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# SOME APPLICATIONS OF FACTORIAL MOMENTS THEOREM

*BY*

AHMED ZOGO MEMON*

## ABSTRACT

Factorial moments play an important role in determining probability distributions of random variables. In some situations it may be tedious to find these moments. A theorem that has its origin in Von R. Mises work, facilitates a relationship between factorial moments and certain probabilities. This paper discusses the relevance of this theorem to such situations, giving its applications to "join counts" in rectangular lattices.

## INTRODUCTION

Let us write $X^{[r]}$ for the factorial expression $X(X-1)(X-2)\ldots\ldots\ldots(X-r+1)$. If X is a random variable, the mathematical expectation of $X^{[r]}$ is called the *r*th factorial moment of X (or of the distribution of X) about the origin. This moment is usually denoted by $\mu'_{[r]}$. It is assumed (when reference is made to the *r*th factorial moment of a particular distribution) the appropriate integral (or sum, as the case may be) converges absolutely for that distribution.

For a two dimensional random variable (X, Y), the mathematical expectation of $(X^{[r]}\,Y^{[s]})$ is its factorial moment of order (r, s); r, s=1,2, 3, . . . . . This definition can be extended on the same lines for the factorial moment of an n – dimensional random variable.

In statistical literature, factorial moments attract our attention for following important reasons. (i) Their calculation is easy for certain

* *Former Professor of Statistics, University of Benghazi, Libya*

discrete distributions and the continuous distributions grouped in intervals. (ii) They provide very concise formulae for distributions of the binomial type. (iii) They are related to ordinary moments; that is, the *r*th moment of a distribution can be obtained from its first r factorial moments. However, there arise situations where even for distributions of the binomial type it becomes tedious to find factorial moments. To tackle such problems we can try the possibility of exploiting the factorial moments theorem in determining factorial moments of the random variable involved. If the conditions of this theorem permit, calculation of these moments may turn to be simple, convenient and quick. This paper attempts to briefly introduce the factorial moments theorem and give its applications in univariate and multivariate situations.

## 2. FACTORIAL MOMENTS THEOREM

The factorial moments theorem has its origin in the paper (6) by Von R. Mises. Later, this theorem has been demonstrated by Maurice Frechet (2), P.V.K. Iyer (4). Memon and David (5) use it to obtain the asymptotic distribution of lattice join counts. Fuchs and David (3) propose following multivariate analogue of this theorem.

### STATEMENT OF THE THEOREM

Consider a finite set $\Omega$ of events, divided in some fashion into m subsets $\Omega_\alpha$ containing respectively $N_\alpha$ events. Let $w_\alpha$ be a particular subset of $\Omega_\alpha$ containing $n(w_\alpha)$ events. Let $P(w) = P(w_1 \ldots\ldots w_m)$ be the probability that $\Sigma n(w_\alpha)$ events in $w_\alpha$ materialize, and let $V(v) = V(v_1 \ldots\ldots, v_m)$ be the class of all $\binom{N_1}{V_1} \ldots\ldots\ldots \binom{N_m}{V_m}$ set vectors $(w_1, \ldots\ldots, w_m)$ that can be formed under the restriction $n(w_\alpha) = v_\alpha, \alpha = 1, \ldots\ldots, m$. Finally, let $I = (I_1, \ldots\ldots, I_m)$ be the *m* dimensional chance variable whose $\alpha$th component counts the number of events of $\Omega_\alpha$ that materialize. If $\mu'_{[v]} = \mu'_{[v_1, \ldots\ldots, v_m]}$ is the factorial moment of order v of I, then

$$\mu'_{[v]} = S(v)\prod_{\alpha=1}^{m}(v_\alpha) \quad ,$$

where $\quad S(v) = \sum_{V(v)} P(w)$.

Its proof is omitted in (3). The theorem can also be proved like as in (5) for the univariate case.

This theorem facilitates a relationship between the factorial moments and certain probabilities. So if it is possible to know these probabilities, the distribution of the random variable considered can be determined.

## 3. APPLICATIONS

We apply the above theorem to find factorial moments in following situations.

### UNIVARIATE PROBLEMS

For such problems, take m = 1.

**(a) Binomial distributions :** Let us define random variables as

$\phi_i \quad = 1$ if the ith event occurs, $i = 1, 2, \ldots\ldots\ldots, n$.

$\quad = 0$, otherwise.

When $\phi_i$s are assumed to be i.i.d., and p is the probability of the occurrence of the event under consideration, the random variable $X = \sum_{i=1}^{n} \phi_i$ has a binomial distribution with parameters n and p. Using the above theorem.

$$\mu'_{[r]} = r!\sum_{r} \Pr.(\phi_{i_1} = 1, \phi_{i_2} = 1, \ldots\ldots\ldots\ldots, \phi_{i_r} = 1)$$

$$= \frac{n!}{(n-r)!}\, p^r.$$

**(b) Poisson distribution :** If we take $\lim_{n\to\infty} np = \lambda$ in the factorial moment for the binomial distribution, the rth factorial moment of the Poisson distribution comes to $\lambda^r$.

**(c) Two dimensional rectangular lattice :** Suppose that at every point of a *lxm* lattice, the events B or W materialize with probabilities p and q respectively, where p + q = 1.

(i) Let X be the total number of horizontal and vertical BB joins. It is indeed difficult to find ordinary moments for the distribution of X. Moran (7) gives first four moments of this distribution. However if we use the factorial moments theorem, calculation of factorial moments is simplified to a great extent; from which the ordinary moments can be obtained. Memon and David (5) determine the factorial moments of X in this manner. To illustrate it, we take a simple case of 2x2 lattice. Here, the first moment is

$\mu'_{[1]} = 1!\,\Sigma_1$ Pr. (one particular BB join).

$\Sigma_1$ comprises all configurations in which only one BB join is possible. The probability of each of these outcomes remains $p^2$. So,

$\mu'_{[1]} = 4p^2.$

The second factorial moment is

$\mu'_{[2]} = 2!\,\Sigma_2$ Pr. (two particular BB joins).

$\Sigma_2$ extends over one two horizontal BB joins', one 'two vertical BB joins', four' one horizontal BB join and one vertical BB join' ; the probabilities for which are $p^4$, $p^4$ and $p^3$ respectively. So,

$\mu'_{[2]} = 4p^4 + 8p^3.$

Following the same approach we can show that

$\mu'_{[3]} = 24p^4.$

and $\mu'_{[4]} = 24p^4$.

**(ii)** If Y is taken as the number of diagonal BB joins in the above situation, the problem of calculation of moments by the above theorem no longer presents any difficulty. Here, it follows rather immediately that $\mu'_{[1]} = 2p^2$, and $\mu'_{[2]} = 2p^4$.

**(iii)** When the events B materialize at more than two points of a 2x2 lattice, triangles are formed. Suppose we consider the distribution of the number of such BBB triangles. For this distribution, one may verify that $\mu'_{[1]} = 4p^3$, $\mu'_{[2]} = 12p^4$, $\mu'_{[3]} = 24p^4$, $\mu'_{[4]} = 24p^4$.

**(d) Binomial sequence :** Consider the distribution of the number of BB joins between successive observations of a binomial sequence where the events B and W occur with probabilities p and q respectively. If n is the number of observations, the above theorem can be applied to find factorial moments for the distribution of the number of BW joins. Iyer (4) gives the factorial moments for this distribution as well as for the distribution of the number of BW and WB joins between the successive observations.

## MULTIVARIATE PROBLEMS

**(a)** Consider a set $\Omega$ comprising five events $E_1$, $E_2$, $E_3$, $E_4$, $E_5$, with probabilities

$$\Pr.\ \{E_1, E_2, E_3, E_4, E_5\} = 1/2,$$

$$\Pr.\ \{\overline{E}_1, \overline{E}_2, \overline{E}_3, \overline{E}_4, E_5\} = 1/6,$$

$$\Pr.\ \{\overline{E}_1, E_2, E_3, E_4, \overline{E}_5\} = 1/6,$$

$$\Pr.\ \{E_1, \overline{E}_2, E_3, E_4, \overline{E}_5\} = 1/6,$$

**(i)** Let $\Omega_1$ and $\Omega_2$ be the subsets consisting of the events $E_1$, $E_2$ ; and $E_3$, $E_4$, $E_5$ respectively. By the above theorem for a two dimensional chance variable $(I_1, I_2)$, the factorial moments of order (2,1) is

$$\mu'_{[2,1]} = S(2,1) \prod_{\alpha=1}^{2} (\nu_\alpha)$$

Here, $\prod_{\alpha=1}^{2} (\nu_\alpha)! = 2!\,1! = 2,$

$$S(2, 1) = \Pr.\{E_1, E_2; E_3\} + \Pr.\{E_1, E_2; E_4\} + \Pr.\{E_1, E_2; E_5\}$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

so that $\mu'_{[2,1]} = 3$

One may verify that $\mu'_{[2,2]} = 6$, $\mu'_{[2,3]} = 6$

**(ii)** Let $\Omega_1 = \{E_1\}, \Omega_2 = \{E_2, E_3\}, \Omega_3 = \{E_4, E_5\}$.

The chance variable $(I_1, I_2, I_3)$ has the factorial moment of order (1,1,1)

$$\mu'_{[1,1,1]} = S(1,1,1) \prod_{\alpha=1}^{3}(\nu_\alpha)$$

$$\prod_{\alpha=1}^{3}(\nu_\alpha)! = 1,$$

and $S(1, 1, 1) = \Pr.\{E_1; E_2; E_4\} + \Pr.\{E_1; E_3; E_4\} + \Pr.\{E_1; E_2; E_5\} + \Pr.\{E_1; E_3; E_5\}$

$$= \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{6}\right) + \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{6}\right)$$

$$= 2\,\frac{1}{3}$$

So that required factorial moment of $(I_1, I_2, I_3)$ is $2\,\frac{1}{3}$. We can also show that its $\mu'_{[1,2,1]} = 2$

(b) Let us take up again the problem of two dimensional rectangular lattice considered above, and suppose that X denotes the number of horizontal and vertical BB joins and Y, the number of diagonal BB joins. For the random (X, Y),

$$\mu'_{[1,1]} = 1!\,1!\,\Sigma\, P(w_1, w_2),$$

where Σ is over eight possible, "one horizontal or vertical BB join" and 'one diagonal BB join'. Consequently, this moment comes to $8p^3$. Similarly, $\mu'_{[2,1]} = 8p^3 + 16p^4$; $\mu'_{[2,2]} = 24p^4$

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