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An Empirical Study on Robustness of Slope Rotatable Central Composite Designs

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Abstract

In this paper, an empirical study of Robustness of Slope Rotatable Central Composite Designs under Intra-class correlation structure is suggested. Here we studied the variance for different values of the Intra-class correlated coefficient (ρ) and the distance from the centre (d) empirically for the factors $2 \leq v \leq 8$ (v-number of factors).

Keywords

Second Order Slope Rotatable Designs (SOSRD), Robust Second Order Slope Rotatable Designs (RSOSRD), Correlated errors

1. Introduction

Box and Hunter (1957) introduced Rotatable Designs for the exploration of Response Surface Designs. Panda and Das (1994) introduced First Order Rotatable Designs with correlated errors. Das (1997) introduced Robust Second Order Rotatable Designs (RSORD).

In Response Surface methodology, good estimation of the derivatives of the Response function may be as important or perhaps more important than estimation of mean Response. Estimation of differences in Responses at two different points in the factor space will often be of great importance. If difference in Responses at two points close together is of interest then estimation of local slope (rate of change) of the Response is required.

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Estimation of slopes occurs frequently in practical situations. For instance, there are cases in which we want to estimate rate of reaction in chemical experiment, rate of change in the yield of a crop to various fertilizer doses, rate of disintegration of radioactive material in animal etc. (Park, 1987).

Hader and Park (1978) introduced Slope Rotatable Central Composite Designs (SRCCD). Park (1987) studied a class of Multi-factor Designs for estimating the slope of Response Surfaces. Victor Babu and Narasimham (1991) constructed Second Order Slope Rotatable Design (SOSRD) using Balanced Incomplete Block Designs (BIBD). Das (2003) introduced Slope Rotatability with correlated errors.

In this paper, following the work of Das (2003), an empirical study of Robustness of Slope Rotatable Central Composite Designs under correlation structure is suggested. Here we studied the variance for different values of the Intra-class correlated coefficient (ρ) and also obtained the distance from the centre (d) empirically for the factors $2 \le v \le 8$.

2. Conditions for SOSRD with Uncorrelated Errors (Hader and Park, 1978 and Victor Babu and Narasimham, 1991)

A Second Order Response Surface Design D = ((x_{iu})) for fitting

$$Y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum_{i=1}^{v} \sum_{i \le j=1}^{v} b_{ij} x_{iu} x_{ju} + e_u \qquad (2.1)$$

where x_{iu} denotes the level of the i^{th} factor (i=1,2,…,v) in the u^{th} run (u=1,2,…,N) of the experiment, e_u's are uncorrelated random errors with mean zero and variance σ^2, is said to be a SOSRD if the variance of the estimate of first order partial derivative of $Y_u(x_1, x_2, \ldots, x_v)$ with respect to each of independent variable (x_i) is only a function of the distance ($d^2 = \sum_{i=1}^{v} x_{iu}^2$) of the point ($x_1, x_2, \ldots, x_v$) from the origin (centre of the Design). Here b_0, b_i, b_{ij}, b_{ii} are the parameters of the Model and Y_u is the Response observed at the u^{th} Design point. A Second Order Response Surface Design D is said to be a SOSRD, if the Design points satisfy the following conditions (Hader and Park (1978) and Victor Babu and Narasimham (1991)):

$$\sum_{u=1}^{N} \prod_{i=1}^{v} x_{iu}^{\alpha_i} = 0 \text{ if any } \alpha_i \text{ is odd, for } \sum \alpha_i \le 4 \qquad (2.2)$$

(i) $\sum_{u=1}^{N} x_{iu}^2$ = constant = $N\lambda_2$

(ii) $\sum_{u=1}^{N} x_{iu}^4$ = constant = $cN\lambda_4$, for all i (2.3)

$$\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \text{constant} = N\lambda_4, \text{ for all } i \neq j \quad (2.4)$$

$$\sum_{u=1}^{N} x_{iu}^4 = c \sum_{u=1}^{N_1} x_{iu}^2 x_{ju}^2 \quad (2.5)$$

where c, λ_2 and λ_4 are constants and the summation is over the Design points.
The variances and covariances of the estimated parameters are,

$$V(\hat{b}_0) = \frac{\lambda_4(c+v-1)\sigma^2}{N[\lambda_4(c+v-1) - v\lambda_2^2]}$$

$$V(\hat{b}_i) = \frac{\sigma^2}{N\lambda_2}$$

$$V(\hat{b}_{ij}) = \frac{\sigma^2}{N\lambda_4}$$

$$V(\hat{b}_{ii}) = \frac{\sigma^2[\lambda_4(c+v-2) - (v-1)\lambda_2^2]}{(c-1)N\lambda_4[\lambda_4(c+v-1) - v\lambda_2^2]}$$

$$Cov(\hat{b}_0, \hat{b}_{ii}) = \frac{-\lambda_2\sigma^2}{N[\lambda_4(c+v-1) - v\lambda_2^2]}$$

$$Cov(\hat{b}_{ii}, \hat{b}_{jj}) = \frac{(\lambda_2^2-\lambda_4)\sigma^2}{(c-1)N\lambda_4[\lambda_4(c+v-1)-v\lambda_2^2]} \quad (2.6)$$

and other covariances vanish.

An inspection of the variance of $\hat{b}_0$ shows that a necessary condition for the existence of a Non-Singular Second Order Design is,

$$[\lambda_4(c+v-1) - v\lambda_2^2] > 0 \text{ i.e., } \frac{\lambda_4}{\lambda_2^2} > \frac{v}{c+v-1} \text{ (non-singularity condition)} \quad (2.7)$$

For the Second Order Model

$$\frac{\partial \hat{y}_x}{\partial x_i} = \hat{b}_i + 2\hat{b}_{ii}x_i + \sum_{j=1;j\neq i}^{v} \hat{b}_{ij}\, x_j \quad (2.8)$$

$$V\left(\frac{\partial \hat{y}_x}{\partial x_i}\right) = V(\hat{b}_i) + 4x_i^2 V(\hat{b}_{ii}) + \sum_{j=1;j\neq i}^{v} x_j^2\, V(\hat{b}_{ij}) \quad (2.9)$$

The condition for right hand side of (2.9) to be a function of $d^2 = \sum_{i=1}^{v} x_{iu}^2$ alone (for Slope Rotatability) is clearly,

$$V(\hat{b}_{ii}) = \frac{1}{4} V(\hat{b}_{ij}) \quad (2.10)$$

(2.2) to (2.6) and (2.10) lead to condition

$$\lambda_4[v(5-c) - (c-3)^2] + \lambda_2^2[v(c-5) + 4] = 0 \quad (2.11)$$

Therefore, equation (2.2) to (2.5), (2.7) and (2.11) give a set of conditions for Slope Rotatability in any general Second Order Response Surface Design.
Further,

$$V\left(\frac{\partial \hat{y}_x}{\partial x_i}\right) = \frac{1}{N}\left(\frac{1}{\lambda_2} + \frac{d^2}{\lambda_4}\right)\sigma^2 \tag{2.12}$$

3. Conditions for Second Order Slope Rotatable Designs with Correlated Errors (Das, 2003)

A Second Order Response Surface Design D = ((x_{iu})) for fitting is,

$$Y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum_{i=1}^{v}\sum_{i\le j=1}^{v} b_{ij} x_{iu} x_{ju} + e_u \tag{3.1}$$

where x_{iu} denotes the level of the i^{th} factor (i=1,2,…,v) in the u^{th} run (u=1,2,…,N) of the experiment, e_u's are correlated random errors, is said to be a RSOSRD, if the variance of the estimate of first order partial derivative of $Y_u(x_1, x_2, \ldots, x_v)$ with respect to each of independent variable (x_i) is only a function of the distance ($d^2 = \sum_{i=1}^{v} x_{iu}^2$) of the point ($x_1, x_2, \ldots, x_v$) from the origin (centre of the Design). Such a spherical variance function for estimation of slopes in the Second Order Response Surface is achieved if the Design points satisfy the following conditions (Das (2003)):

$$\sum_{u=1}^{N_1}\prod_{i=1}^{v} x_{iu}^{\alpha_i} = 0 \quad \text{if any } \alpha_i \text{ is odd, for } \sum \alpha_i \le 4 \tag{3.2}$$

(i) $\sum_{u=1}^{N_1} x_{iu}^2$ = constant = $N_1\lambda_2$

(ii) $\sum_{u=1}^{N_1} x_{iu}^4$ = constant = $N_1\lambda_4$, for all i (3.3)

$\sum_{u=1}^{N_1} x_{iu}^2 x_{ju}^2$ = constant = $N_1\lambda_4$, for all $i \neq j$ (3.4)

$$\sum_{u=1}^{N_1} x_{iu}^4 = c\sum_{u=1}^{N_1} x_{iu}^2 x_{ju}^2 \tag{3.5}$$

where c, λ_2 and λ_4 are constants and the summation is over the Design points.
The variances and covariances of the estimated parameters are,

$$V(\hat{b}_0) = \frac{[\lambda_4(c+v-1)\{1+(N_1-1)\rho\} - v\rho N_1\lambda_2^2]\sigma^2\{1+(N_1-1)\rho\}}{N_1\Delta}$$

$$V(\hat{b}_i) = \frac{\sigma^2(1-\rho)}{N_1\lambda_2}$$

$$V(\hat{b}_{ij}) = \frac{\sigma^2(1-\rho)}{N_1\lambda_4}$$

$$V(\hat{b}_{ii}) = \frac{\sigma^2(1-\rho)[\lambda_4(c+v-2)\{1+(N_1-1)\rho\} - (v-1)\rho N_1\lambda_2^2 - (v-1)\lambda_2^2(1-\rho)]}{(c-1)N_1\lambda_4\Delta}$$

$$Cov(\hat{b}_0, \hat{b}_{ii}) = -\frac{\lambda_2\sigma^2(1-\rho)\{1+(N_1-1)\rho\}}{N_1\Delta}$$

$$Cov(\hat{b}_{ii}, \hat{b}_{jj}) = \frac{\sigma^2(1-\rho)[\lambda_2^2(1-\rho) - \lambda_4\{1+(N_1-1)\rho\} + N_1\rho\lambda_2^2]}{(c-1)N_1\lambda_4\Delta} \tag{3.6}$$

where $\Delta = [\lambda_4(c+v-1)\{1+(N_1-1)\rho\} - v\rho N_1\lambda_2^2 - v\lambda_2^2(1-\rho)]$
and other covariances vanish.

An inspection of the variance of $\hat{b}_0$ shows that a necessary condition for the existence of a Non-Singular Robust Second Order Slope Rotatable Design is,

$$[\lambda_4(c+v-1)\{1+(N_1-1)\rho\} - v\rho N_1\lambda_2^2 - v\lambda_2^2(1-\rho)] > 0 \tag{3.7}$$

i.e., $\frac{\lambda_4}{\lambda_2^2} > \frac{v\{1+(N_1-1)\rho\}}{(c+v-1)\{1+(N_1-1)\rho\}}$ (non-singularity condition) (3.8)

For the Second Order Model

$$\frac{\partial\hat{y}_x}{\partial x_i} = \hat{b}_i + 2\hat{b}_{ii}x_i + \sum_{j=1;j\neq i}^{v}\hat{b}_{ij}\,x_j \tag{3.9}$$

$$V\left(\frac{\partial\hat{y}_x}{\partial x_i}\right) = V(\hat{b}_i) + 4x_i^2V(\hat{b}_{ii}) + \sum_{j=1;j\neq i}^{v}x_j^2\,V(\hat{b}_{ij}) \tag{3.10}$$

The condition for right hand side of equation (3.9) to be a function of $d^2 = \sum_{i=1}^{v}x_{iu}^2$ alone (for Slope Rotatability) is clearly,

$$V(\hat{b}_{ii}) = \frac{1}{4}V(\hat{b}_{ij}) \tag{3.11}$$

Equation (3.2) to (3.6) and (3.11) lead to condition (Das (2003)),

$$\left(\frac{\{1+(N-1)\rho\}cN_1\lambda_4 - \rho N_1^2\lambda_2^2}{(1-\rho)}\right)\left[4N_1 - \left(\frac{\{1+(N-1)\rho\}cN_1\lambda_4 - \rho N_1^2\lambda_2^2}{(1-\rho)}\right)\right.$$
$$\left.+v\left(\frac{N_1\lambda_2^2(1-\rho)}{\lambda_4\{1+(N_1-1)\rho\}}\right) - (v-2)\left(\frac{\{1+(N_1-1)\rho\}N_1\lambda_4 - \rho N_1^2\lambda_2^2}{N_1\lambda_4\{1+(N_1-1)\rho\}}\right)\right]$$
$$+\frac{N_1[\{1+(N_1-1)\rho\}N_1\lambda_4 - \rho N_1^2\lambda_2^2]}{(1-\rho)}\left[4(v-2) + (v-1)\right.$$
$$\left.\left(\frac{\{1+(N_1-1)\rho\}N_1\lambda_4 - \rho N_1^2\lambda_2^2}{N_1\lambda_4\{1+(N_1-1)\rho\}}\right)\right]$$

$$-N_1^2\lambda_2^2\left[4(v-1)+v\left(\frac{\{1+(N_1-1)\rho\}N_1\lambda_4-\rho N_1^2\lambda_2^2}{N_1\lambda_4\{1+(N_1-1)\rho\}}\right)\right]=0 \qquad (3.12)$$

For $\rho=0$, equation (3.11) reduces to SOSRD condition of Victor Babu and Narasimham (1991).

$$\lambda_4[v(5-c)-(c-3)^2]+\lambda_2^2[v(c-5)+4]=0 \qquad (3.13)$$

Therefore, equation (3.2) to (3.5), (3.8) and (3.12) give a set of conditions for Robust Slope Rotatability in any general Second Order Response Surface Design. Further,

$$V\left(\frac{\partial\hat{y}_x}{\partial x_i}\right)=\frac{(1-\rho)}{N_1}\left(\frac{1}{\lambda_2}+\frac{d^2}{\lambda_4}\right)\sigma^2 \qquad (3.14)$$

4. An Empirical Study on Robust Slope Rotatable Central Composite Designs

The widely used Design for fitting a Second Order Model is the Central Composite Design. These Designs are obtained by adding suitable factorial combinations to those obtained from $\frac{1}{2^p}\times 2^v$ Fractional Factorial Design, where $\frac{1}{2^p}\times 2^v$ is a suitable fractional replicate of 2^v, in which no interaction with less than five factors is confounded. The 2v or 2v+1 additional factorial combination, in Central Composite Designs are,

±a	0	0	⋯	0
0	±a	0	⋯	0
0	0	±a	⋯	0
⋯	⋯	⋯	⋯	⋯
⋯	⋯	⋯	⋯	⋯
0	0	0	⋯	±a

and at least one central point (0,0,0, ⋯, 0), if necessary (n_0) central points. Thus the total number of factorial combinations in this Design (N) is $2^{(v-p)}+2v+n_0$.

4.1 Intra-class Structure: Intra-class structure is the variance-covariance structure when errors of any two observations have the same correlation and each has the same variance.

Let ρ $\left(-\frac{1}{N_1-1} < \rho < 1\right)$ be the correlation between errors of any two observations, each having the same variance σ^2.

Following Das (1997, 2003) methods of construction of RSORD and RSOSRD, here we consider a Slope Rotatable Central Composite Design of Hader and Park (1978) having 'n' Non-central Design points involving v-factors. The set of n Design points can be extended to (2n+1) points by incorporating (n+1) central points in the following way. One central point is placed in between each pair of Non-central Design points in the sequence, resulting thereby in (n-1) such central points. The other two central points are placed one at the beginning and one at the end.

Let N^* be the number of Design points of the Robust Slope Rotatable Central Composite Designs in which n be the number of Non-central Design points and m be the number of central points placed in between the each pair of the axial points, i.e., N^*=n+m. Let N_1 be the total number of Design points, where N_1=2n+1 and n= $2^{t(k)}$+2v.

For the Design points generated from the Central Composite Design (CCD), simple symmetry conditions, equation (3.3) to (3.5) are true. Further, from equation (3.4) and (3.5), we have

$$\sum_{u=1}^{N_1} x_{iu}^2 = 2^{t(v)} + 2a^2 = \text{constant} = N_1\lambda_2, 1 \le i \le v \tag{4.1}$$

$$\sum_{u=1}^{N_1} x_{iu}^4 = 2^{t(v)} + 2a^4 = \text{constant} = cN_1\lambda_4, 1 \le i \le v \tag{4.2}$$

$$\sum_{u=1}^{N_1} x_{iu}^2 x_{ju}^2 = 2^{t(v)} = \text{constant} = N_1\lambda_4, 1 \le i \ne j \le v \tag{4.3}$$

Substituting λ_2, λ_4 and c in equation (3.12), we get the following bi-quadratic equation,

$$\begin{aligned}
&[(8v(1-\rho+N_1\rho)-4N_1\{1+(N_1-1)\rho\})]\{1+(N_1-1)\rho\}a^8 \\
&+\left[2^{t(v)+3}v\ \{1+(N_1-1)\rho\}[1-\rho+N_1\rho]a^6-32N_1\rho^2 2^{t(v)}\right]a^6+ \\
&\begin{bmatrix} 2^{t(v)+1}(4-v)N_1(\{1+(N_1-1)\rho\})^2+2^{2t(v)+1}v\{1+(N_1-1)\rho\}[1-\rho+N_1\rho] \\ +2^{t(v)+4}\{1+(N_1-1)\rho\}(1-v-\rho+N_1\rho+v\rho-vN_1\rho) \end{bmatrix} a^4+ \\
&\left[2^{2t(v)+4}(1-v-\rho+N_1\rho+v\rho-vN_1\rho)\{1+(N_1-1)\rho\}\right]a^2+\left[2^{2t(v)+2}(v-1)N_1(\{1+\right. \\
&\left.(N_1-1)\rho\})^2+2^{3t(v)+2}(1-v-\rho+N_1\rho+v\rho-vN_1\rho)\{1+(N_1-1)\rho\}\right]=0
\end{aligned} \tag{4.4}$$

If at least one positive real root exists for the above equation (4.4) then only the Design exists. Solving equation (4.4) we get the Robust Slope Rotatable Central Composite Designs values 'a' for different factors (v).

The Robustness of Slope Rotatable Central Composite Design values 'a' and the variance of estimated slopes for different values of ρ (0 to 1) for the factors $2 \leq v \leq 8$ are given in Table 1.

Note: If ρ=0, then equation (4.4) reduces to Hader and Park (1978) Slope Rotatable Central Composite Design (SRCCD) equations with N_1 Design points as given below:

$$(8v - 4N_1)a^8 + 2^{t(v)+3}va^6 + \left[2^{t(v)+1}(4 - v)N_1 + 2^{2t(v)+1}v + (1 - v)2^{t(v)+4}\right]a^4 \\ +(1 - v)2^{2t(v)+4}a^2 + \left[2^{2t(v)+2}(v - 1)N_1 + (1 - v)2^{3t(v)+2}\right] = 0 \quad (4.5)$$

4.2 Example: An empirical study on Robustness of Slope Rotatable Central Composite Design for v=2 factors is given below. Here N^*=12, N_1=17, n=8, m=4. From equations (4.1), (4.2) and (4.3), we have

$$\sum_{u=1}^{N_1} x_{iu}^2 = 4 + 2a^2 = N_1\lambda_2 \quad (4.6)$$

$$\sum_{u=1}^{N_1} x_{iu}^4 = 4 + 2a^4 = c\,N_1\lambda_4 \quad (4.7)$$

$$\sum_{u=1}^{N_1} x_{iu}^2 x_{ju}^2 = 4 = N_1\lambda_4 \quad (4.8)$$

Substituting v=2, N_1=17 in equation (4.4) and on simplification, we get the following bi-quadratic equation:

$$(1 + 16\rho)^2[52a^8 - 64a^6 - 272a^4 + 256a^2 - 832] = 0 \quad (4.9)$$

Solving equation (4.9) we get a=1.7352, ∀ρ.

5. Study of Dependence of Variance Function of Response at Different Design Points of Robustness of Slope Rotatable Central Composite Designs for Different Values of the Intra-class Correlation Coefficient ρ and Distance 'd' from the Centre

In this section, we study the dependence of variance function of Response at different Design points of Robustness of Slope Rotatable Central Composite Designs for given 'v' and for different values of Intra-class correlation coefficient 'ρ' and distance 'd' from the centre for d=0.1 (0.1) 1. The variance of the estimated derivative is given by, (∴ from equation (3.14)).

$$V\left(\frac{\partial \hat{y}_x}{\partial x_i}\right) = V\left(\hat{b}_i\right) + 4x_i^2 V\left(\hat{b}_{ii}\right) + \sum_{j=1;j\neq i}^{v} x_j^2\, V\left(\hat{b}_{ij}\right) \quad (5.1)$$

$$= \frac{\sigma^2(1-\rho)}{N_1\lambda_2} + \frac{\sigma^2(1-\rho)}{N_1\lambda_4}\, d^2 \quad (5.2)$$

The numerical calculations are appended in Table 2.

5.1 Conclusion: From the Table 2, we observe that,

(i) For given v, ρ and d=0.1 (0.1) 1.0, $V\left(\frac{\partial \hat{y}_x}{\partial x_i}\right)$ is Robustly increasing.

(ii) $V\left(\frac{\partial \hat{y}_x}{\partial x_i}\right)$ is decreasing when the values of ρ are increasing.

5.2 Example:

For v=2 factors

$V\left(\frac{\partial \hat{y}_x}{\partial x_i}\right) = 0.0998\sigma^2 (1-\rho) + 0.2500\sigma^2 (1-\rho)\, d^2$ (using (5.2))

$V\left(\frac{\partial \hat{y}_x}{\partial x_i}\right) = 0.0998\sigma^2 + 0.2500\sigma^2 d^2$

$V\left(\frac{\partial \hat{y}_x}{\partial x_i}\right) = 0.1023$ (by taking d=0.1, ρ=0 and σ=1).

In particular, for v=2 factors almost all ‘d’ values are Robustly decreasing when the values of ‘ρ’ are increasing.

Graphical representation for Robustness of Slope Rotatable Central Composite Designs for v=2 factors is given in Figures 1 and 2. The Figures give the variance of the estimated slope for different values of ρ and distance ‘d’ from the centre for the factors 2≤v≤8.

Table 1: The variance of estimated derivatives (Slopes) for the factors $2 \leq v \leq 8$

ρ	v=2, N_1=17, a=1.7352	v=3, N_1=29, a=1.7096	v=4, N_1=49, a=2.3707	v=5, N_1=53, a=2.3611
	$V\left(\frac{\partial \hat{y}_x}{\partial x_i}\right)$	$V\left(\frac{\partial \hat{y}_x}{\partial x_i}\right)$	$V\left(\frac{\partial \hat{y}_x}{\partial x_i}\right)$	$V\left(\frac{\partial \hat{y}_x}{\partial x_i}\right)$
0	$0.0998\sigma^2+0.2500\sigma^2d^2$	$0.0620\sigma^2+0.1250\sigma^2d^2$	$0.0367\sigma^2+0.0625\sigma^2d^2$	$0.0368\sigma^2+0.0625\sigma^2d^2$
0.1	$0.0898\sigma^2+0.2250\sigma^2d^2$	$0.0558\sigma^2+0.1125\sigma^2d^2$	$0.0330\sigma^2+0.0563\sigma^2d^2$	$0.0331\sigma^2+0.0562\sigma^2d^2$
0.2	$0.0798\sigma^2+0.2000\ \sigma^2d^2$	$0.0496\sigma^2+0.1000\sigma^2d^2$	$0.0294\sigma^2+0.0500\sigma^2d^2$	$0.0295\sigma^2+0.0500\sigma^2d^2$
0.3	$0.0699\sigma^2+0.1750\ \sigma^2d^2$	$0.0434\sigma^2+0.0875\sigma^2d^2$	$0.0257\sigma^2+0.0438\sigma^2d^2$	$0.0258\sigma^2+0.04370\sigma^2d^2$
0.4	$0.0599\sigma^2+0.1500\ \sigma^2d^2$	$0.0372\sigma^2+0.0750\sigma^2d^2$	$0.0220\sigma^2+0.0375\sigma^2d^2$	$0.0221\sigma^2+0.0375\sigma^2d^2$
0.5	$0.0499\sigma^2+0.1250\sigma^2d^2$	$0.0310\sigma^2+0.0625\sigma^2d^2$	$0.0184\sigma^2+0.0313\sigma^2d^2$	$0.0184\sigma^2+0.0312\sigma^2d^2$
0.6	$0.0399\sigma^2+0.1000\sigma^2d^2$	$0.0248\sigma^2+0.0500\sigma^2d^2$	$0.0147\sigma^2+0.0250\sigma^2d^2$	$0.0147\sigma^2+0.0250\sigma^2d^2$
0.7	$0.0299\sigma^2+0.0750\sigma^2d^2$	$0.0186\sigma^2+0.0375\sigma^2d^2$	$0.0110\sigma^2+0.0188\sigma^2d^2$	$0.0110\sigma^2+0.0187\sigma^2d^2$
0.8	$0.0200\sigma^2+0.0500\sigma^2d^2$	$0.0124\sigma^2+0.0250\sigma^2d^2$	$0.0073\sigma^2+0.0125\sigma^2d^2$	$0.0074\sigma^2+0.01250\sigma^2d^2$
0.9	$0.0100\sigma^2+0.0250\sigma^2d^2$	$0.0062\sigma^2+0.0125\sigma^2d^2$	$0.0037\sigma^2+0.9039\sigma^2d^2$	$0.0037\sigma^2+0.0062\sigma^2d^2$

ρ	v=6, N_1=89, a=2.7997	v=7, N_1=157, a=3.3239	v=8, N_1=161, a=3.3240
	$V\left(\frac{\partial \hat{y}_x}{\partial x_i}\right)$	$V\left(\frac{\partial \hat{y}_x}{\partial x_i}\right)$	$V\left(\frac{\partial \hat{y}_x}{\partial x_i}\right)$
0	$0.0213\ \sigma^2+0.0312\sigma^2d^2$	$0.0116\sigma^2+0.0156\sigma^2d^2$	$0.0116\sigma^2+0.0156\sigma^2d^2$
0.1	$0.0189\ \sigma^2+0.0281\sigma^2d^2$	$0.0105\sigma^2+0.0141\sigma^2d^2$	$0.0105\sigma^2+0.0141\sigma^2d^2$
0.2	$0.0168\ \sigma^2+0.0250\sigma^2d^2$	$0.0093\sigma^2+0.0125\sigma^2d^2$	$0.0093\sigma^2+0.0125\sigma^2d^2$
0.3	$0.0147\ \sigma^2+0.0219\sigma^2d^2$	$0.0081\sigma^2+0.0109\sigma^2d^2$	$0.0081\sigma^2+0.0109\sigma^2d^2$
0.4	$0.0126\ \sigma^2+0.0187\sigma^2d^2$	$0.0070\sigma^2+0.0094\sigma^2d^2$	$0.0070\sigma^2+0.0094\sigma^2d^2$
0.5	$0.0105\ \sigma^2+0.0156\sigma^2d^2$	$0.0058\sigma^2+0.0078\sigma^2d^2$	$0.0058\sigma^2+0.0078\sigma^2d^2$
0.6	$0.0084\ \sigma^2+0.0125\sigma^2d^2$	$0.0046\sigma^2+0.0063\sigma^2d^2$	$0.0046\sigma^2+0.0063\sigma^2d^2$
0.7	$0.0063\ \sigma^2+0.0094\sigma^2d^2$	$0.0035\sigma^2+0.0047\sigma^2d^2$	$0.0035\sigma^2+0.0047\sigma^2d^2$
0.8	$0.0042\ \sigma^2+0.0062\sigma^2d^2$	$0.0023\sigma^2+0.0031\sigma^2d^2$	$0.0023\sigma^2+0.0031\sigma^2d^2$
0.9	$0.0021\ \sigma^2+0.0031\sigma^2d^2$	$0.0012\sigma^2+0.0016\sigma^2d^2$	$0.0012\sigma^2+0.0016\sigma^2d^2$

NB: $V\left(\frac{\partial \hat{y}_x}{\partial x_i}\right)$ is obtained by using the equation (5.2).

Table 2: Study of dependence of estimated Slope of Robustness of Slope Rotatable Central Composite Designs at different Design points for 2≤v≤8 for different values of ‘ρ

v = 2

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0633	0.0670	0.0730	0.0820	0.0930	0.1070	0.1233	0.1420	0.1633	0.187
0.1	0.0569	0.0603	0.0660	0.0738	0.0840	0.0963	0.1109	0.1278	0.1469	0.1683
0.2	0.0506	0.0536	0.0590	0.0656	0.0750	0.0856	0.0986	0.1136	0.1306	0.1496
0.3	0.0443	0.0469	0.0510	0.0574	0.0650	0.0749	0.0863	0.0994	0.1143	0.1309
0.4	0.0380	0.0402	0.0440	0.0492	0.0560	0.0642	0.0740	0.0852	0.098	0.1122
0.5	0.0316	0.0335	0.0370	0.0410	0.0470	0.0535	0.0616	0.0710	0.0816	0.0935
0.6	0.0253	0.0268	0.0290	0.0328	0.0370	0.0428	0.0493	0.0568	0.0653	0.0748
0.7	0.0190	0.0201	0.0220	0.0246	0.0280	0.0321	0.0370	0.0426	0.0490	0.0561
0.8	0.0127	0.0134	0.0150	0.0164	0.0190	0.0214	0.0247	0.0284	0.0327	0.0374
0.9	0.0063	0.0067	0.0070	0.0082	0.0090	0.0107	0.0123	0.0142	0.0163	0.0187

v = 3

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.1023	0.1098	0.1220	0.1398	0.1620	0.1898	0.2223	0.2598	0.3023	0.3498
0.1	0.0921	0.0988	0.1100	0.1258	0.1460	0.1708	0.2001	0.2338	0.2721	0.3148
0.2	0.0818	0.0878	0.0980	0.1118	0.1300	0.1518	0.1778	0.2078	0.2418	0.2798
0.3	0.0717	0.0769	0.0860	0.0979	0.1140	0.1329	0.1557	0.1819	0.2117	0.2449
0.4	0.0614	0.0659	0.0730	0.0839	0.0970	0.1139	0.1334	0.1559	0.1814	0.2099
0.5	0.0512	0.0549	0.0610	0.0699	0.0810	0.0949	0.1112	0.1299	0.1512	0.1749
0.6	0.0409	0.0439	0.0490	0.0559	0.0650	0.0759	0.0889	0.1039	0.1209	0.1399
0.7	0.0307	0.0329	0.0370	0.0419	0.0490	0.0569	0.0667	0.0779	0.0907	0.1049
0.8	0.0205	0.0220	0.0250	0.0280	0.0330	0.038	0.0445	0.0520	0.0605	0.0700
0.9	0.0103	0.0110	0.0120	0.0140	0.0160	0.019	0.0223	0.0260	0.0303	0.0350

v = 4

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0373	0.0392	0.0423	0.0467	0.0523	0.0592	0.0673	0.0767	0.0873	0.0992
0.1	0.0336	0.035252	0.0381	0.04201	0.0471	0.0533	0.0606	0.069	0.0786	0.0893
0.2	0.0299	0.0314	0.0339	0.0374	0.0419	0.0474	0.0539	0.0614	0.0699	0.0794
0.3	0.0261	0.027452	0.0296	0.03271	0.0367	0.0415	0.0472	0.0537	0.0612	0.0695
0.4	0.0224	0.0235	0.0254	0.028	0.0314	0.0355	0.0404	0.046	0.0524	0.0595
0.5	0.0187	0.019652	0.0212	0.02341	0.0262	0.0297	0.0337	0.0384	0.0438	0.0497
0.6	0.015	0.0157	0.017	0.0187	0.021	0.0237	0.027	0.0307	0.035	0.0397
0.7	0.0112	0.011752	0.0127	0.01401	0.0157	0.0178	0.0202	0.023	0.0262	0.0298
0.8	0.0074	0.0078	0.0084	0.0093	0.0104	0.0118	0.0134	0.0153	0.0174	0.0198
0.9	0.0038	0.003952	0.0043	0.00471	0.0053	0.006	0.0068	0.0077	0.0088	0.01

v = 5

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0374	0.0393	0.0424	0.0468	0.0524	0.0593	0.0674	0.0768	0.0874	0.0993
0.1	0.0337	0.0353	0.0382	0.0421	0.0472	0.0533	0.0606	0.0691	0.0786	0.0893
0.2	0.0300	0.0315	0.0340	0.0375	0.0420	0.0475	0.0540	0.0615	0.0700	0.0795
0.3	0.0262	0.0275	0.0297	0.0328	0.0367	0.0415	0.0472	0.0538	0.0612	0.0695
0.4	0.0225	0.0236	0.0255	0.0281	0.0315	0.0356	0.0405	0.0461	0.0525	0.0596
0.5	0.0187	0.0196	0.0212	0.0234	0.0262	0.0296	0.0337	0.0384	0.0437	0.0496
0.6	0.015	0.0157	0.017	0.0187	0.021	0.0237	0.027	0.0307	0.035	0.0397
0.7	0.0112	0.0117	0.0127	0.0140	0.0157	0.0177	0.0202	0.0230	0.0261	0.0297
0.8	0.0075	0.0079	0.0085	0.0094	0.0105	0.0119	0.0135	0.0154	0.0175	0.0199
0.9	0.0038	0.0039	0.0043	0.0047	0.0053	0.0059	0.0067	0.0077	0.0087	0.0099

v = 6

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0213	0.0222	0.0238	0.0260	0.0288	0.0322	0.0363	0.0410	0.0463	0.0522
0.1	0.0192	0.0200	0.0214	0.02340	0.0259	0.0290	0.0327	0.0369	0.0417	0.047
0.2	0.0171	0.0178	0.0191	0.0208	0.0231	0.0258	0.0291	0.0328	0.0371	0.0418
0.3	0.0149	0.0156	0.0167	0.0182	0.0202	0.0226	0.0254	0.0287	0.0324	0.0366
0.4	0.0128	0.0133	0.0143	0.0156	0.0173	0.0193	0.0218	0.0246	0.0277	0.0313
0.5	0.0107	0.0111	0.0119	0.0130	0.0144	0.0161	0.0181	0.0205	0.0231	0.0261
0.6	0.0085	0.0089	0.0095	0.0104	0.0115	0.0129	0.0145	0.0164	0.0185	0.0209
0.7	0.0064	0.0067	0.0072	0.0078	0.0087	0.0097	0.0109	0.0123	0.0139	0.0157
0.8	0.0043	0.0044	0.0048	0.0052	0.0058	0.0064	0.0072	0.0082	0.0092	0.0104
0.9	0.0021	0.0022	0.0024	0.0026	0.0029	0.0032	0.0036	0.0041	0.0046	0.0052

v = 7

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0118	0.0122	0.0130	0.0141	0.0160	0.0172	0.0192	0.0216	0.0242	0.0272
0.1	0.0106	0.0111	0.0118	0.0128	0.0140	0.0156	0.0174	0.0195	0.0219	0.0246
0.2	0.0094	0.0098	0.0104	0.0113	0.0120	0.0138	0.0154	0.0173	0.0194	0.0218
0.3	0.0082	0.0085	0.0091	0.0098	0.0110	0.012	0.0134	0.0151	0.0169	0.019
0.4	0.0071	0.0074	0.0078	0.0085	0.0090	0.0104	0.0116	0.013	0.0146	0.0164
0.5	0.0059	0.0061	0.0065	0.0070	0.0080	0.0086	0.0096	0.0108	0.0121	0.0136
0.6	0.0047	0.0049	0.0052	0.0056	0.0060	0.0069	0.0077	0.0086	0.0097	0.0109
0.7	0.0035	0.0037	0.0039	0.0043	0.0050	0.0052	0.0058	0.0065	0.0073	0.0082
0.8	0.0023	0.0024	0.0026	0.0028	0.0030	0.0034	0.0038	0.0043	0.0048	0.0054
0.9	0.0012	0.0013	0.0013	0.0015	0.0020	0.0018	0.002	0.0022	0.0025	0.0028

v = 8

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0118	0.0122	0.0130	0.0141	0.0155	0.0172	0.0192	0.0216	0.0242	0.0272
0.1	0.0106	0.0111	0.0118	0.0128	0.014	0.0156	0.0174	0.0195	0.0219	0.0246
0.2	0.0094	0.0098	0.0104	0.0113	0.0124	0.0138	0.0154	0.0173	0.0194	0.0218
0.3	0.0082	0.0085	0.0091	0.0098	0.0108	0.012	0.0134	0.0151	0.0169	0.019
0.4	0.0071	0.0074	0.0078	0.0085	0.0094	0.0104	0.0116	0.013	0.0146	0.0164
0.5	0.0059	0.0061	0.0065	0.007	0.0078	0.0086	0.0096	0.0108	0.0121	0.0136
0.6	0.0047	0.0049	0.0052	0.0056	0.0062	0.0069	0.0077	0.0086	0.0097	0.0109
0.7	0.0035	0.0037	0.0039	0.0043	0.0047	0.0052	0.0058	0.0065	0.0073	0.0082
0.8	0.0023	0.0024	0.0026	0.0028	0.0031	0.0034	0.0038	0.0043	0.0048	0.0054
0.9	0.0012	0.0013	0.0013	0.0015	0.0016	0.0018	0.002	0.0022	0.0025	0.0028

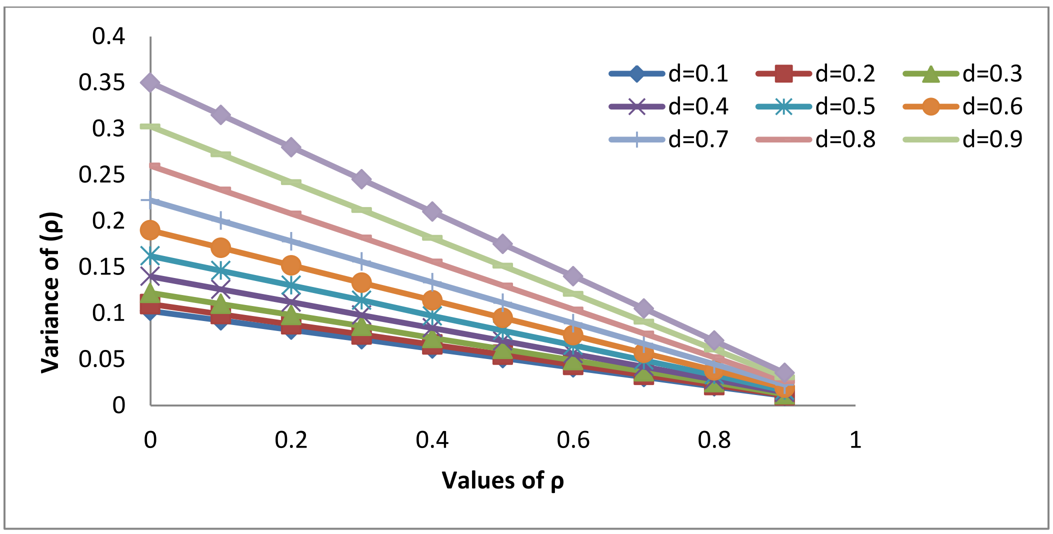

Figure 1: Graphical representation for Robustness of Second Order Slope Rotatable Central Composite Design for v=2 factor

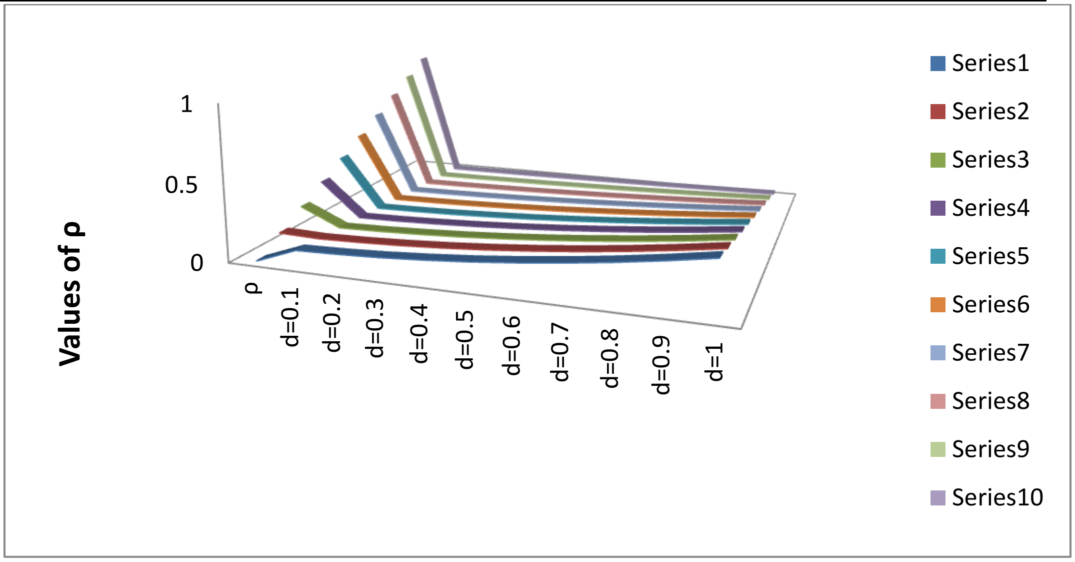

Figure 2: Graphical representation for Robustness of Second Order Slope Rotatable Central Composite Design for v=2 factor

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