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A New Class of Generalized Exponential Ratio and Product Type Estimators for Population Mean using Variance of an Auxiliary Variable

Hina Khan[1] *and Ahmad Faisal Siddiqi*[2]

Abstract

In this paper, a general class of Estimator and Exponential Ratio and Product Type of Estimators for estimating finite population mean in Single Phase Sampling is considered. The expression for the Bias and Mean Square Error (MSE) of the first order approximation of the proposed class of Estimators are given. The properties of the proposed Estimators have been analyzed for independent units under Simple Random Sampling without replacement (SRSWOR) with ignoring finite population Correction Factor. It has been shown that the proposed Ratio and Product Exponential Type Estimators are more efficient than Simple Random Sampling, Classical Ratio and Bhal and Tuteja (1991) Estimator.

Keywords

Mean Square Error, Bias, Efficiency

1. Introduction

History of using Auxiliary information in sample surveys is as old as the history of the applications of survey sampling itself. The Classical Ratio Estimator by Cochran (1954) and Product Estimator by Robson (1957), Murthy (1964) for population mean of estimand variable which are based on some Prior information of corresponding population mean of an Auxiliary variable are well known in the sample survey.

[1] Department of Statistics, GC University, Lahore, Pakistan.
Email: hinamuzaffar_1@hotmail.com
[2] Professor and Associate Dean, School of Business and Economics, University of Management and Technology, Lahore, Pakistan.

The Classical Ratio Type Estimator was first given by Cochran (2007), for estimating population mean based on some Prior information of population $\bar{X}$ of an Auxiliary variable X. Bahl and Tuteja (1991) proposed new Exponential Ratio and Product Type Estimators for estimating the mean of the finite population using information on single Auxiliary variable.

Consider a population of N units. Associated with the i[th] unit are the variables of main interest Y_I and the Auxiliary variables X_i for i= 1,2,3........N. Using symbol's scheme in Cochran (2007), Panse et al. (1961) and Sukhatme et al. (1953), Let $\bar{X}$ and $\bar{Y}$ are respectively the population mean of variables X and Y. The Classical Ratio Estimator of population mean $\bar{Y}$of the study variable Y are respectively, defined as

$$\bar{y}_R = \bar{X}\frac{\bar{y}}{\bar{x}} \tag{1.1}$$

The Estimator, in equation (1.1), is a Biased Estimator with a Bias of the order $\frac{1}{n}$ Cochran (2007). The loss at biasness results in the gain in precision. Its variance, as given in Sukhatme et al. (1953).

$$MSE(\bar{y}_R) = \frac{1-f}{n}\left(\mu_{20} + R^2\mu_{02}(1-2k)\right) \tag{1.2}$$

With k $= \frac{\rho}{R}\sqrt{\frac{\mu_{20}}{\mu_{02}}}$, $\mu_{20\ or\ S_y^2}$ and $\mu_{02\ or\ S_x^2}$ are variances of Y and X respectively, and $R = \frac{\bar{Y}}{\bar{X}}$ is the Ratio of population means.

The idea is, as Sukhatme et al. (1953) puts it, to produce more reliable Estimators by making use of Auxiliary information beside the information contained in the variable in question. This pursuit of developing more and more reliable Estimators keeps such efforts alive all the times. The Estimate is using Auxiliary information in terms of its means. Attempts have been made to replace, or augment, mean with other characteristics including variances, coefficients of variations, kurtosis, etc. If the coefficient of variation of these variables are denoted by $C_x^2 = \frac{s_x^2}{\bar{X}^2}$ and $C_y^2 = \frac{s_y^2}{\bar{Y}^2}$ respectively, Sisodia and Dwivedi (1981) and Singh and Kumar (2011) used these coefficients of variation, as:

$$\bar{y}_{SD} = \bar{y}\frac{\bar{X}+C_x}{\bar{x}+C_x} \tag{1.3}$$

Bhal and Tuteja (1991) suggested Exponential Ratio Type and Product Type

Estimators for population mean $\bar{Y}$ respectively, as:

$$\bar{y}_{Re} = \bar{y} exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \quad (1.4)$$

$$\bar{y}_{Rp} = \bar{y} exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)$$

Olkin (1958) discussed these Ratio Type Estimators with more than one Auxiliary variables. Singh (1965) developed Ratio Type Estimators and discussed the conditions when these are more precise than others. Hanif and Hammd (2010) developed Ratio Type Estimators for Two Phase Sampling. Murthy (1964) compared different Estimators for their relative efficiency and established that the Ratio Type Estimators are more efficient, especially for all cases when the percentage coefficient of variation is more than 50%. Oral and Kadilar (2011) Ratio Type Estimators are extremely useful to estimate the population mean when the correlation between study and Auxiliary variables is positively high. Most of these authors discussed the properties of Estimators along with their first order Bias and Mean Square Error. Sharma and Singh (2013) suggested some class of Estimators for estimating population variance in the presence of measurement errors. Dharmadhikari and Tankou (1991) worked on a multivariate version of the Ratio Type Estimator which, to a first order of approximation, is as efficient as the Regression Estimator. The Estimator is Biased but still more efficient than the conventional. These are only a few efforts in the domain of developing Ratio Type Estimators to give an idea.

For Simplicity, let us assumed that the population size is large compared to sample sizes so that the finite population correction terms may be ignored. Write

$$e_{\bar{y}} = \bar{y} - \bar{Y} \qquad e_{s_x^2} = s_x^2 - S_x^2 \qquad e_{\bar{x}} = \bar{x} - \bar{X} \quad (1.5)$$

$$E(e_{\bar{y}}^2) = \frac{\mu_{20}}{n} \qquad E(e_{\bar{x}}^2) = \frac{\mu_{02}}{n} \qquad E(e_{\bar{y}} e_{s_x^2}) = \frac{\mu_{12}}{n} \quad (1.6)$$

$$E\left(e_{\bar{y}\bar{x}}\right) = \frac{\mu_{11}}{n} E\left(e_{\bar{x}} e_{s_x^2}\right) = \frac{\mu_{03}}{n} E\left(e_{s_x^2}^2\right) = \frac{\mu_{02}^2}{n}[\beta_2 - 1]$$

$$\mu_{rs} = \frac{1}{N}\sum_{i=1}^{N}(Y_i - \bar{Y})^S (X_i - \bar{X})^r$$

2. Proposed Exponential Ratio Type Estimator

In this section, we propose some Exponential Ratio Type Estimator for Single Phase Sampling using single Auxiliary variable for estimating the population mean $\bar{Y}$, we define Exponential Ratio Estimator as:

$$\bar{y}_{ER} = \bar{y}\exp\left[\frac{S_X^2 - s_X^2}{S_X^2 + s_X^2}\right] \tag{2.1}$$

Using the notations given in equation (1.5)

$$\bar{y}_{ER} = \left(\bar{Y} + e_{\bar{y}}\right)exp\left[\frac{-e_{S_x^2}}{S_X^2 + (e_{S_x^2} + S_X^2)}\right] \tag{2.2}$$

Or

$$\bar{y}_{ER} = \left(\bar{Y} + e_{\bar{y}}\right)exp\left[\frac{-e_{S_x^2}}{2S_X^2}\left\{1 + \frac{e_{S_x^2}}{2S_X^2}\right\}^{-1}\right] \tag{2.3}$$

Assuming that sample is large enough, expanding the expression $exp\left[\frac{-e_{S_x^2}}{2S_X^2}\left\{1 + \frac{e_{S_x^2}}{S_X^2}\right\}^{-1}\right]$ and retaining the terms of first order, the

Mean Square Error is,

$$MSE\left(\bar{y}_{ER}\right) = E\left(e_{\bar{y}}^2\right) + \frac{\bar{Y}^2 E(e_{S_x^2}^2)}{4S_x^4} - \frac{\bar{Y}(e_{\bar{y}} e_{S_x^2})}{S_x^2} \tag{2.4}$$

After putting the expected values from equation (1.2) in (2.4), we get

$$MSE\left(\bar{y}_{ER}\right) = \frac{1}{n}\left[\mu_{20} + \frac{\bar{Y}^2}{4}\left(\beta_2 - 1\right) - \frac{\bar{Y}\mu_{12}}{\mu_{02}}\right] \tag{2.5}$$

where $\beta_2 = \frac{\mu_{04}}{\mu_{02}^2}$.

Expanding equation (2.3) up to degree two the Bias at 2nd order approximation is

$$Bias\left(\bar{y}_{ER}\right) = \frac{3\bar{Y}(\beta_2 - 1)}{8} - \frac{\mu_{12}}{2\mu_{02}} \tag{2.6}$$

3. Proposed Generalized Exponential Ratio Type Estimator

Motivated by Upadhyaya et al. (2011) and Sanaullah et al. (2012), a Generalized Exponential Type Estimator is visualized for estimating population mean. We define a Generalized Exponential Type Estimator by introducing variance as an Auxiliary variable is defined as:

$$\bar{y}_{GER} = \bar{y}\exp\left[\frac{S_X^2 - s_X^2}{S_X^2 + (a-1)s_X^2}\right] \tag{3.1}$$

where ‘a’ is an unknown constant not equal to zero.

Equation (3.1) can be written as:

$$(\bar{Y} + e_{\bar{y}})exp\left[\frac{-e_{s_x^2}}{S_x^2 + (a-1)\left(e_{s_x^2} + S_x^2\right)}\right] \tag{3.2}$$

or

$$(\bar{Y} - e_{\bar{y}})exp\left[\frac{-e_{s_x^2}}{aS_x^2}\left\{1 + \frac{(a-1)e_{s_x^2}}{aS_x^2}\right\}^{-1}\right] \tag{3.3}$$

Expanding equation (3.3) and retaining the terms up to first order, assuming a large sample, the mean square error is,

$$E(\bar{y}_{GER} - \bar{Y})^2 = E\left(e_{\bar{y}}^2\right) + \frac{\bar{Y}^2 E\left(e_{s_x^2}^2\right)}{a^2 S_x^4} - \frac{2\bar{Y}E\left(e_y e_{s_x^2}\right)}{aS_x^2} \tag{3.4}$$

Using equation (1.2), we get the MSE as:

$$MSE(\bar{y}_{GER}) = \frac{1}{n}\left[\mu_{20} + \frac{\bar{Y}^2}{a^2}(\beta_2 - 1) - \frac{2\bar{Y}\mu_{12}}{a\mu_{02}}\right] \tag{3.5}$$

Differentiating equation (3.5) with respect to ’a’ and setting equal to zero, we have optimal value after some simplification as:

$$a = \bar{Y}\frac{\mu_{02}}{\mu_{12}}(\beta_2 - 1) \tag{3.6}$$

Finally, the minimum MSE is given as:

$$MSE(\bar{y}_{GER})_{min} = \frac{1}{n}\left[\mu_{20} - \frac{\mu_{12}^2}{\mu_{02}^2(\beta_2 - 1)}\right] \tag{3.7}$$

The Bias of the estimator $\bar{y}_{GER}$ up to the 2nd order of approximation will be obtained as:

$$Bias(\bar{y}_{GER}) = \frac{(a-1)}{a^2}\bar{Y}(\beta_2 - 1) + \frac{\bar{Y}\mu_{02}(\beta_2 - 1)}{2a} - \frac{\mu_{12}}{a\mu_{02}} \tag{3.8}$$

3.1 Family of Exponential Ratio Type Estimators: In Table 1, the MSE's of first order and Bias of second order of proposed Estimator $\bar{y}_{ERE}$ under certain choices of the constant 'a'.

Table 1: MSE's and Bias of the proposed Exponential Ratio Type Estimators

Estimator		**a**	**MSE's and Biases**
T_1	$\bar{y}\exp\left[\frac{S_x^2 - s_x^2}{S_x^2}\right]$	1	$MSE(T_{1R}) = \frac{1}{n}\left[\mu_{20} + \bar{Y}^2(\beta_2 - 1) - \frac{2\bar{Y}\mu_{12}}{\mu_{02}}\right]$ Bias$(\bar{T}_1) = \frac{\bar{Y}(\beta_2-1)}{2} - \frac{\mu_{12}}{\mu_{02}}$
T_2	$\bar{y}\exp\left[\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right]$	2	$MSE(T_{2R}) = \frac{1}{n}\left[\mu_{20} + \frac{\bar{Y}^2}{4}(\beta_2 - 1) - \frac{\bar{Y}\mu_{12}}{\mu_{02}}\right]$ Bias$(\bar{T}_2) = \frac{3\bar{Y}(\beta_2-1)}{8} - \frac{\mu_{12}}{2\mu_{02}}$
T_3	$\bar{y}\exp\left[\frac{S_x^2 - s_x^2}{S_x^2 - \frac{s_x^2}{2}}\right]$	1/2	$MSE(T_{3R}) = \frac{1}{n}\left[\mu_{20} + 4\bar{Y}^2(\beta_2 - 1) - \frac{4\bar{Y}\mu_{12}}{\mu_{02}}\right]$ Bias$(\bar{T}_3) = \bar{Y}(\beta_2 - 1) + 2\frac{\mu_{12}}{\mu_{02}}$

3.2 Proposed Generalized Exponential Product Type Estimator: For estimating the population mean $\bar{Y}$, we define a Generalized Exponential Product Type Estimator as:

$$\bar{y}_{GEP} = \bar{y}\exp\left(\frac{s_x^2 - S_x^2}{S_x^2 + (b-1)s_x^2}\right) = \bar{y}\exp\left(-\left(\frac{S_x^2 - s_x^2}{S_x^2 + (b-1)s_x^2}\right)\right) \tag{3.2.1}$$

After using the procedure in equation (3.2) to (3.8), the expressions for MSE will be obtained as:

$$MSE(\bar{y}_{GEP}) = \mu_{02} + \frac{\bar{Y}^2}{b^2}\left(\frac{\mu_{04}}{\mu_{02}^2} - 1\right) + \frac{2\mu_{12}}{b\mu_{02}} \tag{3.2.2}$$

with the optimum value, for which the MSE of $\bar{y}_{GEP}$ is minimum as:

$$b = -\frac{\bar{Y}\mu_{02}}{\mu_{12}}\left(\frac{\mu_{04}}{\mu_{02}^2} - 1\right) \tag{3.2.3}$$

Similarly, the expression for the bias of $\bar{y}_{GEP}$ will be obtained as:

$$Bias(\bar{y}_{GER}) = \frac{(b-1)}{b^2}\bar{Y}(\beta_2 - 1) + \frac{\bar{Y}\mu_{02}(\beta_2-1)}{2b} + \frac{\mu_{12}}{b\mu_{02}} \tag{3.2.4}$$

3.2.1 Family of Exponential Product Type Estimators: In Table 2, the MSE's of first order of the proposed estimator $\bar{y}_{EPE}$ under certain choices of the constant 'a'.

Table 2: MSE's of the Proposed Product Type Estimators

Estimator		b	MSE's and Biases
T_1	$\bar{y}\exp\left[-\dfrac{S_x^2 - s_x^2}{S_x^2}\right]$	1	$MSE(T_{1P}) = \dfrac{1}{n}\left[\mu_{20} + \bar{Y}^2(\beta_2 - 1) + \dfrac{2\bar{Y}\mu_{12}}{\mu_{02}}\right]$
T_2	$\bar{y}\exp\left[-\dfrac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right]$	2	$MSE(T_{2P}) = \dfrac{1}{n}\left[\mu_{20} + \dfrac{\bar{Y}^2}{4}(\beta_2 - 1) + \dfrac{\bar{Y}\mu_{12}}{\mu_{02}}\right]$
T_3	$\bar{y}\exp\left[-\dfrac{S_x^2 - s_x^2}{S_x^2 - \dfrac{s_x^2}{2}}\right]$	1/2	$MSE(T_{3P}) = \dfrac{1}{n}\left[\mu_{20} + 4\bar{Y}^2(\beta_2 - 1) + \dfrac{4\bar{Y}\mu_{12}}{\mu_{02}}\right]$

4. Mathematical Comparison

The efficiency condition for the Proposed Estimators is as follows:

1. The estimator $\bar{y}_{GER}$ will be more precise estimator over SRS estimator $\bar{y}$ when

$$\frac{\mu_{20}}{n} - \frac{\mu_{12}^2}{n\mu_{02}^2(\beta_2-1)} < \frac{\mu_{20}}{n} \tag{4.1}$$

This holds only if $\beta_2 > 1$, and $MSE(\bar{y}_{GER})_{min}$ (3.7) is less than Var($\bar{y}$) in SRS without replacement.

2. The estimator $\bar{y}_{GER}$ will be more precise estimator over classical ratio estimator $\bar{y}_R$ if,

$$\frac{\mu_{20}}{n} - \frac{\mu_{12}^2}{n\mu_{02}^2(\beta_2-1)} < \frac{1}{n}[\mu_{20} + R^2\mu_{02} - 2R\mu_{11}] \tag{4.2}$$

if $\beta_2 > 1 - \frac{\mu_{12}^2}{(R^2-2R\mu_{11})\mu_{02}^2}$ *than* $MSE(\bar{y}_{GER})_{min}$(3.7) is less than MSE $(\bar{y}_R)$.

3. The estimator $\bar{y}_{GER}$ will be more precise estimator over Bahl and Tuteja (1991) ratio estimator $\bar{y}_{Re}$ if,

$$\frac{\mu_{20}}{n} - \frac{\mu_{12}^2}{n\mu_{02}^2(\beta_2-1)} < \frac{1}{n}\left[\mu_{20} + \frac{\bar{Y}^2\mu_{02}}{4\bar{X}^2} - \frac{\bar{Y}\rho\sqrt{\mu_{20}}\sqrt{\mu_{02}}}{\bar{X}}\right] \tag{4.3}$$

This holds only if $\bar{X}, \bar{Y}$ both are positive and $\rho < \frac{\bar{X}\mu_{12}^2}{\bar{Y}(\beta_2-1)\mu_{02}^{5/2}\mu_{20}^{1/2}} + \frac{\bar{Y}\sqrt{\mu_{02}}}{4\bar{X}\sqrt{\mu_{20}}}$, $MSE(\bar{y}_{GER})_{min}$ (3.7) is less than $MSE(\bar{y}_{Re})$.

4. The $MSE(\bar{y}_{GER})$ (3.5) $< Var(\bar{y})$ if $a > \frac{\bar{Y}(\beta_2-1)\mu_{02}}{2\mu_{12}}$.
5. The $MSE(\bar{y}_{GER})$ (3.5) $< MSE(\bar{y}_R)$ if $a > \frac{A}{\left[B^2+\left[R^2\mu_{02}-2R\mu_{11}\right]\right]^{1/2}+B}$.
6. The $MSE(\bar{y}_{GER})$ (3.5) $< MSE(\bar{y}_{Re})$ if $a > \frac{A}{\left[B^2+\frac{R^2\mu_{02}}{4}-R\rho\sqrt{\mu_{20}\mu_{02}}\right]^{1/2}+B}$.

where $A = \bar{Y}(\beta_2 - 1)^{1/2}$ and $B = \frac{\mu_{12}}{\mu_{02}(\beta_2-1)^{1/2}}$

Similarly, comparisons can be made for Product Estimators.

5. Conclusion

We have shown that the Generalized Exponential Ratio Estimator (GER) is more efficient than Simple Random Sampling (SRS) for populations with $\beta_2>1$. GER estimator is more efficient Ratio Estimator when equation 4.2 holds. Bhal and Tuteja (1991) Estimator is less efficient than GER Estimator when equation 4.3 holds.

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