ISSN 1684-8403
Journal of Statistics
Volume 21, 2014. pp. 75-89

On Characterizations of Certain Continuous Distributions

Mohammad Ahsanullah[1], *Ghulam Hussein G. Hamedani*[2],
B. M. Golam Kibria[3] *and Mohammad Shakil*[4]

Abstract

This paper presents some characterizations and properties of certain new Distributions. First, various characterizations of the Distribution of the ratio of two independent Maxwell and Rayleigh random variables are presented. Then we establish characterizations results related to two Distributions, Modified Burr XII - Geometric Distribution (MBGD) and Folded $t-$Distribution. We also characterize a Distribution due to Bondesson (1979). These characterizations are based on: (i) a simple relationship between two Truncated moments; (ii) conditional expectations of functions of Order Statistics and (iii) conditional expectation of a power of a random variable.

Keywords

Characterizations, Folded *t*-Distribution, Hazard function, Modified Burr XII-Geometric distribution, Ratio of independent random variables

1. Introduction

As pointed out by Nadarajah and Kotz (2006), the Distribution of U / V for independent random variables U and V is of interest in biological and physical sciences, econometrics, and ranking and selection. Examples include Mendelian Inheritance ratios in genetics, mass to energy ratios in nuclear physics, target to control precipitation in meteorology, and inventory ratios in economics.

[1] Department of Management Sciences, Rider University, Lawrenceville, NJ 08648, USA.
[2] Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, WI 53201-1881, USA.
[3] Department of Mathematics and Statistics, Florida International University, University Park, Miami, FL 33199, USA.
[4] Department of Mathematics, Miami Dade College, Hialeah Campus, Hialeah, FL 33012, USA.

They also referred to another important example which is the stress-strength Model in context of Reliability. It describes the life of a component which has a random strength V and is subject to a random stress *U*.

The Distribution of U / V have been studied by several authors when these random variables come from the same family of Distributions. Recently, Shakil and Ahsanullah (2011) also pointed out that the Distribution of the ratio of independent random variables arises in many fields of studies in engineering. For the detailed explanation of the importance of the Distribution of the ratio of independent random variables, we also refer the interested reader to Shakil and Ahsanullah (2011) where they consider the Distributional properties of record values of the ratio of independent Rayleigh random variables. Nadarajah (2010) studied the Distributional properties as well as estimation of the ratio of independent Weibull random variables. For the detailed discussion, domain of applicability and practical examples we refer the interested reader to Nadarajah (2010). A more interesting case, however, is when the random variables have different Distributions. Nadarajah and Kotz (2006) considered the case when the independent random variables U and V have deferent but similar Distributions (Gamma and Weibull) and obtained the exact Distribution of the ratio U /V. Shakil et al. (2007) obtained the Distribution of U / V when U and V are independent Maxwell and Rayleigh random variables.

In the applications where the underlined Distribution is assumed to belong to a certain family of Distributions, the investigator needs to verify that the underlying Distribution is in fact the assumed one. To this end, the investigator has to rely on the characterizations of the assumed Distribution and determine if the corresponding conditions are satisfied. Thus, the problem of characterizing Distributions becomes important and essential. Consequently, the field of characterization of Distributions has attracted the attention of many researchers and hence, various characterizations have been established in many different directions. The goal of the present work is to establish various characterizations of the Distribution of the ratio of certain independent random variables as well as for three other Distributions mentioned in the next paragraph.

In Section 2, we present characterizations of the Distribution of the ratio of two independent Maxwell and Rayleigh (RMR) random variables. Our results in subsections 2.1- 2.2, will be based on a simple relationship between two Truncated moments; and on conditional expectations of certain functions of Order

Statistics, respectively. In Section 3, we present characterizations of the Modified Burr XII - Geometric Distribution (MBGD). Section 4 presents characterizations of the Folded *t*-Distribution. In Section 5, we present a characterization of a Distribution due to Bondesson (1979). The concluding remarks are given in Section 6.

2. Characterizations of the RMR Distribution

Let U and V have Maxwell and Rayleigh Distributions, respectively. The p.d.f.'s (probability density functions) f_U and f_V are given respectively by

$$f_U(u) = \sqrt{\frac{2}{\pi}} a^{3/2} u^2 e^{-\frac{au^2}{2}}, \quad u > 0$$

and

$$f_V(v) = \left(\frac{v}{\sigma^2}\right) e^{-\frac{v^2}{2\sigma^2}}, \quad v > 0$$

where α and σ are positive parameters.

The p.d.f. (f) and c.d.f. (F) of the Distribution of the ratio, $X = \frac{U}{V}$, of two independent Maxwell and Rayleigh random variables U and V (given above) are respectively (see Shakil et al. (2007))

$$f(x) = f(x; \alpha, \sigma) = \frac{3a^{3/2}\sigma^3 x^2}{(1 + a\sigma^2 x^2)^{5/2}}, \quad x > 0 \tag{1.1}$$

and

$$F(x) = \frac{a^{3/2}\sigma^3 x^3}{(1 + a\sigma^2 x^2)^{3/2}}, \quad x \geq 0 \tag{1.2}$$

We denote the random variable X with c.d.f., equation (1.2) by RMR.

As pointed out in the introduction, the Distribution of the ratio of independent random variables has applications in many fields of study. So, an investigator will be vitally interested to know if their model fits the requirements of RMR Distribution. To this end, the investigator relies on characterizations of this Distribution, which provide conditions under which the underlying Distribution is indeed that of RMR. In this section, we will present several characterizations of this Distribution.

Throughout this paper we assume, where necessary, that the Distribution Function F is twice differentiable on its support.

2.1. Characterization based on Two Truncated Moments: In this subsection, we present characterizations of RMR Distribution in terms of Truncated moments. We like to mention here the works of Galambos and Kotz (1978), Glanzel (1987, 1988, 1990), Glanzel et al. (1984), Glanzel and Hamedani (2001), Hamedani (1993, 2002, 2006) and Kotz and Shanbhag (1980), and among others. Our characterization results presented here will employ an interesting result due to Glanzel (1987), (Theorem 1 below).

Theorem 1: Let (Ω, F, P) be a given probability space and let $H = [a, b]$ be an Interval for some a < b (a = -∞, b = ∞ might as well be allowed). Let $X : \Omega \to H$ be a continuous random variable with the Distribution Function F and let 'g' and 'h' be two real functions defined on H such that

$$\mathrm{E}[g(X) \mid X \geq x] = \mathrm{E}[h(X) \mid X \geq x] \quad \eta(x), \qquad x \in H$$

is defined with some real function η. Assume that $g, h \in C^1(H)$, $\eta \in C^2(H)$ and F is twice continuously differentiable and strictly Monotone Function on the set *H*. Finally, assume that the equation $h\eta = g$ has no real solution in the interior of *H*. Then F is uniquely determined by the functions 'g', 'h' and η, particularly $F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u)h(u) - g(u)} \right| \exp(-s(u))\, du$, where the function 's' is a solution of the differential equation $s' = \frac{\eta' h}{\eta h - g}$ and C is a constant, chosen to make $\int_H dF = 1$.

Remarks 2.1.1: (a) In Theorem 1, the Interval H need not be closed. (b) The goal is to have the function η as simple as possible. For a more detailed discussion on the choice of η, we refer the reader to Glanzel and Hamedani (2001) and Hamedani $(1993, 2002, 2006)$.

Proposition 2.1.2: Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $h(x) = x^{-1}$ and $g(x) = x^{-1}(1 + a\sigma^2 x^2)$ for $x \in (0, \infty)$. The p.d.f. of X is equation (1.1) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = 3(1 + a\sigma^2 x^2), \quad x > 0.$$

Proof: Let X have p.d.f., equation (1.1), then

$$(1-F(x))\ \mathrm{E}[h(X)\mid X\geq x]=a^{1/2}\sigma\left(1+a\sigma^2x^2\right)^{-3/2},\quad x>0,$$

and

$$(1-F(x))\ \mathrm{E}[g(X)\mid X\geq x]=3a^{1/2}\sigma\left(1+a\sigma^2x^2\right)^{-1/2},\quad x>0,$$

and finally,

$$\eta(x)h(x)-g(x)=2x^{-1}\left(1+a\sigma^2x^2\right)>0,\quad for\ \ x>0.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\ h(x)}{\eta(x)\ h(x)-g(x)}=3a\sigma^2x\left(1+a\sigma^2x^2\right)^{-1},\quad x>0,$$

and hence $s(x)=\ln(\left(1+a\sigma^2x^2\right)^{3/2})$.

Now, in view of Theorem 1 (with $C=a^{1/2}\sigma$), X has c.d.f., equation (1.2), and p.d.f., equation (1.1).

Corollary 2.1.3: Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $h(x)=x^{-1}$ for $x\in(0,\infty)$. The p.d.f. of X is equation (1.1) if and only if there exist functions g and η defined in Theorem 1 satisfying the differential equation:

$$\frac{\eta'(x)\ x^{-1}}{\eta(x)\ x^{-1}-g(x)}=3a\sigma^2x\left(1+a\sigma^2x^2\right)^{-1},\quad x>0.$$

Remarks 2.1.4: (i) The general solution of the differential equation given in Corollary (2.1.3) is,

$$\eta(x)=\left(1+a\sigma^2x^2\right)^{3/2}\left[-\int 3a\sigma^2x^2\left(1+a\sigma^2x^2\right)^{-5/2}g(x)\ dx+D\right],$$

for $x>0$, where D is a constant. One set of appropriate functions is given in Proposition (2.1.2) with $D=0$.

(ii) Clearly there are other triplet functions (h,g,η) satisfying conditions of Proposition (2.1.2).

2.2. Characterization based on Conditional Expectations of Certain Functions of Order Statistics: Let $X_{1:n}\leq X_{2:n}\leq\ldots\leq X_{n:n}$ be n Order Statistics from a continuous c.d.f. F. We present here characterization results based on some functions of these Order Statistics. We refer the reader to Ahsanullah and Hamedani (2007), Hamedani et al. (2008) and Hamedani (2010) among others, for characterizations of other well-known continuous Distributions in this direction. The proof of the following proposition is similar to that of Theorem 2.5 of Hamedani (2010) in which $k(x)=x^{\gamma}$ for some $\gamma>0$. We give a brief proof, however, for the sake of completeness.

Proposition 2.2.1: Let $X:\Omega\to(a,b)$, $a\geq 0$ be a continuous random variable with c.d.f., F and k(x) be a differentiable function such that $\lim_{x\to a} k(x)(F(x))^n=0$. Let $q(x,n)$ be a real-valued function which is differentiable with respect to x and $\int_a^b \frac{k'(x)}{q(x,n)}dx=\infty$. Then

$$E\lfloor k(X_{n:n})|X_{n:n}<t\rfloor = k(t)-q(t,n), \qquad a<t<b \tag{2.2.1}$$

implies that

$$F(x)=\left(\frac{q(b,n)}{q(x,n)}\right)^{\frac{1}{n}} e^{-\int_x^b \frac{k'(t)}{nq(t,n)}dt}, \quad x\geq a.$$

Proof: Condition $(2.2.1)$ and assumption $\lim_{x\to a} k(x)(F(x))^n=0$ imply that

$$\int_a^t k'(x)\big(F(x)\big)^n dx = q(t,n)(F(t))^n \tag{2.2.2}$$

Differentiating equation $(2.2.2)$ with respect to 't', we have

$$\frac{f(t)}{F(t)}+\frac{\frac{\partial}{\partial t}q(t,n)}{nq(t,n)}=\frac{k\prime(t)}{nq(t,n)} \tag{2.2.3}$$

Integrating equation $(2.2.3)$ with respect to 't' from 'x' to 'b', results in:

$$F(x)=\left(\frac{q(b,n)}{q(x,n)}\right)^{\frac{1}{n}} e^{-\int_x^b \frac{k'(t)}{nq(t,n)}dt}, \quad x\geq a.$$

Remarks 2.2.2: (a) In Proposition (2.2.1), the interval (a,b) is allowed to be unbounded, as we mentioned in the Introduction. (b) In Hamedani (2010), no applications of his Theorem 2.5 was provided. We are pleased to see here that there are Distributions for which Proposition (2.2.1) (or Theorem 2.5) can be employed to characterize them.

We now present characterizations of RMR Distributions based on certain functions of the nth Order Statistic, $X_{n:n}$: For $k(x)=x^3(1+a\sigma^2x^2)^{-3/2}$ and $q(x,n)=\frac{1}{n+1}x^3(1+a\sigma^2x^2)^{-3/2}$, Proposition (2.2.1) gives a characterization of equation (1.2). There are clearly other pairs of functions ‘k’ and ‘q’ which satisfy conditions of Proposition (2.2.1).

Let $X_j,\ j=1,2,...,n$ be ‘n’ independently and identically distributed random variables with c.d.f. (F) and corresponding p.d.f. (f) and let $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ be their corresponding Order Statistics. Let $X^*_{1\,:\,n-i+1}$ be the 1st Order Statistic from a sample of size $n-i+1$, $i>1$ of random variables with c.d.f. $F_t(x)=\frac{F(x)-F(t)}{1-F(t)}$, $x \ge t$ (t is fixed) and corresponding p.d.f. $f_t(x)=\frac{f(x)}{1-F(t)}$, $x \ge t$. Then

$$(X_{i:n} \mid X_{i-1:n}=t) \overset{d}{=} X^*_{1\,:\,n-i+1} \quad (\overset{d}{=} \textit{means equal in distribution}),$$

that is,

$$f_{X_{i\,:\,n} \mid X_{i-1\,:\,n}}(x \mid t) = f_{X^*_{1\,:\,n-i+1}}(x) = (n-i+1)(1-F_t(x))^{n-i}\,\frac{f(x)}{1-F(t)},\ x \ge t,$$

Now we can state the following characterization of RMR Distribution in yet somewhat different direction. The proof is similar to that of Propositions (2.2.1) and hence is omitted.

Corollary 2.2.3: Let $X\ :\ \Omega \to (0,\infty)$ be a continuous random variable with c.d.f. (F). Then

$$\mathrm{E}\left[X^3_{i:n}\ \left(1+a\sigma^2 X^2_{i:n}\right)^{-3/2}\ \mid\ X_{i-1:n}\ =t \right] = \frac{n-i}{n-i+1} t^3\ \left(1+a\sigma^2 t^2\right)^{-3/2}, \quad t>0,$$

for $n>i>1$ if and only if X has c.d.f., equation (1.2).

3. Characterizations of the MBGD Distribution

The p.d.f. (f), c.d.f. (F) and Hazard Function λ_F of the MBGD Distribution are given, respectively, by

$$f(x)=f(x;p,\theta,\beta,\lambda)=\theta p\ (\beta+\lambda x)x^{\beta-1}e^{\lambda x}\left(p+\theta x^{\beta}e^{\lambda x}\right)^{-2}, \quad x>0 \tag{3.1}$$

$$F(x)=1-p\left(p+\theta x^{\beta}e^{\lambda x}\right)^{-1}, \quad x \ge 0 \tag{3.2}$$

and

$$\lambda_F(x)=\left(\beta x^{-1}+\lambda\right)F(x)=\left(\beta x^{-1}+\lambda\right)\theta x^{\beta}e^{\lambda x}\left(p+\theta x^{\beta}e^{\lambda x}\right)^{-1}, \quad x>0 \tag{3.3}$$

where $0<p\le 1$, θ, $\lambda>0$ and $\beta \ge 0$ are parameters.

3.1. Characterization based on Two Truncated Moments:

Proposition 3.1.1: Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $h(x)=1$ and $g(x)=\left(p+\theta x^{\beta}e^{\lambda x}\right)^{1/2}$ for $x \in (0,\infty)$. The p.d.f. of X is equation (3.1) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=2\left(p+\theta x^{\beta}e^{\lambda x}\right)^{1/2}, \quad x>0.$$

Proof: Let X have p.d.f., equation (3.1), then

$$(1-F(x))\ \mathrm{E}[h(X)\mid X\geq x]=p\left(p+\theta x^{\beta}e^{\lambda x}\right)^{-1}, \quad x>0,$$

and

$$(1-F(x))\ \mathrm{E}[g(X)\mid X\geq x]=2p\left(p+\theta x^{\beta}e^{\lambda x}\right)^{-1/2}, \quad x>0,$$

and finally

$$\eta(x)h(x)-g(x)=\left(p+\theta x^{\beta}e^{\lambda x}\right)^{1/2}>0.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\ h(x)}{\eta(x)\ h(x)-g(x)}=\theta\left(\beta x^{-1}+\lambda\right)x^{\beta}e^{\lambda x}\left(p+\theta x^{\beta}e^{\lambda x}\right)^{-1}, \quad x>0,$$

and hence

$$s(x)=\ln\left(p+\theta x^{\beta}e^{\lambda x}\right)+c_1,$$

where c_1 is a constant. Now, in view of Theorem 1 (with $C=pe^{c_1}$), X has c.d.f., equation (3.2), and p.d.f., equation (3.1).

Corollary 3.1.2: Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $h(x)=1$ for $x \in (0,\infty)$. The *pdf* of X is equation (3.1) if and only if there exist functions g and η defined in Theorem 1 satisfying the differential equation:

$$\frac{\eta'(x)}{\eta(x)-g(x)}=\theta\left(\beta x^{-1}+\lambda\right)x^{\beta}e^{\lambda x}\left(p+\theta x^{\beta}e^{\lambda x}\right)^{-1}, \quad x>0.$$

Remark 3.1.3: The general solution of the differential equation given in Corollary (3.1.2) is,

$$\eta(x)=\left(p+\theta x^{\beta}e^{\lambda x}\right)\left[-\int g(x)\ \theta\left(\beta x^{-1}+\lambda\right)x^{\beta}e^{\lambda x}\left(p+\theta x^{\beta}e^{\lambda x}\right)^{-2}dx+D\right],$$

for $x > 0$, where D is a constant. One set of appropriate functions is given in Proposition 3.1.1 with $D = 0$.

3.2. Characterization based on Hazard Function: For the sake of completeness, we state the following simple fact. Let F be an absolutely continuous Distribution with the corresponding p.d.f. (f). The Hazard Function corresponding to F is,

$$\lambda_F(x) = \frac{f(x)}{1 - F(x)}, \quad x \in Supp\ F \tag{3.2.1}$$

where $Supp\ F$ is the support of F.

It is obvious that the Hazard Function of a twice differentiable Distribution Function satisfies the first Order differential equation:

$$\frac{\lambda_F'(x)}{\lambda_F(x)} - \lambda_F(x) = k(x) \tag{3.2.2}$$

where $k(x)$ is an appropriate integratable function. Although this differential equation has an obvious form since

$$\frac{f'(x)}{f(x)} = \frac{\lambda_F'(x)}{\lambda_F(x)} - \lambda_F(x),$$

for many Uni-variate Continuous Distributions, equation $(3.2.2)$ seems to be the only differential equation in terms of the Hazard Function. The goal here is to establish a differential equation which has as simple form as possible and is not of the trivial form equation $(3.2.2)$. For some general families of Distributions this may not be possible. Here is our characterization result for the MBGD Distribution.

Proposition 3.2.1: Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The p.d.f. of X is equation (3.1) if and only if its hazard function λ_F satisfies the differential equation:

$$\lambda_F'(x) - p\left(\beta x^{-1} + \lambda\right)\left(p + \theta x^{\beta} e^{\lambda x}\right)^{-1} \lambda_F(x) = -\beta\theta x^{\beta-2} e^{\lambda x}\left(p + \theta x^{\beta} e^{\lambda x}\right)^{-1}, \quad x > 0 \tag{3.2.3}$$

Proof: If X has p.d.f., equation (3.1), then obviously equation $(3.2.3)$ holds. If λ_F satisfies equation $(3.2.3)$, then

$$\left(px^{-\beta} e^{-\lambda x} + \theta\right)\lambda_F'(x) - p\left(\beta x^{-1} + \lambda\right)x^{-\beta} e^{-\lambda x} \lambda_F(x) = -\beta\theta x^{-2},$$

or

$$\frac{d}{dx}\left(\left(px^{-\beta}e^{-\lambda x}+\theta\right)\lambda_F(x)\right)=\beta\theta x^{-2},$$

or

$$\left(px^{-\beta}e^{-\lambda x}+\theta\right)\lambda_F(x)=-\beta\theta x^{-1}+C$$

where C is a constant. Then

$$\lambda_F(x)=\frac{f(x)}{1-F(x)}=\left(\beta x^{-1}+\frac{C}{\theta}\right)\theta x^{\beta}e^{\lambda x}\left(p+\theta x^{\beta}e^{\lambda x}\right),\quad x>0.$$

Integrating both sides of the above equation with respect to x from 0 to x and after some computations, we arrive at equation (3.2) with $C=\lambda\theta$.

4. Characterizations of the Folded t-Distribution

The p.d.f. (f), c.d.f. (F) and Hazard Function λ_F of a Folded $t-$Distribution are given, respectively, by

$$f(x)=2\left(x^2+2\right)^{-3/2},\quad x>0, \tag{4.1}$$

$$F(x)=x\left(x^2+2\right)^{-1/2},\quad x\geq 0, \tag{4.2}$$

and

$$\lambda_F(x)=\left(x^2+2\right)^{-1/2}+x\left(x^2+2\right)^{-1},\quad x>0, \tag{4.3}$$

4.1. Characterization based on Two Truncated Moments:

Proposition 4.1.1: Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $h(x)=x$ and $g(x)=x\left(x^2+2\right)^{-1/2}$ for $x\in(0,\infty)$. The p.d.f. of X is equation (4.1) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(x^2+2\right)^{-1/2},\quad x>0.$$

Proof: Let X have p.d.f., equation (4.1), then

$$(1-F(x))\,\mathrm{E}[h(X)\mid X\geq x]=2\left(x^2+2\right)^{-1/2},\qquad x>0,$$

and

$(1-F(x))\ \mathrm{E}[g(X)\mid X\geq x]=(x^2+2)^{-1},\quad x>0,$

and finally

$\eta(x)h(x)-g(x)=-\frac{1}{2}x(x^2+2)^{-1/2}<0.$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\ h(x)}{\eta(x)\ h(x)-g(x)}=x(x^2+2)^{-1},\quad x>0,$$

and hence

$s(x)=\ln\left(c_1(x^2+2)^{1/2}\right),\quad x>0,$

where c_1 is a constant.

Now, in view of Theorem 1 (with $C=\alpha$), X has c.d.f., equation (4.2), and p.d.f., equation (4.1).

Corollary 4.1.2: Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $h(x)=x$ for $x\in(0,\infty)$. The p.d.f. of X is equation (4.1) if and only if there exist functions g and η defined in Theorem 1 satisfying the differential equation:

$$\frac{\eta'(x)x}{\eta(x)x-g(x)}=x(x^2+2)^{-1},\quad x>0.$$

Remark 4.1.3: The general solution of the differential equation given in Corollary (4.1.2) is,

$\eta(x)=(x^2+2)^{1/2}\left[-\int g(x)\ (x^2+2)^{-3/2}dx+D\right],$

for $x>0$, where D is a constant. One set of appropriate functions is given in Proposition (4.1.1) with $D=0.$

4.2. Characterization based on Hazard function:

Proposition 4.2.1: Let $X:\Omega\to(0,\infty)$ be a continuous random variable. The p.d.f. of X is equation (4.1) if and only if its Hazard Function λ_F satisfies the differential equation:

$$\lambda_F'(x)+x\left(x^2+2\right)^{-1/2}\lambda_F(x)=2\left(x^2+2\right)^{-2},\quad x>0 \qquad (4.2.1)$$

with the boundary condition $\lambda_F(0)=1/\sqrt{2}$.

Proof: If X has c.d.f., equation (4.1), then obviously equation $(4.2.1)$ holds. If λ_F satisfies equation $(4.2.1)$, then

$$\frac{d}{dx}\left(\left(x^2+2\right)^{1/2}\lambda_F(x)\right)=2\left(x^2+2\right)^{-3/2},$$

or

$$\left(x^2+2\right)^{1/2}\lambda_F(x)=x\left(x^2+2\right)^{-1/2}+C,$$

where $C=1$ in view of the boundary condition on λ_F. Hence

$$\lambda_F(x)=x\left(x^2+2\right)^{-1}+\left(x^2+2\right)^{-1/2}=\left(x^2+2\right)^{-1/2}[x\left(x^2+2\right)^{-1/2}+1]$$

Integrating both sides of the above equation with respect to x from 0 to x we arrive at equation (4.2).

4.3. Characterization based on Truncated Moment of certain Functions of Order Statistics: In view of Proposition (2.2.1) for $k(x)=x\left(x^2+2\right)^{-1/2}$ and $q(x,n)=\frac{1}{n+1}x\left(x^2+2\right)^{-1/2}$, we have the following characterization of equation (4.2) which is similar to Corollary (2.2.3).

Corollary 4.3.1: Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable with c.d.f. (F) . Then

$$\mathrm{E}\left[X_{i:n}\left(X_{i:n}^2+2\right)\mid X_{i-1:n}=t\right]=\frac{n-i+1}{n-i}\,t\left(t^2+2\right)^{-1/2},\quad t>0 \qquad (4.3.1)$$

for $n>i>1$ if and only if X has c.d.f., equation (4.2).

5. Characterization of a Distribution due to Bondesson (1979)

Bondesson (1979) considered a Distribution with p.d.f. (f) of the form

$$f(x)=f(x;\alpha,\beta,\mu,\nu,p)=C\,x^{\mu-1}\left(\alpha+\beta\,x^{p}\right)^{-\nu},\quad x>0, \qquad (5.1)$$

where the parameters α , β , μ , ν are all positive, $0<p\le 1$, $\mu<p\nu$ and C is a normalizing constant.

This Distribution has also appeared in Shakil and Kibria (2010) who discussed some of its properties. We have the following characterization of equation (5.1) based on power of a random variable X with p.d.f., equation (5.1).

Proposition 5.1: Let $X : \Omega \to (0, \infty)$ be a continuous random variable with a differentiable p.d.f. (f) Then 'f' is given by equation (5.1) if and only if,

$$E\left[X^{p} \mid X \le t\right] = \frac{\alpha\mu}{\beta(p\nu - \mu - p)} - \frac{\alpha\, t + \beta\, t^{p+1}}{\beta(p\nu - \mu - p)}\, s(t), \tag{5.2}$$

where $s(t) = \frac{f(t)}{F(t)}$ and F is c.d.f. corresponding to p.d.f. (f).

Proof: If X has p.d.f., equation (5.1), then

$$\frac{f'(x)}{f(x)} = -\frac{\alpha(1-\mu) + \beta(p\nu - \mu + 1)\, x^{p}}{\alpha\, x + \beta\, x^{p+1}}, \tag{5.3}$$

from which we obtain,

$$x^{p} f(x) = \frac{\alpha\mu}{\beta(p\nu - \mu - p)} f(x) - \frac{1}{\beta(p\nu - \mu - p)} \frac{d}{dx}\left[\left(\alpha x + \beta x^{p+1}\right) f(x)\right].$$

Integrating both sides of the above equation with respect to x from 0 to t, we arrive at,

$$\int_0^t x^{p} f(x)dx = \frac{\alpha\mu}{\beta(p\nu - \mu - p)} F(t) - \frac{1}{\beta(p\nu - \mu - p)}\left[\left(\alpha t + \beta t^{p+1}\right) f(t)\right].$$

Dividing both sides of this equation by $F(t)$, we obtain

$$\frac{\int_0^t x^{p} f(x)dx}{F(t)} = \frac{\alpha\mu}{\beta(p\nu - \mu - p)} - \frac{1}{\beta(p\nu - \mu - p)}\left[\left(\alpha t + \beta t^{p+1}\right) s(t)\right],$$

which is equation (5.2). Conversely, if equation (5.2) holds, then

$$\int_0^t x^{p}\, f(x)dx = \frac{\alpha\mu}{\beta(p\nu - \mu - p)} F(t) - \frac{\alpha t + \beta t^{p+1}}{\beta(p\nu - \mu - p)}\, f(t).$$

Upon differentiating both sides of this equation with respect to 't', we have

$$t^{p} f(t) = \frac{\alpha\mu}{\beta(p\nu - \mu - p)} f(t) - \frac{\alpha t + \beta t^{p+1}}{\beta(p\nu - \mu - p)} f'(t) - \frac{\alpha + \beta(p+1)t^{p}}{\beta(p\nu - \mu - p)} f(t),$$

from which we obtain equation (5.3) and hence the result.

6. Concluding Remarks

To check the suitability of the underlying Distribution of a model the investigator will be looking for conditions under which the desired Distribution is characterized. In this paper, various characterizations of certain Distributions are presented in different directions. It is hoped that the findings of the paper will be useful for the practitioners in various fields of statistics and applied sciences that are looking for right Distribution for their Model.

Acknowledgements

The authors are grateful to two referees for their comments and suggestions which greatly improved the presentation of the content of this work.

References

1. Ahsanullah, M. and Hamedani, G.G. (2007). Certain characterizations of the Power Function and Beta Distributions based on Order Statistics. *Journal of Statistical Theory and Applications*, **6**, 220 – 225.
2. Bondesson, L. (1979). A general result on infinite divisibility. *The Annals of Probability*, **7,** 965 – 979.
3. Galambos, J. and Kotz, S. (1978). *Characterizations of Probability Distributions. A unified approach with an emphasis on exponential and related models*. Lecture Notes in Mathematics, **675**, Springer, Berlin.
4. Glanzel, W. (1987). A characterization theorem based on Truncated Moments and its application to some Distribution families. *Mathematical Statistics and Probability Theory*, (Bad Tatzmannsdorf, 1986), **B**, Reidel, Dordrecht, 75 – 84.
5. Glanzel, W. (1988). A characterization of the Normal Distribution. *Studia Scientiarum Mathematcarum Hungarica*, **23**, 89 – 91.
6. Glanzel, W. (1990). Some consequences of a characterization theorem based on Truncated Moments, *Statistics*, **21**, 613 – 618.
7. Glanzel, W. and Hamedani, G. G. (2001). Characterizations of univariate continuous Distributions. *Studia Scientiarum Mathematcarum Hungarica*, **37**, 83 – 118.
8. Glanzel, W., Telcs, A. and Schubert (1984). A Characterization by Truncated Moments and its application to Pearson-Type Distributions. *Z. Wahrsch. Verw. Gebiete*, **66**, 173 – 183.

9. Hamedani, G.G. (1993). Characterizations of Cauchy, Normal and Uniform Distributions. *Studia Scientiarum Mathematcarum Hungarica*, **28**, 243 – 247.
10. Hamedani, G.G. (2002). Characterizations of univariate continuous Distributions II. *Studia Scientiarum Mathematcarum Hungarica*, **39**, 407 – 424.
11. Hamedani, G.G. (2006). Characterizations of univariate continuous Distributions III. *Studia Scientiarum Mathematcarum Hungarica*, **43**, 361 – 385.
12. Hamedani, G.G. (2010). Characterizations of continuous univariate Distributions based on the Truncated Moments of Functions of Order Statistics. *Studia Scientiarum Mathematcarum Hungarica*, **47**, 462 – 484.
13. Hamedani, G.G., Ahsanullah, M. and Sheng, R. (2008). Characterizations of certain continuous univariate Distributions based on Truncated Moment of the first Order Statistic. *Aligrah Journal of Statistics*, **28**, 75 – 81.
14. Kotz, S. and Shanbhag, D.N. (1980). Some new approaches to probability Distributions. *Advances in Applied Probability*, **12**, 903 – 921.
15. Nadarajah, S. (2010). Distribution properties and estimation of the ratio of independent Weibull random variables. *Advances in Statistical Analysis*, **94**, 231 – 246.
16. Nadarajah, S. and Kotz, S. (2006). On the product and ratio of Gamma and Weibull random variables. *Econometric Theory*, **22**, 338 – 344.
17. Shakil, M. and Ahsanullah, M. (2011). Record values of the ratio of Rayleigh random variables, *Pakistan Journal of Statistics*, **27**, 307 – 325.
18. Shakil, M. and Kibria, B.M.G. (2010). On a family of life Distributions based on generalized Pearson differential equation with applications in health statistics. *Journal of Statistical Theory and Applications*, **9**, 255 – 282.
19. Shakil, M., Kibria, B.M.G. and Chang, K-C. (2007). Distributions of the product and ratio of Maxwell and Rayleigh random variables. *Statistical Papers*, **49**, 729 – 747.