ISSN 1684-8403
Journal of Statistics
Volume 20, 2013. pp. 88-101

Economic Reliability Test Plans Based on Rayleigh Distribution

Rani Raja Lakshmi Kantam[1] *and Balantrapu Sriram*[2]

Abstract

Sampling plans in which items are put to test, to collect the life of the items in order to decide upon accepting or rejecting a submitted lot, are called Reliability Test Plans. The basic probability model of the life of the product is specified as the well-known Rayleigh Distribution. For a given Producer's Risk sample size, termination number, the waiting time to terminate the test plan and the operating characteristics are presented. The preferability of the test plan over similar plans existing in the literature with respect to cost and time of the experiment with a numerical example and comparison with Gamma (2) is established.

Keywords

Consumer risk, Operating characteristic, Producer risk, Reliability test plan, Termination number, Waiting time

1. Introduction

A life test is an experiment that is conducted to determine whether or not a product needs the specified requirements for average life. Generally, in such a test, a fixed number of products are taken as a sample out of a submitted lot of those products. To decide upon the acceptance or otherwise of the lot on the basis of the observed life times of the sampled test procedure requires a specification of sample size, a terminating rule to arrive at a decision, the criterion that defines the preferability or otherwise of the lot and, above all, the risks associated with the decisions.

[1] Department of Statistics, Acharya Nagarjuna University, Nagarjuna Nagar-522 510, A.P., India.
Email: kantam_rrl@rediffmail.com
[2] Department of Statistics, AAR and BMR Degree College, Nunna Vijayawada (Rural),Vijayawada-521212, Krishna District, A.P., India.
Email: sriram_stat@rediffmail.com

In reliability studies, Constant Failure Rate (CFR) is the rate at which the system fails is independent of its age and for continuously operating systems this implies a Constant Failure Rate. The Constant Failure Rate model for continuously operating system leads to an Exponential Distribution. The CFR model is the central distribution in reliability studies, Epstein and Sobel (1954) developed Reliability Test Plans for Exponential Distribution. Gupta and Groll (1961) constructed similar sampling plans based on Gamma Life Test sample data. Goode and Kao (1961) constructed sampling plans based on Weibull Distribution. Kantam and Rosaiah (1998) suggested acceptance sampling plans based on life tests when the Failure Density model of the products is Half-logistic Distribution. Acceptance sampling based on life tests when the Failure Density model of the products is a Log-logistic Distribution was studied by Kantam et al. (2001). In this paper, we construct sampling plans following Braverman (1981, Ch.11), an approach from that of Goode and Kao (1961), by considering the well-known Rayleigh model as the Failure Density governing the life times of the products in the submitted lot and have made an attempt to construct the necessary test plan that can be used to decide upon accepting or otherwise rejection of the submitted lot of products and make a relative comparison of the two approaches. Such test plan is suggested by Kantam et al. (2006) for Log-logistic Distribution. Kantam and Sriram (2010) constructed Reliability Test Plans for Gamma (2). The necessary theory of the present plans is given in Section 2, the Operating Characteristics (OC) are given in Section 3, comparative study is presented in Section 4, and numerical example in Section 5.

2. Reliability Test Plans

Let a lot of products of infinitely large size be submitted for sampling inspection and decision to reject or accept. Let us assume that the probability density function of life of a product is a Rayleigh Distribution with scale parameter σ, whose probability density function f(x;σ), cumulative distribution function F(x;σ) are given by the equations

$$f(x/\sigma) = \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right) \qquad x, \sigma > 0 \tag{2.1}$$

$$F(x/\sigma) = 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) \tag{2.2}$$

Also the probability density function of life of a product is a Gamma (2) Distribution with scale parameter σ, whose probability density function f(x;σ), cumulative distribution function F(x;σ) are given by the equations

$$f(x) = \frac{x}{\sigma^2}\exp\left(-\frac{x}{\sigma}\right) \tag{2.3}$$

$$F(x) = 1 - \{\exp\left(-\frac{x^2}{2\sigma^2}\right)\}(1 + \frac{x}{\sigma}) \tag{2.4}$$

Let α be the probability of rejecting the submitted lot that is truly good in some sense-known as Producer's Risk. Naturally, α should be as small as possible then we can think of the decision making in two different ways.

(i) Let σ_o be a specified value of σ representing the mean life of the product and t_o be a pre-assigned time at which the life testing experiment of sample products is designed to be terminated. Hence t_o may be called Terminating time. Gupta and Groll (1961) suggested the minimum sample size required n and an acceptance number c such that if c or less failures occur out of n before the time t_o then the lot would be accepted with a probability (1-α). This approach is basically counting number of failures out of n within the terminating time t_o and hence the life testing experiment would be stopped as soon as the time t_o is reached or $(c+1)^{st}$ failure is realized whichever is earlier. Goode and Kao (1961) constructed sampling plans based on Weibull Distribution. A typical Table of Goode and Kao (1961) is reproduced in Table 1, and Table 2 is illustrated by an example.

Suppose, an experimenter wishes (Goode and Kao (1961)) to know that the true mean life $2\sigma_o$ is at least 5000 hours with probability 0.95 and the experiment is designed to stop at 1000 hours after starting. For an acceptance number c=2 from the Table 1, the minimum sample size required is the entry corresponding to c=2, $t/2\sigma_o$=0.4 and this is 41. Hence, it is suggested that if 41 products are put to test at time '0' with an aim of stopping the test at the 1000^{th} hour we can accept the lot with probability of 0.95, if the number of failures before the 1000^{th} hour is less than or equal to 2, otherwise the lot shall be rejected if the number of failures within 1000^{th} hour is 3 or more.

(ii) Alternatively, one can think of another Reliability Test Plan. Let n stand for the number of sampled items to be inspected. Let r be natural number such that if r failures are realized before the termination time t_o the lot would be rejected, that is the experiment is stopped as soon as r^{th} failure is reached or termination time t_o is reached whichever is earlier. In this sense, r is called Termination Number. The sample size naturally depends on cost considerations and expected waiting time to reach a decision. Larger sample sizes may decrease expected waiting time but increase cost of experimentation. As a balance between these two aspects, let us consider the sample size as a multiple of termination number.

We know that the probability of r failures out of n tested items is given by $\binom{n}{r}p^r q^{n-r}$ where p=F(x;σ) is the cumulative distribution function of the Rayleigh Distribution. Hence, acceptance of probability of lot is

$$P_a = \sum_{i=0}^{r-1}\binom{n}{i}p^i q^{n-i} \tag{2.5}$$

If α is denoted as the Producer's Risk and n is assumed to be a multiple of the termination number, it would be n=kr, where k be any natural number, hence, we can write the above equation as

$$\sum_{i=0}^{r-1}\binom{kr}{i}p^i q^{kr-i} = 1-\alpha \tag{2.6}$$

Using the cumulative probability of Binomial Distribution, the above equation can be solved for p. Equating F(x;σ) to p we can get the value of x/σ corresponding to p, that is x/σ is the solution of

$$F(x;\sigma) = 1 - \exp\left(-\frac{x^2}{\sigma^2}\right) = p \tag{2.7}$$

We have tabulated the values of x/σ as obtained from Equations (2.6), (2.7) for α=0.05, 0.01, r=1(1) 10, k=2(1) 10 in Tables 3 and 4. These Tables can be considered as another Reliability Test Plans.

As an example, suppose we have to construct a Life Test sampling plan with an acceptance probability of 0.95 for lots with an acceptable mean life of 1000 hours and termination number 5, sample size 10. From Table 3 the entry against r=5 under the column 2r is 0.50159. Since the acceptable mean life is given to be 1000 hours for a Rayleigh Distribution this implies $2\sigma_o$=1000, therefore, σ_o=500. If the termination time is given by t_o the Table value says that t_o/σ_o=0.50159, that is t_o= 500×0. 50159=250.795

This Test Plan will be implemented as follows. Select 10 items from the submitted lot and put them to test if the fifth failure is realized before 251st hour of the test reject the lot otherwise accept the lot, in either case terminating the experiment as soon as the fifth failure is reached or 251st hour of the test time is reached whichever is earlier. In the case of acceptance, the assurance is that the average life of the submitted products is at least 1000 hours.

3. Operating Characteristic Curve

If the true but unknown life of the product deviates from the specified life of the product, it should result in a considerable change in the probability of acceptance of the lot based on the sampling plan. Hence, the probability of acceptance can be regarded as a function of the deviation of specified average from the true average. This function is called Operating Characteristic Function of the Sampling Plan. Hence, the Operating Characteristic (OC) lies between 0 and 1. Specifically, if F (T/σ) is the cumulative distribution function of the life time random variable of the product, σ_o corresponds to specified life, we can write

$$F(T/\sigma) = F\left[\frac{T}{\sigma_0}.\frac{\sigma_0}{\sigma}\right] \tag{3.1}$$

where σ corresponds to true but unknown average life. The ratio σ_o/σ in the right hand side of above equation can be taken as a measure of changes between true and specified lives. For instance, $\sigma_o/\sigma <1$ implies true mean life is more than the declared life leading to more acceptance probability or less Failure Risk. Similarly, σ_o/σ more than 1 implies less acceptance probability or more failure risk. Hence, giving a set of hypothetical values, say σ_o/σ =0.1(0.1) 0.9, we can have the corresponding acceptance probabilities for the given sampling plan. The graphs between σ_o/σ, probability of acceptance given by Equation (2.5) for a sampling plan forms the OC curve of the plans are given in Figures 1 to 4. Here we have selected some plans and OC values of these plans are given in Tables 5 and 6.

4. Comparative Study

The sampling plans constructed for the chosen model with two different approaches can be compared at some common entries of the two approaches within a given model. It may be noted in the first approach (given Consumer Risk) an optimum sample size is solved for fixing scaled termination time, acceptance number. In the second approach the earliest termination time is calculated fixing sample size and acceptance number/termination number. The respective approaches of these two approaches have common entries at some values of sample size, acceptance number/termination number. These are presented in Table 7 which reveals that the scale termination time given by approach II at the common entries in the Table is uniformly smaller than that by the first approach. Hence, the second approach is preferable to the first with respect to savings in experiment time. While applying these test plans for a live data, the goodness of fit of the model for the data is to be confirmed and then the respective Tables of the confirmed model constructed on approach II are to be used.

4.1 Rayleigh vs Gamma (2): The sampling plans constructed for the chosen models with the same approaches can be compared at some common entries of the two approaches between the models. For both Rayleigh and Gamma (2) of Kantam and Sriram (2010), the earliest termination time is calculated fixing sample size and acceptance number/termination number based on approach II. The respective models of the approach II have common entries at some values of sample size, acceptance number/termination number. These are presented in Table 8. The first entry in Table 8 corresponds to termination time by approach II of Rayleigh Distribution and the second one relates to that of Gamma (2) which reveals that the scale termination time given by approach I of Rayleigh at the common entries in the Table is uniformly smaller than that by the Gamma (2). Hence, the sampling plans based on Rayleigh Distribution are preferable to Gamma (2) with respect to savings in experimental time.

4.1.1 Numerical Example: Consider the following ordered failure times of the release of a software given in terms of hours from starting of the execution of the software upto the time at which a failure of the software occurs (Wood, 1996). This data can be regarded as an ordered sample of size n = 12 with observations

$$\{x_{i\,/}\, i = 1,\ldots..12\} = \{519, 968, 1430, 1893, 2490, 3058, 3625, 4422, 5218, 5823, 6539, 7083\}$$

On the basis of approach I, let the required average lifetime be 1000 hours and the testing time be t = 1000 hours, this leads to a sample size n = 12 with a corresponding ratio of $t_o/2\sigma_o = 1$ and an acceptance number c = 3 which are obtained from Table 2 for a risk probability of 0.01. Therefore, the sampling plan for the above sample data is (n = 12, c = 3, $t_o/2\sigma_o = 1.0$). Based on the observations, we have to decide whether to accept the product or reject it. We accept the product only, if the number of failures before 1000 hours is less than or equal to 3. From the given ordered sample we notice that the earliest failures of the software product are at 519 and 968 hours, which are less than 1000 hours.

On the basis of approach II, for a given ordered sample of size 12, for a termination number 4, at a risk probability of 0.01 the value of t_o/σ_o from Table 4 is t_o/σ_o =0.28129, that is the termination time is 281 hours. The second plan means – "The number of failures earlier than 281 hours of the experiment should not be more than 3". From the given ordered data, we notice two failures at 519, 968 which is not earlier than the termination time 281 hours. That is, in a sample of 12 failure time instants there are two failures at 519 and 968 hours before the termination time t = 1000 hours, and this number is less than the acceptance number of the plan c = 3. From the two approaches I and II the decision on the first approach can be reached at the 1000^{th} hour and that in the second approach the decision is taken at 281^{st} hour. Hence, the waiting time due to second approach to come to a decision is less than that of first approach. Hence the second approach is preferred.

5. Conclusion

In this paper, an acceptance decision rule is developed based on the Life Test when the life distribution of test items follows a Rayleigh Distribution, for the use of plans by the practitioners.

Table 1: Minimum sample size necessary to assert the average life to exceed specified average life σ_o with probability P^*=0.95 and the corresponding acceptance number C, using Binomial Probabilities

c \ $t_o/2\sigma_o$	P^*=0.95						
	2.0	**1.5**	**1**	**0.40**	**0.20**	**0.15**	**0.10**
0	1	2	3	19	75	134	300
1	2	3	6	31	120	212	475
2	4	5	8	41	159	281	631
3	5	6	10	50	196	347	777
4	6	7	12	60	231	409	918
5	7	8	14	69	266	470	1054
6	8	10	16	78	300	530	1188
7	9	11	18	86	333	588	1319
8	10	12	19	95	365	646	1448
9	11	13	21	103	398	703	1576
10	12	15	23	112	430	759	1702
11	13	16	25	120	461	815	1827
12	14	17	27	128	493	871	1951
13	15	18	29	136	524	926	2074
14	16	19	30	144	555	980	2196
15	17	21	32	153	586	1035	2318

Table 2: Minimum sample size necessary to assert the average life to exceed specified average life σ_o with probability P^*=0.99 and the corresponding acceptance number C, using Binomial Probabilities

c \ $t_o/2\sigma_o$	P^*=0.99						
	2.0	**1.5**	**1.0**	**0.40**	**0.20**	**0.15**	**0.1**
0	2	3	5	29	116	205	461
1	3	4	8	42	167	296	665
2	4	5	10	54	212	375	841
3	5	7	12	65	253	448	1007
4	6	8	14	75	293	518	1163
5	7	10	16	85	331	586	1314
6	8	11	18	95	368	651	1461
7	10	12	20	104	404	715	1604
8	11	13	22	113	440	778	1745
9	12	15	24	122	475	840	1884
10	13	16	26	132	509	901	2020
11	14	17	28	140	543	961	2155
12	15	18	30	148	577	1021	2289
13	16	20	32	158	611	1080	2421
14	17	21	34	167	644	1138	2552
15	18	22	36	175	677	1197	2683

Table 3: Rayleigh Distribution Life Test Termination Time in units of scale parameter ($1-\alpha=0.95$)

r\n	2r	3r	4r	5r	6r	7r	8r	9r	10r
1	0.16082	0.13136	0.11341	0.10183	0.09307	0.08576	0.08050	0.07595	0.07211
2	0.32118	0.25536	0.21809	0.19357	0.17642	0.16294	0.15165	0.14323	0.13555
3	0.40841	0.32118	0.27311	0.24214	0.21954	0.20292	0.18927	0.17785	0.16861
4	0.46375	0.36269	0.30755	0.27236	0.24727	0.22753	0.21230	0.19932	0.18856
5	0.50159	0.39092	0.33186	0.29325	0.26569	0.24507	0.22826	0.21447	0.20292
6	0.53094	0.41241	0.34953	0.30906	0.27981	0.25757	0.23994	0.22535	0.21375
7	0.55386	0.42932	0.36347	0.32118	0.29101	0.26792	0.24947	0.23409	0.22172
8	0.57180	0.44316	0.37518	0.33109	0.30002	0.27608	0.25683	0.24140	0.22826
9	0.58727	0.45465	0.38461	0.33952	0.30679	0.28278	0.26348	0.24727	0.23409
10	0.60018	0.46375	0.39250	0.34644	0.31360	0.28802	0.26866	0.25241	0.23848

Table 4: Rayleigh Distribution Life Test Termination Time in units of scale parameter ($1-\alpha=0.99$)

r\n	2r	3r	4r	5r	6r	7r	8r	9r	10r
1	0.07147	0.05789	0.05043	0.04513	0.04128	0.03789	0.03573	0.03347	0.03172
2	0.20724	0.16506	0.14113	0.12582	0.11410	0.10522	0.09845	0.09241	0.08775
3	0.29776	0.23482	0.20004	0.17713	0.16082	0.14813	0.13834	0.12998	0.12305
4	0.35959	0.28129	0.23921	0.21157	0.19142	0.17642	0.16506	0.15517	0.14673
5	0.40442	0.31512	0.26717	0.23628	0.21447	0.19716	0.18355	0.17286	0.16365
6	0.43826	0.34106	0.28876	0.25535	0.23118	0.21302	0.19860	0.18641	0.17642
7	0.46541	0.36192	0.30604	0.27088	0.24507	0.22535	0.21013	0.19716	0.18713
8	0.48802	0.37832	0.32043	0.28278	0.25609	0.23555	0.21954	0.20652	0.19501
9	0.50672	0.39250	0.33186	0.29325	0.26569	0.24433	0.22753	0.21375	0.20219
10	0.52311	0.40441	0.34259	0.30227	0.27311	0.25167	0.23409	0.22027	0.20868

Table 5: Rayleigh Distribution OC values of Reliability Test Plans of approach I

	n=2, r=1		n=3, r=1	
	T/σ_o=1.5, 2		T/σ_o=1,1.5	
σ_o/σ	P_a		P_a	
	1-α=0.95	1-α=0.99	1-α=0.95	1-α=0.99
0.1	0.95609	0.92312	0.97045	0.93487
0.2	0.83521	0.72614	0.88692	0.76330
0.3	0.66699	0.48675	0.76337	0.54474
0.4	0.48678	0.37531	0.61877	0.33963
0.5	0.32467	0.13533	0.47238	0.18499
0.6	0.19794	0.05613	0.33960	0.08806
0.7	0.11029	0.01984	0.22993	0.03663
0.8	0.05612	0.00598	0.14660	0.01329
0.9	0.02611	0.00153	0.08804	0.00422
1.0	0.01111	0.00033	0.04979	0.00117
1.1	0.00432	0.00006	0.02652	0.00028
1.2	0.00154	0.00001	0.01330	0.00006
1.3	0.00049	0	0.00628	0.00001
1.4	0.00015	0	0.00279	0
1.5	0.00004	0	0.00117	0
1.6	0.00001	0	0.00046	0
1.7	0	0	0.00017	0
1.8	0	0	0.00006	0
1.9	0	0	0.00002	0

Table 6: Rayleigh Distribution OC values of Reliability Test Plans of approach II

	n=2, r=1		n=3, r=1	
	T/σ_o=0.16082,0.07147		T/σ_o=0.13136,0.05789	
σ_o/σ	P_a		P_a	
	1-α=0.95	1-α=0.99	1-α=0.95	1-α=0.99
0.1	0.99948	0.99990	0.99949	0.99991
0.2	0.99794	0.99960	0.99793	0.99961
0.3	0.99536	0.99908	0.99536	0.99910
0.4	0.99176	0.99836	0.99174	0.99838
0.5	0.98716	0.99744	0.98715	0.99748
0.6	0.98154	0.99632	0.98154	0.99640
0.7	0.97498	0.99503	0.97495	0.99509
0.8	0.96743	0.99349	0.96742	0.99359
0.9	0.95897	0.99176	0.95893	0.99189
1.0	0.94959	0.98984	0.94954	0.99001
1.1	0.93933	0.98772	0.93929	0.98789
1.2	0.92822	0.98539	0.92817	0.98564
1.3	0.91631	0.98289	0.91623	0.98314
	n=2, r=1		n=3, r=1	

σ_o/σ	T/σ_o=0.16082,0.07147		T/σ_o=0.13136,0.05789	
	P_a		P_a	
	1-α=0.95	1-α=0.99	1-α=0.95	1-α=0.99
1.4	0.90358	0.98018	0.90353	0.98048
1.5	0.89013	0.97727	0.89005	0.97764
1.6	0.87598	0.97418	0.87589	0.97459
1.7	0.86115	0.97091	0.86106	0.97136
1.8	0.84570	0.96745	0.84559	0.96798
1.9	0.82966	0.96381	0.82955	0.96434

Table 7: Rayleigh Distribution: Common entries of scale termination time (T / σ_o)

α = 0.05

c	r	n: 2r	n: 3r
0	**1**	0.16082 1.5	0.13136 1.0
1	**2**		0.25536 1.0

α = 0.01

c	r	n: 2r	n: 3r	n: 4r	n: 5r
0	**1**	0.07147 2.0	0.05789 1.5		0.04513 1.0
1	**2**	0.20724 1.5		0.14113 1.0	
3	**4**		0.28129 1.0		

The first entry in each Table corresponds to termination time by approach II and the second one relates to that of Goode and Kao (1961).

Table 8: Common entries of scaled termination time (T/ σ_o) for Rayleigh and Gamma (2)

c	r	n: 3r	n: 4r
α=0.05			
1	**2**	0.25536 0.40583	
α=0.01			
1	**2**		0.14113 0.21304

Figure 1: Rayleigh Distribution OC curves of Table 5 approach I (1-α=0.95)

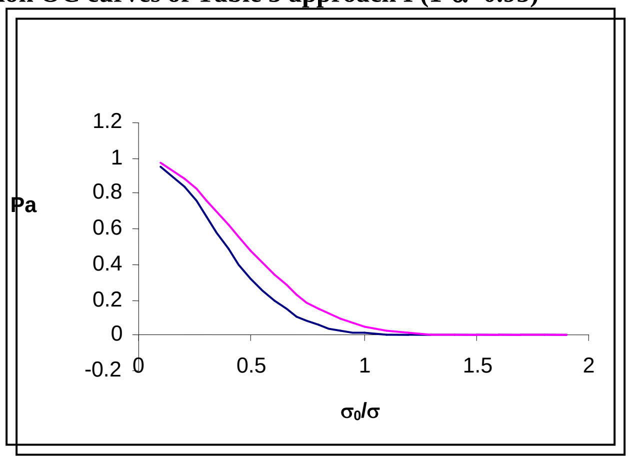


Figure 2: Rayleigh Distribution OC Curves of Table 5 approach I (1-α=0.99)

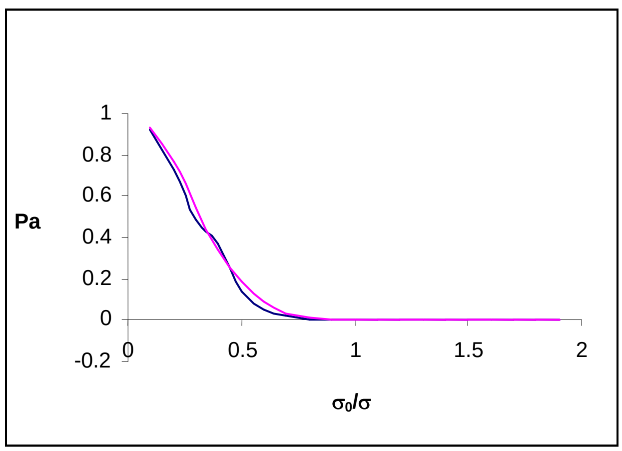


Figure 3: Rayleigh Distribution OC curves of Table 6 approach II (1-α=0.95)

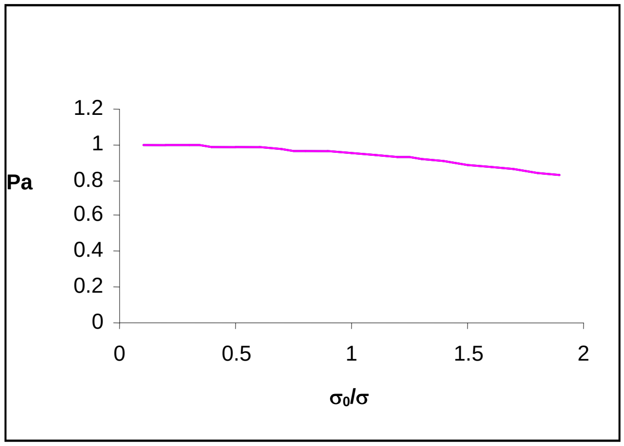
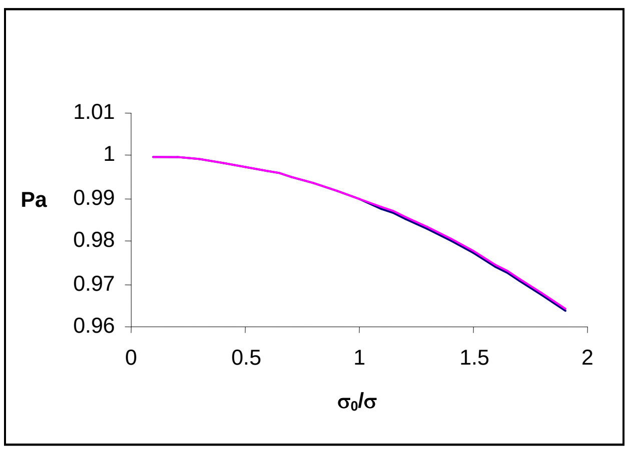


Figure 4: Rayleigh Distribution OC curves of Table 6 approach II (1-α=0.99)



References

1. Braverman, J.D. (1981). *Fundamentals of Statistical Quality Control.* Reston Publishing House Co., Inc. A Prentice-Hall Company, Reston, Virginia.
2. Epstein, B. and Sobel, M. (1954). Some theorems relevant to life testing from an Exponential Distribution. *Annuals of Mathematical Statistics,* **25**, 373-381.
3. Goode, H.P. and Kao, J. H. K. (1961). Sampling plans based on the Weibull Distribution, Proceedings of Seventh National Symposium on Reliability and Quality Control, Philadelphia, Pennsylvania, 20-40.

4. Gupta, S.S. and Groll, P.A. (1961). Gamma Distribution in acceptance sampling based on Life Tests. *Journal of the American Statistical Association,* **56**, 942-970.
5. Kantam, R. R. L. and Rosaiah, K. (1998). Half-logistic Distribution in acceptance sampling based on Life Tests. *IAPQR Transactions*, **23(2)**, 117-125.
6. Kantam, R. R. L., Rosaiah, K. and Srinivas Rao, G. (2001). Acceptance sampling based on Life Tests: Log-logistic Model. *Journal of Applied Statistics*, **28(1)**, 121-128.
7. Kantam, R. R. L., Srinivas Rao, G., and Sri-Ram, B. (2006). An Economic Reliability Test Plan: Log-logistic Distribution. *Journal of Applied Statistics*, **33(3)**, 291-296.
8. Kantam, R. R. L. and Sri-Ram, B. (2010). Reliability Test Plans - Gamma Variate. *Ancharya Nagarjuna Uuniversity Journal of Physical Sciences*, **2(2)**, 30-41.
9. Wood, A. (1996). Predicting software reliability. *IEEE Transactions on Software Engineering*, **22**, 69-77.