 95% Confidence Intervals (C.I) and Posterior Predictive Intervals (P.P.I) for θ = 6

n	Priors	95% C.I			95% P.P.I		
		LL	UL	UL – LL	LL	UL	UL – LL
50	U.P	4.5659	8.5872	4.0213	0.1277	1.0000	0.8723
	J.P	4.4669	8.3338	3.8669	0.1395	1.0000	0.8605
	I.G.P	4.3732	8.0958	3.7227	0.1412	1.0000	0.8588
	I.CS.P	4.3814	8.1111	3.7297	0.1413	1.0000	0.8587
	I.E.P	4.3413	8.1647	3.8235	0.1367	1.0000	0.8633
70	U.P	4.7080	8.0060	3.2980	0.1416	1.0000	0.8584
	J.P	4.6336	7.8417	3.2081	0.1393	1.0000	0.8607
	I.G.P	4.5622	7.6850	3.1227	0.1417	1.0000	0.8583
	I.CS.P	4.5684	7.6954	3.1270	0.1418	1.0000	0.8582
	I.E.P	4.5426	7.7247	3.1822	0.1370	1.0000	0.8630
100	U.P	4.8765	7.5900	2.7135	0.1433	1.0000	0.8567
	J.P	4.8215	7.4835	2.6620	0.1400	1.0000	0.8600
	I.G.P	4.7683	7.3807	2.6124	0.1433	1.0000	0.8567
	I.CS.P	4.7728	7.3877	2.6149	0.1436	1.0000	0.8564
	I.E.P	4.7566	7.4034	2.6468	0.1379	1.0000	0.8621

Table 8: 95% Confidence Intervals (C.I) and Posterior Predictive Intervals (P.P.I) for θ = 9

n	**Priors**	**95% C.I**			**95% P.P.I**		
		LL	**UL**	**UL – LL**	**LL**	**UL**	**UL – LL**
50	**U.P**	6.8666	12.9142	6.0476	0.1857	1.0000	0.8143
	J.P	6.7177	12.5330	5.8153	0.2022	1.0000	0.7978
	I.G.P	6.5763	12.1744	5.5981	0.2045	1.0000	0.7955
	I.CS.P	6.5845	12.1897	5.6051	0.2046	1.0000	0.7954
	I.E.P	6.5269	12.2754	5.7484	0.1983	1.0000	0.8017
70	**U.P**	7.0782	12.0366	4.9584	0.2052	1.0000	0.7948
	J.P	6.9663	11.7895	4.8232	0.2019	1.0000	0.7981
	I.G.P	6.8587	11.5533	4.6946	0.2052	1.0000	0.7948
	I.CS.P	6.8649	11.5637	4.6988	0.2054	1.0000	0.7946
	I.E.P	6.8281	11.6114	4.7832	0.1987	1.0000	0.8013
100	**U.P**	7.3316	11.4113	4.0796	0.2075	1.0000	0.7925
	J.P	7.2490	11.2512	4.0022	0.2029	1.0000	0.7971
	I.G.P	7.1687	11.0962	3.9275	0.2075	1.0000	0.7925
	I.CS.P	7.1732	11.1032	3.9299	0.2078	1.0000	0.7922
	I.E.P	7.1504	11.1291	3.9787	0.2000	1.0000	0.8000

Table 9: Posterior probabilities and Bayes factors under Uniform and Jeffrey's Priors

Null Hyp.	**Alt. Hyp.**	**Uniform Prior**			**Jeffrey's Prior**		
		Posterior Probabilities		**Bayes Factor**	**Posterior Probabilities**		**Bayes Factor**
H_0	**H_1**	**$P(H_0)$**	**$P(H_1)$**	**B**	**$P(H_0)$**	**$P(H_1)$**	**B**
$\theta \leq 2.0$	$\theta > 2.0$	0.0015	0.9985	0.0015	0.0025	0.9975	0.0025
$\theta \leq 2.5$	$\theta > 2.5$	0.0802	0.9198	0.0871	0.1057	0.8943	0.1182
$\theta \leq 3.0$	$\theta > 3.0$	0.4130	0.5870	0.7035	0.4758	0.5242	0.9078
$\theta \leq 3.5$	$\theta > 3.5$	0.7661	0.2339	3.2762	0.8132	0.1868	4.3523
$\theta \leq 4.0$	$\theta > 4.0$	0.9342	0.0658	14.2080	0.9530	0.0470	20.2974

Table 10: Posterior probabilities and Bayes factors under Inverse Gamma and Inverse Chi-square Priors

Null Hyp.	Alt. Hyp.	Inverse GammaPrior			Inverse Chi Square Prior		
		Posterior Probabilities		Bayes Factor	Posterior Probabilities		Bayes Factor
H_0	H_1	$P(H_0)$	$P(H_1)$	B	$P(H_0)$	$P(H_1)$	B
$\theta \le 2.0$	$\theta > 2.0$	0.0038	0.9962	0.0039	0.0035	0.9965	0.0036
$\theta \le 2.5$	$\theta > 2.5$	0.1358	0.8642	0.1572	0.1304	0.8696	0.1500
$\theta \le 3.0$	$\theta > 3.0$	0.5377	0.4623	1.1632	0.5283	0.4717	1.1199
$\theta \le 3.5$	$\theta > 3.5$	0.8529	0.1471	5.8000	0.8477	0.1523	5.5653
$\theta \le 4.0$	$\theta > 4.0$	0.9670	0.0330	29.2911	0.9654	0.0346	27.8709

Table 11: Posterior probabilities and Bayes factors under Inverse Exponential Prior

Null Hyp.	Alt. Hyp.	Inverse Exponential Prior		
		Posterior Probabilities		Bayes Factor
H_0	H_1	$P(H_0)$	$P(H_1)$	B
$\theta \le 2.0$	$\theta > 2.0$	0.0037	0.9963	0.0038
$\theta \le 2.5$	$\theta > 2.5$	0.1340	0.8660	0.1548
$\theta \le 3.0$	$\theta > 3.0$	0.5346	0.4654	1.1486
$\theta \le 3.5$	$\theta > 3.5$	0.8512	0.1488	5.7205
$\theta \le 4.0$	$\theta > 4.0$	0.9665	0.0335	28.8080

Table 12: Bayesian Estimation under different Loss functions using real life data

Priors	Loss Functions					
	SELF	QLF	PLF	ELF	SLLF	WLF
U.P	1.6992	1.4732	1.7272	1.5149	1.6430	1.5499
	(0.1386)	(0.1242)	(0.1359)	(0.0606)	(0.1039)	(0.1372)
J.P	1.5432	1.3967	1.6974	1.5307	1.5204	1.5380
	(0.1398)	(0.1313)	(0.1378)	(0.0695)	(0.1179)	(0.1278)
I.G.P	1.5154	1.2965	1.5111	1.3954	1.4357	1.3943
	(0.1133)	(0.1165)	(0.1170)	(0.0533)	(0.1066)	(0.1125)
I.CS.P	1.4846	1.4062	1.5288	1.5027	1.4846	1.4270
	(0.1248)	(0.1313)	(0.1119)	(0.0567)	(0.1114)	(0.1188)
I.E.P	1.6972	1.3945	1.7253	1.4931	1.6031	1.4401
	(0.1296)	(0.1240)	(0.1176)	(0.0629)	(0.1138)	(0.1248)

Table 13: 95% Credible Intervals (C.I) and Posterior Predictive Intervals (P.P.I) under different Priors using real life data

Priors	95% C.I			95% P.P.I		
	LL	**UL**	**UL – LL**	**LL**	**UL**	**UL – LL**
U.P	0.7542	2.1252	1.3709	0.0956	0.9310	0.8354
J.P	0.7631	2.2296	1.4664	0.1114	0.9432	0.8318
I.G.P	0.8769	2.1705	1.2936	0.1089	0.8736	0.7647
I.CS.P	0.7756	2.1846	1.4090	0.1047	0.8944	0.7898
I.E.P	0.8076	2.0517	1.2441	0.0995	0.8856	0.7860

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