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SIZE AND POWER PROPERTIES OF ASYMPTOTICALLY ROBUST TESTS FOR EQUALITY OF TWO COVARIANCE MATRICES

By

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ABSTRACT

Some asymptotically robust test statistics for equality of two covariance matrices are discussed. The standard error test based on combined and separate estimator of asymptotic covariance matrices of vectors of second-order sample moments, is estimated with and without transformations. The untransformed test based on separate estimator is equally good as Layard (1972, 1974) [3,4] proposed transformed test based on combined estimator. The effect of transformations on the tests is examined. The size and power performance of the untransformed tests is compared. The standard error test based on separate estimator is found reasonable for moderate size of non-normal samples.

Key words: *Robust, asymptotic covariance matrix, bivariate distribution, convergence in distribution, consistent estimator, kurtosis parameter, bias.*

1. INTRODUCTION

Layard (1972) [3] described some asymptotically robust test statistics for the equality of two covariance matrices, i.e. standard error, grouping, and jackknife for bivariate distributions. Layard (1974) [4] compared the size and power properties of these tests. He proposed transformations, i.e. Log_e transformation of variances and $\text{Tan}h^{-1}$ transformation of sample correlation co-efficient. Furthermore he assumed that the transformed vectors of second-order sample moments have same asymptotic covariance matrix and preferred to use combined estimator of it, for standard error test. Layard (1974) [4] and Pervaiz (1986) [5] concluded that standard error test based on combined estimator is better than grouping and jackknife tests as regards size and power for non-normal distributions. Like Layard (1972, 1974) [3,4] Tiku and Balakrishnan (1985) [7] treated the problem as a test for equality of mean vectors and proposed a T^2 test.

The aim of the paper is to look at:

(a) The effect of combined and separate estimates of asymptotic covariance matrix of vectors of second-order sample moments on size and power performance of standard error test, with and without transformations.

(b) The effect of transformations on size and power properties of the tests.

(c) The performance of the untransformed asymptotically robust tests.

The test statistics are defined in section 3. The sampling experiments are discussed in section 4. Simulations were carried out on ICL 2976 Computer at the University of the Southampton, United Kingdom. The random number generator used was Gϕ5DDF, Gϕ5DBF and Gϕ5CAF from the NAG library through NAG Limited. The programs were written in FORTRAN-IV.

The test statistics computed, are compared with the 5% and 1% points of the approximate null distributions. The results for the 1% case, essentially corroborate those of the 5% case, so these are not reported here. The section 5 provides discussion of empirical results. The conclusions are given in section 6.

2. PROPERTIES OF SAMPLE COVARIANCE MATRICES FOR INDEPENDENTLY AND IDENTICALLY DISTRIBUTED SAMPLES

Suppose two bivariate populations with distribution functions F and G, covariance matrices $\underline{\Sigma}_i, i = 1, 2$ and finite fourth moments.

Where

$$\underline{\Sigma}_i = \begin{bmatrix} \mu_{i,20} & \mu_{i,11} \\ \mu_{i,11} & \mu_{i,02} \end{bmatrix}$$

$$\mu_{i,rs} = \mathrm{E}\left[\left(X_{i1} - \mathrm{E}(X_{i1})\right)^r \left(X_{i2} - \mathrm{E}(X_{i2})\right)^s\right] \tag{2.1}$$

$$X_i = (X_{i1}, X_{i2})^T$$

The sample covariance matrices $\underline{S}_i$ are

$$\underline{S}_i = \begin{bmatrix} s_{i,20} & s_{i,11} \\ s_{i,11} & s_{i,02} \end{bmatrix} \tag{2.2}$$

Where

$$s_{i,rs} = \frac{1}{n_i}\sum_{e=1}^{n_i}\left(x_{ie1} - \bar{x}_{i.1}\right)^r\left(x_{ie2} - \bar{x}_{i.2}\right)^s \tag{2.3}$$

$$\bar{x}_{i.1} = \frac{1}{n_i}\sum_{e=1}^{n_i} x_{ie1}$$

$$\bar{x}_{i.2} = \frac{1}{n_i}\sum_{e=1}^{n_i} x_{ie2} \tag{2.4}$$

Let

$$\underline{S}_1^v = \left(s_{1,20}\ ,\ s_{1,02}\ ,\ s_{1,11}\right)^T$$

$$\underline{S}_2^v = \left(s_{2,20}\ ,\ s_{2,02}\ ,\ s_{2,11}\right)^T \tag{2.5}$$

and $\underline{\Sigma}_i^v$ are determined similarly from (2.1) i.e. vectors of second-order population moments. Following Cramer (1946, p.365) [1], Layard (1972) [3] showed that

$$n_1^{\frac{1}{2}}\left(\underline{S}_1^{*} - \underline{\Sigma}_1^{*}\right) \xrightarrow[\xi]{} N_3\left(\underline{o}, \underline{\Gamma}_1\right) \quad as \quad n_1 \to \infty$$
$$n_2^{\frac{1}{2}}\left(\underline{S}_2^{*} - \underline{\Sigma}_2^{*}\right) \xrightarrow[\xi]{} N_3\left(\underline{o}, \underline{\Gamma}_2\right) \quad as \quad n_2 \to \infty \tag{2.6}$$

(The symbol $\xrightarrow[\xi]{}$ denotes convergence in distribution). Where $\underline{\Gamma}_i$ are:

$$\underline{\Gamma}_i = \begin{bmatrix} \mu_{i,40} - \mu_{i,20} & \mu_{i,22} - \mu_{i,20}\,\mu_{i,02} & \mu_{i,31} - \mu_{i,20}\,\mu_{i,11} \\ & \mu_{i,04} - \mu_{i,02}^{2} & \mu_{i,13} - \mu_{i,02}\,\mu_{i,11} \\ & & \mu_{i,22} - \mu_{i,11}^{2} \end{bmatrix} \tag{2.7}$$

Layard (1972, 1974) [3,4] proposed transformations to hasten convergence to normality. These are:

$$\phi\left[\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right] = \begin{bmatrix} \ln v_1 \\ \ln v_2 \\ \frac{1}{2}\ln\left(\frac{1+\rho}{1-\rho}\right) \end{bmatrix} \tag{2.8}$$

Where $\ell = v_3\left(v_1 v_2\right)^{-\frac{1}{2}}$. Following Layard (1974) [4]

$$n_1^{\frac{1}{2}}\left[\phi\left(\underline{S}_1^{*}\right) - \phi\left(\underline{\Sigma}_1^{*}\right)\right] \xrightarrow[\xi]{} N_3\left(0, \underline{\Omega}_1\right) \quad as \quad n_1 \to \infty$$

$$n_2^{\frac{1}{2}}\left[\phi\left(\underline{S}_2^{*}\right) - \phi\left(\underline{\Sigma}_2^{*}\right)\right] \xrightarrow[\xi]{} N_3\left(0, \underline{\Omega}_2\right) \quad as \quad n_2 \to \infty \tag{2.9}$$

Where

$$\underline{\Omega}_i = \underline{A}_i^{T}\ \underline{\Gamma}_i\ \underline{A}_i \tag{2.10}$$

($\underline{A}$ is matrix of first partial of ϕ evaluated at $\underline{\mu}$)

$$\underline{A}_i = \begin{bmatrix} \mu_{i,20}^{-1} & 0 & -\frac{\rho_i}{2\mu_{i,20}}(1-\rho_i^2) \\ 0 & \mu_{i,20}^{-1} & -\frac{\rho_i}{2\mu_{i,02}}(1-\rho_i^2) \\ 0 & 0 & \frac{1}{(\mu_{i,20}\,\mu_{i,02})^{\frac{1}{2}}}(1-\rho_i^2) \end{bmatrix} \tag{2.11}$$

and $\underline{\Gamma}_i$ as given in (2.7). $\rho_i = \mu_{i,11}(\mu_{i,20}\,\mu_{i,02})^{\frac{1}{2}}$. Layard (1972) [3] suggested that consistent estimators of the asymptotic covariance matrices of $\phi(\underline{S}_i^v)$ can be obtained from (2.10) by substituting sample quantities $s_{i,rs}$ for $\mu_{i,rs}$ population moments.

3. TEST STATISTICS

The problem is to test:

$$H_o\colon F(x_1,x_2) = G(x_1+\xi_1, x_2+\xi_2) \quad vs \quad H_A\colon \underline{\Sigma}_1 \neq \underline{\Sigma}_2$$

Where ξ_1 and ξ_2 are unspecified constants. The choice of H_o ensures that the fourth moments of the distributions are equal.

The tests used in the sampling experiments are as follows:

(1) Standard error

Because we are interested in the comparison of the performance of standard error test based on combined and separate estimator, and with and without transformations, therefore the test is described with these respects.

(a) Untransformed based on Separate Estimator

From (2.6) under H_o the test statistics:

$$(\underline{S}_1^v - \underline{S}_2^v)^T \left[n_1^{-1}\hat{\underline{\Gamma}}_1 + n_2^{-1}\hat{\underline{\Gamma}}_2\right]^{-1}(\underline{S}_1^v - \underline{S}_2^v) \tag{3.1}$$

is approximately distributed as χ_3^2, provided $\hat{\underline{\Gamma}}_1$ and $\hat{\underline{\Gamma}}_2$ are consistent estimators of the asymptotic covariance matrix of $\underline{S}_1^v$ and $\underline{S}_2^v$

respectively. These can be obtained from (2.7) by using sample quantities as defined by (2.3)

(b) Untransformed based on Combined Estimator

From (2.6) the test statistic:

$$\frac{n_1 n_2}{n_1+n_2}[(\underline{S}_1^v-\underline{S}_2^v)^T\hat{\underline{\Gamma}}^{-1}(\underline{S}_1^v-\underline{S}_2^v)]$$

has the distribution of (3.1) under H_O. The $\hat{\underline{\Gamma}}$ can be obtained from (2.7) by using s_{rs} in place of $\mu_{i,rs}$, where

$$s_{rs}=\frac{1}{n_1+n_2}\sum_{i=1}^{2}\sum_{e=1}^{n_i}\left(x_{ie1}-\bar{x}_{1.1}\right)^r\left(x_{ie2}-\bar{x}_{1.2}\right)^s \tag{3.2}$$

The $\bar{x}_{1.1}$ and $\bar{x}_{1.2}$ are defined by (2.4)

(c) Transformed based on Separate Estimator

From (2.8) and (2.9), under H_O the test statistic:

$$[\phi(\underline{S}_1^v)-\phi(\underline{S}_2^v)]^T[n_1^{-1}\hat{\underline{\Omega}}_1+n_2^{-1}\hat{\underline{\Omega}}_2]^{-1}[\phi(\underline{S}_1^v)-\phi(\underline{S}_2^v)] \tag{3.3}$$

has the distribution of (3.1), provided $\hat{\underline{\Omega}}_1$ and $\hat{\underline{\Omega}}_2$ are consistent estimators of the asymptotic covariance matrices of $\phi(\underline{S}_1^v)$ and $\phi(\underline{S}_2^v)$ respectively. These can be obtained from (2.10) by using sample quantities in place of population moments.

(d) Transformed based on Combined Estimator

Under H_O, $\phi(\underline{S}_1^v)$ and $\phi(\underline{S}_2^v)$ have same asymptotic covariance matrix, i.e

$$\underline{\Omega}=\underline{A}^T\underline{\Gamma}\underline{A} \tag{3.4}$$

Therefore Layard (1974) [4], preferred the test statistic:

$$\frac{n_1 n_2}{n_1+n_2}\left[\left(\left(\phi\,\underline{S}_1^v\right)-\phi\left(\underline{S}_2^v\right)\right)^T\hat{\underline{\Omega}}^{-1}\left(\left(\phi\,\underline{S}_1^v\right)-\phi\left(\underline{S}_2^v\right)\right)\right]$$

having the distribution of (3.1). The $\underline{\widehat{\Omega}}$ can be obtained from (3.4) by using s_{rs} as defined by (3.2) for population quantities in (2.7) and (2.11).

(II) Grouping

Each sample is divided randomly into $n_i^{'}\ i = 1,2;$ groups of size $L, i.e.\ n_i = Ln_i^{'}\ for\ L \geq 2$ (assumption is that n_i are divisible by L)

Let

$$\underline{S}_{i,g} = \begin{bmatrix} s_{i,20g} & s_{i,11g} \\ s_{i,11g} & s_{i,02g} \end{bmatrix} \quad g = 1,2,\ldots\ldots,n_i^{'}$$

sample variance - covariance matrices within groups. The $\underline{S}_{i,g}^{v}$, vectors of second-order sample moments from groups of first and second samples, are independent and have approximately the multivariate normal distribution with equal mean vectors and covariance matrices, under H_0, being so the test statistic:

$$(\bar{\underline{S}}_1^{v} - \bar{\underline{S}}_2^{v})^T\ [n_1^{'-1}\underline{\widetilde{\Gamma}}_1 + n_2^{'-1}\underline{\widetilde{\Gamma}}_2]^{-1}\ (\bar{\underline{S}}_1^{v} - \bar{\underline{S}}_2^{v}) \qquad (3.5)$$

has approximately Hotelling's T^2 distribution with 3 and $n_1^{'} + n_2^{'} - 2$ degrees of freedom. Where

$$\underline{\widetilde{\Gamma}}_i = \frac{1}{n_i^{'} - 1}\sum_{g=1}^{n_i^{'}}(\underline{S}_{i,g}^{v} - \bar{\underline{S}}_i^{v})\ (\underline{S}_{i,g}^{v} - \bar{\underline{S}}_i^{v})^T$$

$$\bar{\underline{S}}_i^{v} = \frac{1}{n_i^{'}}\sum_{g=1}^{n_i^{'}}\underline{S}_{i,g}^{v}$$

(III) Jackknife

Let $\underline{S}_{i-e}$, sample covariance matrices, defined as:

$$\underline{S}_{i,e} = \begin{bmatrix} s_{i,20e} & s_{i,11e} \\ s_{i,11e} & s_{i,02e} \end{bmatrix} \quad e = 1,2,\ldots\ldots,n_i$$

The elements of the matrix are estimated second-order moments from the samples by using (n_i-1) observations, with the e-th observation omitted, Let

$$\underline{S}^{v}_{i,e} = n_i \underline{S}^{v}_{i} - (n_i - 1)\underline{S}^{v}_{i-e}$$

The jackknife estimators are the average of $\underline{S}^{v}_{i,e}$; i,e,

$$\underline{S}^{*v}_{i} = n_i \underline{S}^{v}_{i} - \frac{n_i - 1}{n_i}\sum_{e=1}^{n_i}\underline{S}^{v}_{i-e}$$

The $\underline{S}^{v}_{i,e}$ are approximately independent and have, under H_O, approximately equal mean vectors and covariance matrices. Thus test statistic:

$$(\underline{S}^{*v}_{1} - \underline{S}^{*v}_{2})^T\,[n_1^{-1}\underline{\Gamma}^{*}_{1} + n_2^{-1}\underline{\Gamma}^{*}_{2}]^{-1}\,(\underline{S}^{*v}_{1} - \underline{S}^{*v}_{2})$$

has approximately Hotelling's T^2 distribution with 3 and (n_1+n_2-2) degrees of freedom under H_O: Where

$$\underline{\Gamma}^{*}_{i} = \frac{1}{n_i - 1}\sum_{e=1}^{n_i}(\underline{S}^{v}_{i,e} - \underline{S}^{*v}_{i})(\underline{S}^{v}_{i,e} - \underline{S}^{*v}_{i})^T$$

(IV) Tiku and Balakrishnan T^2

Let

$$U_{1e} = \left(x_{1e1} - \bar{x}_{1.1}\right)^2 \; and \; U_{2e} = \left(x_{1e2} - \bar{x}_{1.2-}\right)^2;$$

$$V_{1e} = \left(x_{2e1} - \bar{x}_{2.1}\right)^2 \; and \; V_{2e} = \left(x_{2e2} - \bar{x}_{2.2-}\right)^2;$$

Where

$$x_{ie2-} = x_{ie2} - \hat{b}x_{ie1},$$

$\bar{x}_{i.1}$ and $\bar{x}_{i.2-}$ are usual means, while $\hat{b}$ is the pooled regression coefficient.

Tiku and Balakrishan (1985) [7] suggested that the test statistic:

$$T^2 = \frac{n_1 n_2}{n_1 + n_2} \begin{bmatrix} \bar{w}_1 & \bar{w}_2 \end{bmatrix} \begin{bmatrix} \hat{\phi}_1^2 & 0 \\ 0 & \hat{\phi}_2^2 \end{bmatrix}^{-1} \begin{bmatrix} \bar{w}_1 \\ \bar{w}_2 \end{bmatrix}$$

is distributed approximately as Hotelling's T^2 with 2 and (n_1+n_2-2) degrees of freedom under H_0: Where

$$\bar{w}_1 = \bar{U}_1 - \bar{V}_1$$

$$\bar{w}_2 = \bar{U}_2 - \bar{V}_2$$

$$\hat{\phi}_1^2 = \frac{\sum_{e=1}^{n_1} (U_{1e} - \bar{U}_1)^2 + \sum_{e=1}^{n_2} (V_{1e} - \bar{V}_1)^2}{n_1 + n_2 - 2}$$

$$\hat{\phi}_2^2 = \frac{\sum_{e=1}^{n_1} (U_{2e} - \bar{U}_2)^2 + \sum_{e=1}^{n_2} (V_{2e} - \bar{V}_2)^2}{n_1 + n_2 - 2}$$

4. SAMPLING EXPERIMENTS

Four hypothetical distributions, the normal, the gamma, the double exponential and the contaminated normal are sampled. For details see Layard (1974) [4] and Pervaiz (1986) [5]. Furthermore the set of covariance matrices chosen is the same as Layard (1974) [4]. The covariance matrices which represent the null hypotheses are:

(a) Both I_{2x2}

(b) Both with unit variances and correlation co-efficient 0.9, and the alternative hypotheses are:

(c) I_{2x2} and 2.25 I_{2x2},

(d) I_{2x2} and variances 4 and correlation co-efficient 0.3,

(e) I_{2x2} and the matrix of (b)

5. DISCUSSION OF EMPIRICAL RESULTS

In considering the results it should be noted that the standard deviation of the estimated binomial proportion for a true proportion of 0.05 with samples of size 1000 is 0.07 and with samples of size 500 is 0.010. Therefore for 1000 replication observed proportions lying in (3.6, 6.4)%, and for 500 replications lying in (3.0, 7.0)% do not differ significantly from a true proportion of 5% at 95% level.

Table 1 provides the proportion of rejection observed for distribution-matrix-transformation combinations by using the combined and separate estimators of the asymptotic covariance matrices. Under transformations, the standard error test based on the combined estimator produced much better sizes than the separate estimator. The effect is becoming more significant with the increase of the kurtosis co-efficient of the parent distribution. But in the untransformed case the standard error test based on the combined estimator rejected the null hypothesis too infrequently for (b) in the case of the contaminated normal distribution. The observed size was 1.3% as opposed to the nominal 5% level. While the test based on separate estimator produced reasonable sizes.

To look at the asymptotic convergence of the transformed standard error test based on separate estimator, samples of size $n_1=n_2=20,40....,100$ and 250 are considered. The proportion of rejections observed for elliptical distributions (normal and contaminated normal)-matrix combinations are recorded in Table 2. The asymptotic convergence is not very good and still does not appear to have occurred for the contaminated normal distribution. While the untransformed test based on separate estimator produced very reasonable sizes with samples of size $n_1=n_2=25$. Consequently the separate estimator is used in the untransformed case.

The proportion of rejections observed for distribution-untransformed test- matrix combinations, with samples of size 25, are recorded in Table 3. All tests produced reasonable sizes for the normal distribution. The standard error test produced reasonable sizes for the non-normal distributions as well. The test has decreasing trend in observed sizes with the increasing kurtosis parameter of the parent distribution. The observed sizes for the gamma, the double exponential and the contaminated normal distributions were (7.9, 4.3 & 3.9 %) and (8.0, 4.8 and 3.4%)

The grouping test performed well for the gamma distribution as regards observed sizes, but rejected the null hypothesis too infrequently for (b) in the case of the double exponential and for (a &b) in the case of the contaminated normal distribution. The observed sizes were [(2.8)%] and [(3.3 & 2.5)%] for the respective distributions.

Gross (1976) [2] found the jackknife disappointing in confidence interval terms. Rocke and Downs (1981) [6] empirical study concludes, the jackknife method of variance estimation produces upward bias for the contaminated normal distribution. The upward bias in jackknife variance estimation may cause two infrequent rejections of the null hypothesis in the problem. Therefore, the test rejected the null hypothesis too infrequently for the double exponential and the contaminated normal distributions. The observed sizes were [(2.3 & 2.8) %] and [(2.2 & 1.3)%] for the respective distributions. Under transformations the test was rejecting the null hypothesis too infrequently for the double exponential and the contaminated normal distributions---(cf. Layard, 1974) [4].

The Tiku and Balakrishnan T^2 test produced sizes for the normal and the non-normal distributions ranging from a minimum of 2.5% to a maximum of 5.4%. It rejected the null hypothesis too infrequently for the double exponential and the contaminated normal distributions. The observed sizes were [(3.5 & 3.3%)] and [(2.5 & 2.6%)] for the respective distributions.

The standard error test is better in power than the grouping and the jackknife tests for the normal and the non-normal distributions. The Tiku and Balakrishanan T^2 test has comparable power with the standard error test for (c) and (d), but for (e) the test in inferior in power even than the grouping test.

To analyse the effect of increase in sample size on the performance of the tests the samples of size 50 are considered. The proportions of rejections observed are recorded in Table 4.. The standard error test maintained very good nominal levels for the distributions sampled, and worst being for (b) in the case of the contaminated normal distribution. The observed size was 3.0% as opposed to the nominal 5% level, not too bad.

There is no improvement as regards observed sizes from the grouping test. For (b) for the contaminated normal distribution the situation is very poor now. The observed size had fallen down to 1.6% from 2.5% in Table 3. The jackknife test is improved, and the improvement is very significant for (b) for the contaminated normal distribution. The observed size raised upto 2.3% from 1.3% in Table 3. But the test is still unable to achieve nominal levels for the double exponential and the contaminated normal distributions. Under transformations the jackknife test rejected the null hypothesis too frequently for the contaminated normal distribution---- (cf. Pervaiz, 1986) [5]. The Tiku and Balakrishnan T^2 test produced sizes from a minimum of 2.2% to a maximum of 4.6% for the distributions sampled; no improvement with the increase in sample size.

Of course the power of the tests increased with the increase in sample size.

To be more certain about the performance of the jackknife test, with and without transformations, samples of size $n_1=n_2=250$ are considered. The observed significance levels for distribution matrix combinations are recorded in Table 5. The untransformed jackknife test was rejecting the null hypothesis too infrequently for (a) for the double exponential distribution. The observed size was 2.2%. For all other situations the test maintained very good nominal levels. Under transformations the test rejected the null hypothesis too frequently for the contaminated normal distribution. The observed sizes were (9.4 & 8.0%).

6. CONCLUSIONS

For the standard error test it is not essential to apply transformations and to use combined estimator of asymptotic covariance matrix of vectors of second-order sample moments as suggested by Layard (1972, 1974) [3,4]. The untransformed test based on separate estimator is equally good as regards size and power. Therefore a strong assumption of equal asymptotic covariance matrices can be relaxed. The transformations does not play any significant role for the jackknife test as well.

When transformations are not applied:

(a) The standard error test based on separate estimator performs better than the grouping, the jackknife, and the Tiku and Balakrishnan T^2 tests, as regards size and power, for the non-normal distributions sampled.

(b) The grouping test is the worst in power, but for (e) the Tiku and Balakrishnan T^2 test.

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APPENDIX

Table 1

Empirical size based on 1000 replications for the standard error test by using combined and separate estimators of the asymptotic covariance matrices of vectors of second-order sample moments.

I all cases, $n_1=n_2=25$

	Nominal 5% level			
	Transformed		Untransformed	
Matrix pairs-----(c.f.section 4)	(a)	(b)	(a)	(b)
Combined				
Normal	0.064	0.066	0.044	0.039
Gamma	0.082	0.099	0.045	0.054
Double exponential	0.072	0.082	0.023	0.027
Contaminated normal	0.070	0.068	0.025	0.013
Separate				
Normal	0.131	0.138	0.060	0.064
Gamma	0.172	0.207	0.079	0.080
Double exponential	0.215	0.232	0.043	0.048
Contaminated normal	0.277	0.293	0.039	0.034

Table 2

Empirical size based on 1000 replications from the standard error test using separate estimators of asymptotic covariance matrices for equality to two covariance matrices.

	Nominal 5% level			
	Normal		Contaminated normal	
Matrix pairs---(c.f.section 4)	(a)	(b)	(a)	(b)
Sample size				
$n_1=n_2=20$	0.153	0.160	0.322	0.378
40	0.102	0.103	0.234	0.256
60	0.093	0.090	0.173	0.201
80	0.073	0.088	0.161	0.170
100	0.060	0.073	0.136	0.144
250	0.059	0.063	0.094	0.078

Table 3

Empirical size and power based on 1000 replications for tests of equality of two covariance matrices.

In all cases, $n_1=n_2=25$.

Nominal 5% level

Matrix pairs---(c.f.section 4)

	(a)	(b)	(c)	(d)	(e)
	Normal				
Standard error	0.060	0.064	0.559	0.961	0.962
Grouping (L=5)	0.048	0.051	0.266	0.524	0.494
Jackknife	0.047	0.042	0.481	0.930	0.925
Tiku & Balakrishnan T^2	0.041	0.042	0.580	0.967	0.485
	Gamma				
Standard error	0.079	0.080	0.526	0.886	0.952
Grouping (L=5)	0.044	0.044	0.203	0.453	0.474
Jackknife	0.048	0.055	0.431	0.850	0.921
Tiku & Balakrishnan T^2	0.054	0.047	0.510	0.908	0.456
	Double Exponential				
Standard error	0.043	0.048	0.280	0.659	0.906
Grouping (L=5)	0.043	0.028	0.119	0.325	0.391
Jackknife	0.023	0.028	0.209	0.559	0.855
Tiku & Balakrishnan T^2	0.035	0.033	0.256	0.646	0.298
	Contaminated Normal				
Standard error	0.039	0.034	0.227	0.541	0.747
Grouping (L=5)	0.033	0.025	0.114	0.242	0.291
Jackknife	0.022	0.013	0.169	0.450	0.680
Tiku & Balakrishnan T^2	0.025	0.026	0.247	0.598	0.242

Table 4

Empirical size and power based on 1000 replications for tests of equality of two covariance matrices.

In all cases, $n_1=n_2=50$.

Matrix pairs---------,cf section 4)	Nominal 5% level				
	Size			Power	
	(a)	(b)	(c)	(d)	(e)
	Normal				
Standard error	0.051	0.050	0.908	0.999	0.999
Grouping (L=5)	0.047	0.042	0.690	0.973	0.965
Jackknife	0.041	0.040	0.893	0.999	0.999
Tiku & Balakrishnan T^2	0.039	0.040	0.933	1.000	0.887
	Gamma (replications=500)				
Standard error	0.057	0.061	0.834	0.999	1.000
Grouping (L=5)	0.037	0.032	0.583	0.947	0.964
Jackknife	0.046	0.046	0.812	0.998	0.999
Tiku & Balakrishnan T^2	0.041	0.046	0.861	1.000	0.826
	Double exponential.				
Standard error	0.044	0.039	0.512	0.947	0.994
Grouping (L=5)	0.035	0.032	0.335	0.753	0.922
Jackknife	0.029	0.029	0.469	0.930	0.991
Tiku & Balakrishnan T^2	0.022	0.033	0.583	0.961	0.594
	Contaminated normal				
Standard error	0.038	0.030	0.370	0.795	0.923
Grouping (L=5)	0.034	0.016	0.215	0.584	0.765
Jackknife	0.026	0.023	0.325	0.759	0.910
Tiku & Balakrishnan T^2	0.033	0.034	0.467	0.853	0.448

Table 5

Empirical size based on 500 replications for the jaclkknife test of equality of two covariance matrices.

In all cases, $n_1=n_2=250$.

Matrix pairs (cf.section4)	Nominal 5% level			
	Untransformed		Transformed	
	(a)	(b)	(a)	(b)
Normal	0.042	0.062	0.046	0.068
Double exponential	0.022	0.060	0.034	0.058
Contaminated normal	0.038	0.038	0.094	0.080

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SOME APPLICATIONS OF FACTORIAL MOMENTS THEOREM

BY

AHMED ZOGO MEMON*

ABSTRACT

Factorial moments play an important role in determining probability distributions of random variables. In some situations it may be tedious to find these moments. A theorem that has its origin in Von R. Mises work, facilitates a relationship between factorial moments and certain probabilities. This paper discusses the relevance of this theorem to such situations, giving its applications to "join counts" in rectangular lattices.

INTRODUCTION

Let us write $X^{[r]}$ for the factorial expression $X(X-1)(X-2)\ldots\ldots\ldots.(X-r+1)$. If X is a random variable, the mathematical expectation of $X^{[r]}$ is called the *r*th factorial moment of X (or of the distribution of X) about the origin. This moment is usually denoted by $\mu'_{[r]}$. It is assumed (when reference is made to the *r*th factorial moment of a particular distribution) the appropriate integral (or sum, as the case may be) converges absolutely for that distribution.

For a two dimensional random variable (X, Y), the mathematical expectation of $(X^{[r]} Y^{[s]})$ is its factorial moment of order (r, s); r, s=1,2, 3, ... This definition can be extended on the same lines for the factorial moment of an n – dimensional random variable.

In statistical literature, factorial moments attract our attention for following important reasons. (i) Their calculation is easy for certain

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discrete distributions and the continuous distributions grouped in intervals. (ii) They provide very concise formulae for distributions of the binomial type. (iii) They are related to ordinary moments; that is, the *r*th moment of a distribution can be obtained from its first r factorial moments. However, there arise situations where even for distributions of the binomial type it becomes tedious to find factorial moments. To tackle such problems we can try the possibility of exploiting the factorial moments theorem in determining factorial moments of the random variable involved. If the conditions of this theorem permit, calculation of these moments may turn to be simple, convenient and quick. This paper attempts to briefly introduce the factorial moments theorem and give its applications in univariate and multivariate situations.

2. FACTORIAL MOMENTS THEOREM

The factorial moments theorem has its origin in the paper (6) by Von R. Mises. Later, this theorem has been demonstrated by Maurice Frechet (2), P.V.K. Iyer (4). Memon and David (5) use it to obtain the asymptotic distribution of lattice join counts. Fuchs and David (3) propose following multivariate analogue of this theorem.

STATEMENT OF THE THEOREM

Consider a finite set Ω of events, divided in some fashion into m subsets Ω_α containing respectively N_α events. Let w_α be a particular subset of Ω_α containing $n(w_\alpha)$ events. Let $P(w) = P(w_1 \ldots\ldots w_m)$ be the probability that $\Sigma n(w_\alpha)$ events in w_α materialize, and let $V(v) = V(v_1 \ldots\ldots, v_m)$ be the class of all $\binom{N_1}{V_1} \ldots\ldots\ldots \binom{N_m}{V_m}$ set vectors $(w_1, \ldots\ldots, w_m)$ that can be formed under the restriction $n(w_\alpha) = v_\alpha, \alpha = 1, \ldots\ldots, m$. Finally, let $I = (I_1, \ldots\ldots, I_m)$ be the *m* dimensional chance variable whose αth component counts the number of events of Ω_α that materialize. If $\mu'_{[v]} = \mu'_{[v_1,\ldots\ldots,v_m]}$ is the factorial moment of order v of I, then

$$\mu'_{[v]} = S(v)\prod_{\alpha=1}^{m}(v_\alpha) \quad ,$$

where $\quad S(v) = \sum_{V(v)} P(w)$.

Its proof is omitted in (3). The theorem can also be proved like as in (5) for the univariate case.

This theorem facilitates a relationship between the factorial moments and certain probabilities. So if it is possible to know these probabilities, the distribution of the random variable considered can be determined.

3. APPLICATIONS

We apply the above theorem to find factorial moments in following situations.

UNIVARIATE PROBLEMS

For such problems, take m = 1.

(a) Binomial distributions : Let us define random variables as

$$\phi_i \quad = 1 \text{ if the ith event occurs, } i = 1, 2, \ldots\ldots, n.$$
$$= 0, \text{ otherwise.}$$

When ϕ_is are assumed to be i.i.d., and p is the probability of the occurrence of the event under consideration, the random variable $X = \sum_{i=1}^{n} \phi_i$ has a binomial distribution with parameters n and p. Using the above theorem.

$$\mu'_{[r]} = r!\sum_{r} \Pr.(\phi_{i_1} = 1, \phi_{i_2} = 1, \ldots\ldots\ldots, \phi_{i_r} = 1)$$

$$= \frac{n!}{(n-r)!}\, p^r.$$

(b) Poisson distribution : If we take $\lim_{n\to\infty} np = \lambda$ in the factorial moment for the binomial distribution, the rth factorial moment of the Poisson distribution comes to λ^r.

(c) Two dimensional rectangular lattice : Suppose that at every point of a *lxm* lattice, the events B or W materialize with probabilities p and q respectively, where $p + q = 1$.

(i) Let X be the total number of horizontal and vertical BB joins. It is indeed difficult to find ordinary moments for the distribution of X. Moran (7) gives first four moments of this distribution. However if we use the factorial moments theorem, calculation of factorial moments is simplified to a great extent; from which the ordinary moments can be obtained. Memon and David (5) determine the factorial moments of X in this manner. To illustrate it, we take a simple case of 2x2 lattice. Here, the first moment is

$\mu'_{[1]} = 1!\ \Sigma_1$ Pr. (one particular BB join).

Σ_1 comprises all configurations in which only one BB join is possible. The probability of each of these outcomes remains p^2. So,

$\mu'_{[1]} = 4p^2$.

The second factorial moment is

$\mu'_{[2]} = 2!\ \Sigma_2$ Pr. (two particular BB joins).

Σ_2 extends over one two horizontal BB joins', one 'two vertical BB joins', four' one horizontal BB join and one vertical BB join' ; the probabilities for which are p^4, p^4 and p^3 respectively. So,

$\mu'_{[2]} = 4p^4 + 8p^3$.

Following the same approach we can show that

$\mu'_{[3]} = 24p^4$.

and $\mu'_{[4]} = 24p^4$.

(ii) If Y is taken as the number of diagonal BB joins in the above situation, the problem of calculation of moments by the above theorem no longer presents any difficulty. Here, it follows rather immediately that $\mu'_{[1]} = 2p^2$, and $\mu'_{[2]} = 2p^4$.

(iii) When the events B materialize at more than two points of a 2x2 lattice, triangles are formed. Suppose we consider the distribution of the number of such BBB triangles. For this distribution, one may verify that $\mu'_{[1]} = 4p^3$, $\mu'_{[2]} = 12p^4$, $\mu'_{[3]} = 24p^4$, $\mu'_{[4]} = 24p^4$.

(d) Binomial sequence : Consider the distribution of the number of BB joins between successive observations of a binomial sequence where the events B and W occur with probabilities p and q respectively. If n is the number of observations, the above theorem can be applied to find factorial moments for the distribution of the number of BW joins. Iyer (4) gives the factorial moments for this distribution as well as for the distribution of the number of BW and WB joins between the successive observations.

MULTIVARIATE PROBLEMS

(a) Consider a set Ω comprising five events E_1, E_2, E_3, E_4, E_5, with probabilities

$$\Pr.\ \{E_1, E_2, E_3, E_4, E_5\} = 1/2,$$

$$\Pr.\ \{\overline{E}_1, \overline{E}_2, \overline{E}_3, \overline{E}_4, E_5\} = 1/6,$$

$$\Pr.\ \{\overline{E}_1, E_2, E_3, E_4, \overline{E}_5\} = 1/6,$$

$$\Pr.\ \{E_1, \overline{E}_2, E_3, E_4, \overline{E}_5\} = 1/6,$$

(i) Let Ω_1 and Ω_2 be the subsets consisting of the events E_1, E_2 ; and E_3, E_4, E_5 respectively. By the above theorem for a two dimensional chance variable (I_1, I_2), the factorial moments of order (2,1) is

$$\mu'_{[2,1]} = S(2,1)\prod_{\alpha=1}^{2} (\nu_\alpha)$$

Here, $\prod_{\alpha=1}^{2} (\nu_\alpha)! = 2!\,1! = 2,$

$$S(2, 1) = \Pr.\{E_1, E_2; E_3\} + \Pr.\{E_1, E_2; E_4\} + \Pr.\{E_1, E_2; E_5\}$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

so that $\mu'_{[2,1]} = 3$

One may verify that $\mu'_{[2,2]} = 6$, $\mu'_{[2,3]} = 6$

(ii) Let $\Omega_1 = \{E_1\}$, $\Omega_2 = \{E_2, E_3\}$, $\Omega_3 = \{E_4, E_5\}$.

The chance variable (I_1, I_2, I_3) has the factorial moment of order $(1,1,1)$

$$\mu'_{[1,1,1]} = S(1,1,1)\prod_{\alpha=1}^{3}(\nu_\alpha)$$

$$\prod_{\alpha=1}^{3}(\nu_\alpha)! = 1,$$

and $S(1, 1, 1) = \Pr.\{E_1; E_2; E_4\} + \Pr.\{E_1; E_3; E_4\} + \Pr.\{E_1; E_2; E_5\} + \Pr.\{E_1; E_3; E_5\}$

$$= \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{6}\right) + \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{6}\right)$$

$$= 2\frac{1}{3}$$

So that required factorial moment of (I_1, I_2, I_3) is $2\frac{1}{3}$. We can also show that its $\mu'_{[1,2,1]} = 2$

(b) Let us take up again the problem of two dimensional rectangular lattice considered above, and suppose that X denotes the number of horizontal and vertical BB joins and Y, the number of diagonal BB joins. For the random (X, Y),

$$\mu'_{[1,1]} = 1!\,1!\sum P(w_1, w_2),$$

where Σ is over eight possible, "one horizontal or vertical BB join" and 'one diagonal BB join'. Consequently, this moment comes to $8p^3$. Similarly, $\mu'_{[2,1]} = 8p^3 + 16p^4$; $\mu'_{[2,2]} = 24p^4$

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STATISTICAL INFORMATION

(Some ethical aspects)

By

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Statistics in its infancy was deemed nothing but a mockery of figures. Gradually, it started establishing its roots firmly in various public and private organizations, educational and research institutions. The early statisticians who made contributions in the development of statistical theory might not have visualized the extent of its future involvement in a community's life. Even when Statistics was known to many laymen through the phrase "there are lies, damn lies, and statistics", this discipline of knowledge kept on experiencing a rapid, honourable and effective growth due to its ever-increasing importance. Although some people still raise their eyebrows with odd suspicion at the name of this innocent science, their cynical attitude has much to do with the ethics of a statistician.

Statistical information where in the form of numerical data or a statistical concept evaluated, is a product generally not produced by just one hand. In whatever manner the information may be procured, it is regarded now-a-days an indispensable equipment of an educated mind for decision making. The important consideration is that statistical information should be useable so as to provide a basis on which a confident decision can be taken. It is primarily this reason that scientists and administrators are attracted to make use of Statistics in their respective fields.

Statistical information may not necessarily proceed from a single statistician or an individual qualified also in Statistics. At times it may be too erroneous, or contaminated by numerous errors and biases. As long as there are no deliberate errors, or manipulation of data to suit a particular interest, there is nothing unethical about it.

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1. CLASSES OF STATISTICIANS

Marriage of Statistics with any other science creates possibly a new speciality. Although there exist numerous specialities in Statistics, statisticians may be classified into three broad groups (teaching not included) in view of different levels of ethical problems springing up in their relationship with those with whom they have to come in contact for professional purposes. We have:

(i) Theorists who create concepts, theories, methods for collection and analysis of statistical data.

(ii) Specialists in agriculture, economics, accounting, education, health or other fields of knowledge. They are mostly concerned with theoretical or applied research for the advancement of their subject matter through statistical applications.

(iii) Those who render useful services to a community by the use of statistical tools in collection, tabulation, analysis and interpretation of numerical data. They may have a speciality in an area that bears relationship with the inquiry to be organized and managed.

The last two groups consist of what we term applied statisticians, but the distinction in them is motivated for ethical reasons.

The ethics of a statistical theorist do not worry a community in general. He is a researcher like any other theoretical scientist engaged in pursuit of profound discoveries. Nor the problem of statistical conduct becomes often acute or alarming in the case of the second group of statisticians. The scope of their statistical activities is distinctly defined and it is too narrow to bring them in sharp confrontation with the special interests of others. But the professionals collecting and analysing data carry the overall responsibility of organizing and executing statistical inquiries. Since in extensive inquiries there are many problems of management, the ethical challenge for them is in nature quite unlike that of their fellows in other classes.

2. MANAGEMENT OF STATISTICAL INQUIRES

Statistical inquires are fairly complicated when the material to be covered is large. The census of population is one example. An inquiry is normally made through a sample survey due to economic and technical reasons. Whether it is based on sampling or complete enumeration, its management involves administrative, financial and professional work (planning of sample surveys, personnel training, data collection, analysis and report writing). This task is executed some how. He may have to tackle a variety of complex situations; he has to be a good manager and a capable professional. His conduct may invite quite a criticism from people. Sometimes he is put to enormous inconvenience too. Even when he has a weak public concern, he is to be mentally ready to brave their anger, aggression, indifference and even humiliation. And, he must be honest in his dealings. Should he be not willing to make an assidous effort honestly, the results of his inquiry remain confounded by errors including those which can be eliminated otherwise.

3. OUR EXPECTATIONS FROM A STATISTICIAN IN (GROUP (iii))

Of course, the professional competence promotes the quality, value and respect of his statistical work, and has a positive effect on its reliability. But assuming that the statistician undertaking a certain statistical mission possesses the required qualifications, let us have glance at one's expectations from a fact-finding statistician as well as at the nature of his ethical problems. Perhaps, this is all the more important when intriguing influences about the subject of Statistics still flicker in the minds of people, or tend to invade or abuse their trust in its practitioners.

To all human endeavour, it is true that honesty, loyalty and dependability apply imperiously with more or less equal force. But a statistician has to accomplish his truth discovery task patiently and boldly through a chain of constantly treating barriers of limited time, trained personnel, equipment and budget. And then, the respondents too are being involved in this affair. Weather is yet another factor that may add problems and effect his task adversely. Any of these factors could offer him a temptation to follow an easy-go path and pollute his fact finding mission. To unfold the truth concerning the parameters in a statistical inquiry, he seeks to look for the best possible course of action, both theoretically permissible and practically feasible, within the specified limits of resources. Whether it is collection, analysis, or interpretation of a statistical data, all these stages of statistical work need thorough care, meticulous attention and intellectual honesty of the statistician and his field and other staff involved. Since the basic information has to be the ground on which the whole monument of sophisticated analysis and conclusions is to be erected, it is imperative for him to make an impartial attempt to have this information measured as accurately as possible. If his respondent does not want to be quoted, referred to or identified, this anonymity should be firmly preserved at all costs. If his employer or client wants the findings of a study to stay confidential, this trust must not be betrayed. A great moral responsibility lies on the statistician, and to fulfill it he must be sure that his staff shares it to his satisfaction. It ought to be the bounding duty of a statistician to train his staff not only in `how to execute a statistical project' but also in `why to be intellectually honest'. Such training should be designed so as to hopefully expect to relieve his staff members from the pitfalls of figurative boredom, make them realize of enormous urgency and usefulness of their service, keep their interest alive in pursuit of `truth and nothing but truth', and indicate them with moral obligations in relation to their respondents. They are very important obligations in relation to their respondents. They are very important participants in this whole business. The statistician concerned must not be incapable of creating in them this kind of human feeling.

4. STATISTICIAN'S ETHICAL PROBLEMS

For a public, private or scientific survey the statistician has to mainly carry out dealing with (i) the client who hires him, (ii) subordinate statistical staff for assistance in organizing a survey, (iii) the respondents as subjects in an inquiry. So we should consider his ethical problems with respect to clients, subordinate staff, and respondents separately.

4.1 His Clients

Not all the clients of a statistician use his services for some impartial motive. Sometimes, he is faced with clients who want to do poorly by exploiting their statistician, getting his sanction or even forcing him to project the prefabricated conclusions suiting their needs. When a client comes to a physician for treatment, his attitude is to cooperate, otherwise, he knows his chances of getting well remain obscure. But with a statistician this equation could turn to be different. The client of a statistician may wish to maintain certain convictions or theories by cooking up data with the aid of his statistician's connivance, participation or sanction as his involvement is liable to give the minimal chance to others of catching lies. He may make unreasonable requests or exercise on him an improper pressure or influence to prevent certain features of an inquiry to serve his special interests. This is in fact a very difficult situation for the statistician, and it is here that he has to stay clear or comply with and bring disgrace to his profession. The situation as such directly tests his normal courage. He has to either submit, or use diplomacy to survive or leave his job. Remember, the client could be a government too. So , I will avoid any discussion on this point because of its sensitive nature. But the misfortune is the statistician is made to wear this ugly coat of statistical responsibility. Writing in a leading American journal, a statistician says: " I became aware of this early in my medical consulting career when in a cooperative venture to organize some data for presentation in a legal case a physician suggested calculating the average survival time of a group of cancer patients using the data from......". Not only this, the statistician may also get threats from those whose interests are exposed to risks because of his discoveries. In the same journal, another statistician narrates: "When our quantitative bio-statistical epidemiological studies on the hazards of diagnostic medical X-rays hit the headlines, I got a call from an irate Rochester Radiologist. He complained bitterly that our findings had reduced the business of radiologists by 40%. He then told me he was calling me up before a medical ethics committee that would take away my M.D. He was disappointed when he came to learn that I didn't have one."

The misuse of a statistician is an irritable offense for which the responsibility anyway lies on both, but the main culprit is the statistician who submits and behaves immorally or even criminally. It is sad that all professions even the noblest ones, suffer from this disease.

4.2 His Field Staff

In undertaking public or private surveys of moderate or big size, a statistician has to depend a lot on the field staff for collection of data. Under his control, advice or guidance this staff is to grapple with the difficulties awaiting them in the information collection process. The statistical conduct of these interviews or enumerators is important in achieving and preserving the purity of the initial information. Their job is hard indeed which consists in frequent traveling devoid of any consideration for the type of season, hot or cold, dry or rainy, searching for and knocking at the doors of the persons selected, explaining the purpose of their visit and drawing gently the correct information. They are the first to-be truth seekers. Generally, they are not qualified enough to appreciate the comparative significance of numbers in relation to statistical analysis. For them the inquiry may mean nothing but numbers, a mess of numbers. Even when they do not want to be dishonest; they could make certain mistakes in using their common sense in abnormal situations. As long as they do not commit these errors deliberately or carelessly, there is nothing unethical about it no matter how bad the inquiry's objectives are hit; we cannot forget the maxim, in moral philosophy, "Error destroys action", and so a mission. Sometimes the dishonesty of an enumerator may be due to some kinds of fear and there he may be prone to inventing figures or display carelessness. Once I was asked a question by a statistical investigator in a West African country, "What would you do", Sir," If, like me, you expect to pass a tortuous night in a village where you know the cannibals also dwell, and who may crave to perform some rituals on your blood." In another African country, I happened to meet a field reporter who was discovered to have reported the measurements of sample fields from the number of paces that his horse was making through the fields to avoid snake biting.

Some enumerators slink out is doing hard work, or exploit their simple respondents for their selfish motives. Instead of creating a friendly atmosphere and putting their respondents at ease they may attempt to frighten them or even engage in temporary romance with opposite sexes. In several areas, the people in developing countries are generally ignorant, and an enumerator can force his respondents into submission by posing as a special agent of ruling government. His exploitation may range from securing food to anything including adultery. In latter situation, an enumerator could lose not only a limb but life too; incidents of this type have come to our notice. Anyway, the effects of such an undesirable behaviour of enumerators become manifest in the results of the survey. A strict vigilance of the statistician incharge is essential during collection of basic data. When the concepts or definitions used in a survey are complex, the enumerators' unethical attitude is also attributable to their incompetence and ignorance. Not only that the staff ought to know the relevant meanings behind the figures, the statistician must acquaint them with the real value, usefulness and importance of his statistical inquiry. This serves a great motivating force in reminding them of their genuine responsibilities. A feeling is developing in the Western countries that the statistician should indoctrinate his field and other staff with some sort of ethical code in order to minimize the possibility of their falling susceptible to unwelcome temptations detrimental to the inquiry. By example as well as by formal or informal training he should instill in them the principles of such a code.

4.3 His Respondents

Other than his field staff, fact-finding statistician has to depend a lot on the respondents selected in his statistical project. Their cooperation is also a major factor in improving its accuracy. The problems confronted in data collection and analysis could swell enormously to a formidable extent when the survey includes respondents who for one or other reason cannot give satisfactory information, or happen to be obstinately erroneous, tricky and misleading, or do not wish to cooperate simply. The number of such respondents may not be reasonably large but they are not only an unpleasant source of wasting already limited time of the field staff, their responses are liable to provoke changes in the use of selected statistical techniques. Why should such a respondent behave like that? Is it just his nature, or are there some genuine reasons too for an abnormal attitude? Is he too, much conscious of his right to be let alone for maintaining and preserving his privacy? Experience shows that this kind of behaviour as much to do with the way he is approached for information. He may have an inflated image or prestige, and wishes it to go on. He may have a fear that the release of true information could harm his interests. He may be scared of being blackmailed too. `Why me in a sample? is often a puzzling question in his mind - a very common feeling among the respondents. He may pose this question to the interviewer to calm his fears, but if he remains unsatisfied, naturally he might not take any chance in disclosing the truth. The unwillingness on the part of the respondents may also be due to inconvenience that they are subjected during interviews, or experimentation in medical surveys to some sort of risk too.

In general, the respondents have a tendency to remain suspicious with respect to possible unwarranted disclosure of information about them. What guarantee, they ask, do they have against the misuse of their information. It is only the verbal assurance, desperate but solemn, given to them by a statistician and his staff. Certainly the purpose of a statistical inquiry is not to misuse such information and make a blatant invasion of their privacy.

With the development of informational technology as being made possible by the electronic computers, the threat to the privacy of information about individuals or companies is expected to increase. When it comes to ethics, "There are certain acts which when performed on similar occasions have consequences more than times as great as those resulting from one performance."

5. Possible Remedial Measures

In order to enhance the efficacy of decisions there is an urgent need to promote, or even arouse a mass awareness about the importance of statistical information in the community - a responsibility that normally devolves on the national statistical associations. To achieve this purpose partially, the frequency of popular lectures / seminars / consultancy in applied statistics ought to be increased. A larger emphasis on applications in the statistics courses offered by the teaching institutions would also contribute to this cause.

Perhaps, it is time that the central and provincial statistical offices and statistics department of educational institutions should collaborate to evolve for practicing statisticians a framework of ethical norms enunciating their responsibilities to the clients and respondents.

Let the students be formally acquainted with the statistical ethics for their future obligations to the society before they leave their educational institutions with degrees in Statistics.

The above measures can be useful in significantly improving the quality of statistical information; respectfully upholding public confidence in the wise use of statistical data; and finally in elevating a statistician's integrity in the community.

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EFFICIENCY AND SUPER – EFFICIENCY OF ESTIMATES

BY

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ABSTRACT

Asymptotic theory of estimation based on the criteria of consistency and efficiency is considered and certain methods are shown to yield estimators satisfying this criteria. some results with examples are discussed in which the idea of super – efficiency is not statistically important. We also discuss asymptotic properties of some methods by considering two – piece normal distribution.

1. INTRODUCTION

Much of the work on efficiency was given by an attempt to understand how well the maximum likelihood estimate performs as n goes to infinity. The principle of maximum likelihood proposed by Fisher consists in adopting as an estimate of a parameter θ, the particular value of the parameter which maximizes the probability of the facts actually observed.

Let $X_1, X_2, \ldots\ldots X_n$ be i. i. d. according to a density $f(x\,;\theta)$ and suppose that $T_n = T_n(\underline{x})$ be an estimator of θ. Our object is to examine the asymptotic properties of the estimator T_n as $n\to\infty$. The existence of the first two derivatives of f(x ; θ) with respect to θ allows us to introduce Fisher's amount of information. Let us define

$$I(\theta) = E_\theta\left\{-\frac{\partial^2 \log f(x;\theta)}{\partial^2\theta^2}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (1.1)$$

as the Fisher information function associated with density f(x ; θ). If the density f(x ; θ) satisfies suitable regularity conditions, the celebrated Cramer – Rao inequality states that the variance of any unbiased estimator T_n of g(θ) satisfies

$$Var_\theta(T_n) \geq \{g'(\theta)\}^2/n\, I(\theta) \quad \ldots\ldots\ldots\ldots\ldots \quad (1.2)$$

Suppose that $T_n(\underline{x})$ is asymptotically normal :

$$n^{1/2}\{T_n - g(\theta)\} \rightarrow N\{0, v(\theta)\},\ v(\theta) > 0 \quad \ldots\ldots\ldots \quad (1.3)$$

with $v(\theta) \geq \{g(\theta)\}^2 / I(\theta)$;

If (1.3) holds, T_n will be said to satisfy the Fisher's idea of efficiency. A sequence (T_n) satisfying (1.3) with

$$v(\theta) = \{g'(\theta)\}^2 / I(\theta) \quad \ldots\ldots\ldots\ldots\ldots \quad (1.4)$$

is said to be asymptotically efficient. For g(θ) = θ,

we have

$$v(\theta) \geq I^{-1}(\theta) \quad \ldots\ldots\ldots\ldots\ldots \quad (1.5)$$

For a long time, it was believed that for a consistent asymptotically normal (CAN) estimator the asymptotic variance v(θ) satisfied (1.5) subject only to regularity conditions on the density f(x ; θ). Unfortunately, as everyone knows today that the result (1.5) is not true in general as shown by the examples due to Hodges (Le Cam, 1953). Therefore, there is no lower bound to the asymptotic variance of a CAN estimator, so there does not exist any best CAN estimator, without any further conditions on the estimator Le Can (1953) proved the remarkable result that (1.3) does entail $v(\theta) \geq I^{-1}(\theta)$ almost everywhere (with respect to Lebesgue measure).

<u>Example 1.1</u>

Let $X_1, \ldots . X_n$ be i. i. d. according to the normal distribution $N(\theta, 1)$. In this case $I(\theta) = 1$, and equation (1.5) reduces to $v(\theta) \geq 1$. On the other hand, consider the sequence of estimators,

$$T_n = \begin{pmatrix} \overline{X} & |\overline{X}| > n^{-1/4} \\ C\overline{X} & |\overline{X}| > n^{-1/4} \end{pmatrix}$$

Where $\overline{X}$ is the average of the n observations and C is an arbitrary constant, Then

$$n^{1/2}(T_n - \theta) \rightarrow N\{0, v(\theta)\},$$

with $v(\theta) = 1$ when $\theta \neq 1$ and $v(\theta) = C^2$ when $\theta = 0$. If $C < 1$, inequality (1.3) is therefore violated at $\theta = 0$.

2. SUPER – EFFICIENCY

For a long time it was believed that for consistent asymptotically normal (CAN) estimator the asymptotic variance $v(\theta)$ satisfied (1.3) subject only to regularity conditions on the density $f(x\ ;\ \theta)$. Unfortunately, this is not strictly true without any restrictions on the estimating function. This belief was finally exploded by the examples of Hodges to show that the results (1.3) is not true in general. We can call an estimator super–efficient if for all parameter values the estimator is asymptotically normal around the true value with a variance never exceeding and sometimes less than the Cramer – Rao lower bound.

Since Hodge's examples became known many attempts have been made to rescue the Fisher program. All the serious attempts fall under the following two approaches: In the first approach the authors prove that , for any competing estimators (T_n) which satisfy (1.3), the set of points of super – efficiency have Lebesgue measure Zero. That the points of super – efficiency constitute a set of measure zero was stated in LeCam (1953). In the second approach one imposes conditions on the competing estimators which (conditions) are sufficient to eliminate the super – efficient estimators from competition. A number of attempts have been made to rule out super – efficient estimators by imposing regularity conditions which the competing estimators must satisfy.

Rao (1963) established the results that the asymptotic variance of consistent uniformly asymptotically normal (CUAN) estimator has Fisher's lower bound $I^{-1}(\theta)$ when the probability density satisfies some regularity conditions. It appears then that in the examples of Hodges and LeCam, super – efficiency in the sense of having asymptotic variance less than $I^{-1}(\theta)$ has been achieved at the sacrifice of uniform convergence.

Since there is no lower bound to the asymptotic variance of a CAN estimator, it may be thought that an improvement is possible by constructing a statis T_n with a uniformly lower asymptotic variance and thereby increasing the concentration at every value of the parameter, as at $\theta = 0$ in example (1.1). It is no doubt true that an estimator having a higher concentration than another for every value of θ is more useful in drawing inferences on θ from an observed estimate. From the point of view of a statistician, the most appealing sufficient condition is the one due to Rao (1963) who strengthens (1.3) by requiring that it be uniform on compacts. Assuming $n^{1/2}(T_n - \theta)$ is A. N. $\{0, v(\theta)\}$ uniformly on compacts, Rao proves (1.5).

Rao's proof by using Neyman – peasons lemma states that for testing the simple hypothesis $\theta = \theta_0$ against the simple alternative $\theta=\theta_0+n^{1/2}$, the test based on the m. l. estimator is asymptotically most powerful.

LeCam(1953) has shown that the set of super-efficiency must be of Lebesgue measure zero. However, no every set of measure zero can be a set of super efficiency. For certain classes of loss functions and probability densities, he has proved that if an estimate is super-efficient at a given parameter value θ_0, then there must exist an infinite sequence$\{\theta_n\}$ of values at which this estimate is worse than the M.L estimate. Let $\{\sigma_n^2(\theta)\}$ be the asymptotic variance of the M.L. estimate of the parameter θ and $\{\sigma_n^*(\theta)$ that of an alternative estimate T_n. Then T_n is super-efficient if

$$n \xrightarrow{\lim} \infty\, SUP \frac{\sigma_n^*(\theta)}{\sigma_n(\theta)} < 1 \qquad (2.1)$$

or equivalently with slightly stronger condition,

$$\lim_{n \to \infty} \frac{\sigma_n^*(\theta)}{\sigma_n(\theta)} \leq 1 - \delta \tag{2.2}$$

where $\delta > 0$. If an estimate T_n satisfies condition (2.2), then we shall say that T_n is uniformly super-efficient.

Example (2.1)

Let $\{T_n\}$ be a consistent asymptotically normal (CAN) estimate of θ. Let $\{\sigma_n^2(\theta)\}$ be the asymptotic variance of T_n. If there exists a non-negative sequence of numbers $\{\beta_n\}$ satisfying the conditions:

$$\left.\begin{array}{ll} (i) & \lim_{n \to \infty} \beta_n = 0 \\ (ii) & \lim_{n \to \infty} \dfrac{\beta_n}{\sigma_n(\theta)} = \infty \end{array}\right\} \tag{2.3}$$

and let X be a normal random variable with an unknown mean θ and with variance 1. Then the M.L. estimate of θ based on n independent observations is the arithmatic mean, say $\overline{X}_n$ of the observations. Its variance is $\sigma_n^2 = 1/n$. To produce an estimate T_n uniformly super-efficient, it is sufficient to produce a sequence $\{\beta_n\}$. Because of the particular form of the variance σ_n^2, it is obvious that we can take $\{\beta_n\} = n^{-1/4}$. Suppose that T_n is an estimate for which (2.3) are satisfied. Then they are also satisfied for any estimate which is more efficient than T_n in Fisher's sense. Hence the existence of at-least one estimate T_n satisfying the conditions (2.3) implies the inexistance of an estimate efficient in Fisher's sense.

According to Cox and Hinkley (1974), super-efficiency is not a statistically important because such estimators give no improvement when regarded as test statistics. For any fixed n the reduction in mean square error for parameter points near to the point of super-efficiency is balanced by an increase in mean square error at points a moderate distance away.

3. EFFICIENCY OF M L ESTIMATORS

Asymptotic properties of ML estimates have been discussed in the literature under two separate lines. Some authors, including Cramer (1946) and Gurland (1954), have considered the roots of the likelihood equation, while others, including Wald (1949) and Wolfowitz (1949), have discussed the parameter value which yields the absolute maximum of the likelihood function. Cramer has proved that, under certain regularity conditions, the MLE of θ is consistent and asymptotically efficient. One of his conditions is that for every $\theta \varepsilon \Theta$

$$\left.\begin{array}{l} \left|\dfrac{\partial^{3} \log f(x;\theta)}{\partial\theta^{3}}\right| < G(x) \quad and \\ \int_{-\infty}^{\infty} G(x)\, f(x;\theta)\,dx < \infty \end{array}\right\} \qquad (3.1)$$

Kulldorff (1957) replaced this condition with much weaker one. He introduces the following two conditions in the replacement of (3.1) and shows that MLE is asymptotically efficient under the set of new conditions.

(i) There exists a positive function $g(\theta)$

and $\theta_1, \theta_2 \varepsilon \Theta$ such that

$$\frac{1}{\theta_1-\theta_2}\int_{-\infty}^{\infty}\left(\frac{\partial \log f(x;\theta)}{\partial\theta}\right)_{\theta=\theta_1} f(x;\theta_2) < 0$$

where $0 < |\theta_1 - \theta_2| < g(\theta_2)$ (3.2)

(ii) There exists a positive differentiable function p(θ), then for every θ ε θ

$$\frac{\partial}{\partial\theta}\left\{P(Q)\,\frac{\partial \log f(x;\theta)}{\partial\theta}\right\}$$

is continuous function of uniformly in x. (3.3)

Example (3.1)

Consider the following density function

$$f(x;\theta_1,\theta_2) = (2\pi\theta_2)^{1/2} \quad \exp\left\{-\frac{(x-\theta_1)^2}{2\theta_2}\right\}$$

we have the unique roots for θ_1 , and θ_2 are

$$\hat{\theta}_1 = \frac{1}{n}\sum_{i=1}^{n} x_i \quad and \quad \hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

These estimates are consistent, asymptotically normal and jointly asymptotically efficient. Now let the random variable X, is normally distributed with mean zero and variance θ. We have likelihood function.

$$L = (2\pi\theta)^{-n/2} \quad \exp\left(-\sum_{i=1}^{n}\frac{x_i^2}{2\theta}\right)$$

and the likelihood equation for θ has the unique root $\hat{\theta}_n = \frac{1}{n}\sum_{i-1}^{n} x_i^2$.

This estimate is consistent and asymptotically efficient. However, we find that the expression

$$\frac{\partial^3 \log f(x;\theta)}{\partial\theta^3} = -\frac{1}{\theta^3} + \frac{3x^2}{\theta^4}$$

tends to infinity as $\theta \to 0$, and is not bounded in the open interval $0 < \theta < \infty$ showing that condition (3.1) is not fulfilled. Now checking conditions (3.2) and (3.3), we find that

$$\frac{1}{\theta_1 - \theta_2} \int_{-\infty}^{\infty} \left(\frac{\partial \log f(x;\theta)}{\partial\theta} \right) \theta = \theta_1 \; f(x;\theta_2) dx = -\frac{1}{2\theta_1^2}$$

which is less than zero and so satisfies condition (3.2). Now introducing positive and differentiable function $p(\theta) = \theta^2$.

We have

$$\frac{\partial}{\partial\theta} \left\{ p(\theta) \frac{\partial \log f(x;\theta)}{\partial\theta} \right\} = -\frac{1}{2}$$

which is continuos of θ uniformly in x, and condition (3.3) is also satisfied. DOSS (1962) also presented a set of conditions which are satisfied by the above example. He shows that under these conditions, $\hat{\theta}_1$ and $\hat{\theta}_2$ are consistent, unique, asymptotically normal and jointly asymptotically efficient.

Most of the authors discuss large sample estimation which requires regularity conditions on the second derivative of the likelihood for the MLE to be asymptotically efficient. However, cases are known which are not covered by these regularity conditions. For example, in the density function $f(x;\theta) = \frac{1}{2} e^{-|x-\theta|}$, the sample median is MLE of θ, which is asymptotically normal with variance equal to Cramer-Rao lower bound. But $\partial \log f(x;\theta) / \partial\theta$ is discontinuous and

$\frac{\partial^2 \log f(x;\theta)}{\partial \theta^2} =$ for almost all x. Daniels (1961) applied some weaker conditions for asymptotic efficiency. He proved asymptotic efficiency of $\hat{\theta}$ under the following conditions:

(i) *log f(x;θ)* is continuous in θ through out θ . At every θ_O there is a neighbourhood such that for all θ, θ` in it,

$$|\log f(x;\theta) - \log f(x;\theta')| < S(x,\theta_o)\,|\theta - \theta_o'| \quad and$$
$$E(S^2/\theta_o) < \infty$$

(ii) At every $\theta, \partial \log f(x;\theta)/\partial\theta$ exists and is continuous for almost all x. It is not almost everywhere zero. It is a no where increasing and some where decreasing function of θ.

According to Daniels, it is enough to apply the condition that the function *log f(x;θ)* is convex in θ for almost all x. A sequence of ML estimates is then under θ asymptotically distributed according to $N\{\theta, I^{-1}(\theta)\}$. This result is of interest because it states the asymptotic normal distribution of a sequence of ML estimates without explicit assumptions on the existence of the second derivative of the function *log f(x;θ)*. On the other hand, it is well known that the second derivative of a continuous and convex function exists upto a set of Lebesgue-measure zero.

Example (3.2)

A random variable is said to have the two-piece normal (TPN) distribution with parameters $\theta_1, \theta_2, \theta_3 > 0$ if it has probability density function

$$f(x;\theta_1,\theta_2,\theta_3) = \begin{cases} \left(\frac{2}{\Pi}\right)^{\frac{1}{2}} (\theta_2+\theta_3)^{-1} \exp\left\{\frac{-(x-\theta_1)^2}{2\theta_2^2}\right\}, & x \le \theta_1 \\ \left(\frac{2}{\Pi}\right)^{\frac{1}{2}} (\theta_2+\theta_3)^{-1} \exp\left\{\frac{-(x-\theta_1)^2}{2\theta_3^2}\right\}, & x > \theta_1 \end{cases} \tag{3.4}$$

This distribution was introduced as the joined half-Gaussian by Gibbons and Mylroie (1973) who found it to be a very good fit to impurity profiles data in ion-implantation research. John (1982) discusses estimation of the parameters of the TPN distribution by the method of maximum likelihood and the method of moments. In this paper we examine the asymptotic properties of some methods for TPN distribution.

Let $X_1, X_2, \ldots\ldots\ldots\ldots, X_n$ be a random sample from the TPN distribution with density (3.4). For the MLE's $\hat{\theta}_1, \hat{\theta}_2$ *and* $\hat{\theta}_3$ of θ_1, θ_2 *and* θ_3 respectively, differentiating log-likelihood of (3.4) with respect to θ_1, θ_2 *and* θ_3 we have MLE's of θ_1, θ_2 *and* θ_3

$$\hat{\theta}_1 = \frac{\Sigma x_i}{n}$$

$$\hat{\theta}_2 = \left\{\frac{\Sigma(x_i-\bar{x})^2}{n-\Sigma(x_i-\bar{x})^2}\right\}^{\frac{1}{2}}$$

$$\hat{\theta}_3 = n\left(\frac{\left\{\Sigma\left(x_i-\bar{x}\right)^2\right\}^{\frac{1}{2}}-\left\{\Sigma\left(x_i-\bar{x}\right)^2\right\}^{\frac{3}{2}}}{\left\{n-\Sigma\left(x_i-\bar{x}\right)^2\right\}^{\frac{3}{2}}}\right)$$

Large sample approximations for the variance-covariance matrix of the MLE's are obtained by the inverse Fisher information matrix. The sample information matrix is

$$B_n = \begin{pmatrix} n(\theta_2^{-2}+\theta_3^{-2}) & 2\theta_2^{-3}\Sigma(x_i-\theta_1) & 2\theta_3^{-3}\Sigma(x_i-\theta_1) \\ 2\theta_2^{-3}\Sigma(x_i-\theta_1) & 3\theta_2^{-4}\Sigma(x_i-\theta_1)^2-n(\theta_2+\theta_3)^{-2} & -n(\theta_2+\theta_3)^{-2} \\ 2\theta_3^{-3}\Sigma(x_i-\theta_1) & -n(\theta_2+\theta_3)^{-2} & 3\theta_3^{-4}\Sigma(x_i-\theta_1)^2-n(\theta_2+\theta_3)^{-2} \end{pmatrix}$$

Large sample variance approximations of the MLE's are most useful when the estimators are asymptotically normal. This is not immediately obvious in the case of the TPN distribution. To show that the MLE's are asymptotically normal it is sufficient to consider the case of estimating θ_1 with θ_2 and θ_3 known. In this example, we see that all the regularity conditions given in the earlier sections for the asymptotic normality of $\hat{\theta}_1$ are satisfied except that involving the third derivative of the log-likelihood of (3.4).

According to John (1982), to prove this normality result it is sufficient to check that

$$d_n = n^{-1}\left[\ell''(\theta_1 + kn^{-\frac{1}{2}}\,\ell''(\theta_1)\right] \xrightarrow{P} 0$$

as $n \to \infty$, where k is a constant and ρ" denotes the second derivative with respect to θ_1 of the log-likelihood. Thus $d_n \to 0$ in probability and hence

$n^{-\frac{1}{2}}(\hat{\theta}_1 - \theta_1) \xrightarrow{D} N[o, I^{-1}(\theta)]$ *as* $n \to \infty$, where $I(\theta)$ in this case is equal to $(\theta^2\theta^3)^{-1}$.

The MM estimates of θ_1, θ_2 and θ_3 are

$$\theta_1^+ = \bar{x} - (2/\pi)^{\frac{1}{2}} (\theta_3 - \theta_2), \qquad \theta_2, \theta_3 \text{ known}$$

$$\theta_2^+ = \theta_3 + (\pi/2)^{\frac{1}{2}} (\bar{x} - \theta_1), \qquad \theta_1, \theta_3 \text{ known}$$

By using large sample properties of MLE's, the efficiencies of these estimators relative to the MLE's are

$$E_1 = \frac{Var\,(\hat{\theta}_1)}{Var\,(\theta_1^+)} = \left[1 + \frac{(\pi+2)}{\pi}\left\{\frac{\theta_2}{\theta_3} - 2 + \frac{\theta_3}{\theta_2}\right\}\right]^{-1}$$

$$E_2 = \frac{Var\,(\hat{\theta}_2)}{Var\,(\theta_2^+)} = 2\left(\frac{\theta_3}{\theta_2} + 1\right)^2\left(\frac{3\theta_3}{\theta_2} + 2\right)^{-1}\left[(\pi - 2)\left(\frac{\theta_3}{\theta_2} - 1\right)^2 + \frac{\pi\theta_3}{\theta_2}\right]^{-1}$$

Numerical values of these efficiencies are obtain by some given parameter values. The plots of the efficiencies against some parameters values are shown in figure-1.

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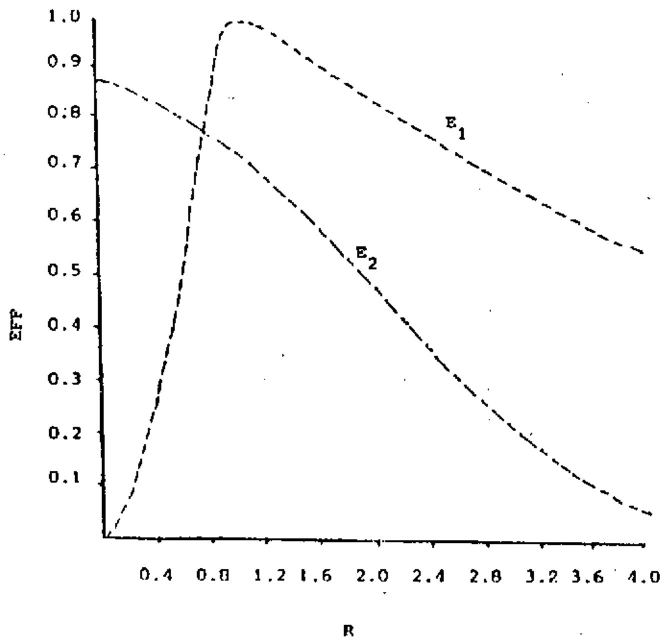

Fig.1 Plots for the efficiencies of MM estimators relative to the MLE's against some paramaters values ($R = \theta_2 / \theta_3$).