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Construction of Measure of Second Order Slope Rotatable Designs Using Balanced Incomplete Block Designs

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Abstract

In this paper, a new method of construction measure of Second Order Slope Rotatable Designs using Balanced Incomplete Block Designs is suggested which enables us to assess the degree of slope-rotatability for a given Response Surface Design.

Keywords

Second order response surface designs, Second order slope rotatable designs (SOSRD), Measure of second order slope rotatable designs

1. Introduction

Response Surface methodology is a statistical technique that is very useful in design and analysis of scientific experiments. In many experimental situations, the experimenter is concerned with explaining certain aspects of a functional relationship

$$Y=f(x_1,x_2,\dots,x_v)+e$$

where Y is the response and $x_1,x_2,\dots,x_v$ are the levels of v-quantitative variables or factors and e is the random error. Response Surface methods are useful where several independent variables influence a dependent variable. The independent variables are assumed to be continued and controlled by the experimenter.

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The response is assumed to be as random variable. For example, if a chemical engineer wishes to find the temperature (x_1) and pressure (x_2) that maximizes the yield (response) of his process then the observed response Y may be written as a function of the levels of the temperature (x_1) and pressure (x_2) as

$$Y=f(x_1,x_2)+e$$

The concept of rotatability, which is very important in Response Surface Designs, was proposed by Box and Hunter (1957). The study of rotatable designs is mainly emphasized on the estimation of differences of yields and its precision. Estimation of differences in responses at two different points in the factor space will often be of great importance. If differences in responses at two points close together are of interest then estimation of local slope (rate of change) of the response is required. Estimation of slopes occurs frequently in practical situations. For instance, there are cases in which we want to estimate rate of reaction in chemical experiment, rate of change in the yield of a crop to various fertilizer doses, rate of disintegration of radioactive material in an animal etc. (Park, 1987).

Hader and Park (1978) introduced Slope Rotatable Central Composite Designs (SRCCD). Victorbabu and Narasimham (1991) studied in detail the conditions to be satisfied by a general Second Order Slope Rotatable Designs (SOSRD) and constructed SOSRD using Balanced Incomplete Block Designs (BIBD). Victorbabu (2007) suggested a review on SOSRD. Park and Kim (1992) suggested a measure of slope rotatability for Second Order Response Surface Designs. Jang and Park (1993) suggested a measure and a graphical method for evaluating slope rotatability in Response Surface Designs. These measures are useful to enable us to assess the degree of slope rotatability for a given Second Order Response Surface Designs.

2. Conditions for Second Order Slope Rotatable Designs

Suppose we want to use the Second Order Response Surface Design D = (x_{iu}) to fit the surface

$$Y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum_{i<j}\sum b_{ij} x_{iu} x_{ju} + e_u \tag{2.1}$$

where x_{iu} denotes the level of the i^{th} u^{th}

of the experiment, e_u's are uncorrelated Random Errors with mean zero and variance σ^2. The design is said to be SOSRD if the variance of the estimate of first order partial derivative of $Y_u(x_1,x_2,\ldots,x_v)$ with respect to each of independent variables (x_i) is only a function of the distance ($d^2=\sum_{i=1}^{v} x_i^2$) of the point (x_1, x_2, $_v$) from the origin (center) of the design. Such a spherical variance function for estimation of slopes in the Second Order Response Surface is achieved if the design points satisfy the following conditions (Hader and Park, 1978; Victorbabu and Narasimham, 1991).

- $\sum x_{iu}=0$, $\sum x_{iu}x_{ju}=0$, $\sum x_{iu}x_{ju}^2=0$, $\sum x_{iu}\,x_{ju}\,x_{ku}=0$, $\sum x_{iu}^3=0$, $\sum x_{iu}x_{ju}^3=0$, $\sum x_{iu}x_{ju}\,x_{ku}^2=0$, $\sum x_{iu}x_{ju}x_{ku}x_{lu}=0$; for $i\neq j\neq k\neq l$;
- (i) $\sum x_{iu}^2=\text{Constant}=N\lambda_2$; for all i; (ii) $\sum x_{iu}^4=\text{Constant}=cN\lambda_4$; for all i
- $\sum x_{iu}^2x_{ju}^2=\text{Constant}=N\lambda_4$; for $i\neq j$
- $\dfrac{\lambda_4}{\lambda_2^2}>\dfrac{v}{(c+v-1)}$.
- $[v(5-c)-(c-3)^2]\,\lambda_4+[v(c-5)+4]\,\lambda_2^2=0$ (2.2)

where c, λ_2 and λ_4 are constants.

The variances and covariances of the estimated Parameters are

$$V(b_0)=\frac{\lambda_4(c+v-1)\sigma^2}{N\left[\lambda_4(c+v-1)-v\lambda_2^2\right]};\quad V(b_i)=\frac{\sigma^2}{N\lambda_2},\qquad V(b_{ij})=\frac{\sigma^2}{N\lambda_4},$$

$$V(b_{ii})=\frac{\sigma^2}{(c-1)N\lambda_4}\left[\frac{\lambda_4(c+v-2)-(v-1)\lambda_2^2}{\lambda_4(c+v-1)-v\lambda_2^2}\right],\ Cov(b_0,b_{ii})=-\frac{\lambda_2\sigma^2}{N[\lambda_4(c+v-1)-v\lambda_2^2]}$$

$$Cov(b_{ii},b_{jj})=\frac{(\lambda_2^2-\lambda_4)\sigma^2}{(c-1)N\lambda_4[\lambda_4(c+v-1)-v\lambda_2^2]}\quad \text{and other covariances vanish.}\quad (2.3)$$

3. Second Order Slope Rotatable Designs using Balanced Incomplete Block Designs

A Balanced Incomplete Block Design (BIBD) denoted by is an arrangement of v treatments in b blocks each containing k(<v) treatments, if (i)

every treatment occurs at most once in a block, (ii) every treatment occurs in exactly blocks, and (iii) every pair of treatments occurs together in blocks.

Let denote a Balanced Incomplete Block Design, $2^{t(k)}$ denote a fractional replicate of 2^k in ±1 levels, in which no interaction with less than five factors is confounded. $[1-(v,b,r,k,\lambda)]$ denote the design points generated from the transpose of incidence matrix of BIBD. $[1-(v,b,r,k,\lambda)]2^{t(k)}$ are the b $2^{t(k)}$ design (Raghavarao, 1971).

^{1} denote the design points generated from (a,0,0...,0) point set, and $\bigcup$ denotes combination of the design points generated from different sets of points. Let (n_a) denote replication of axial points, n_0 denote the number of central points. The method of construction of SOSRD using BIBD is given below (Victorbabu and Narasimham, 1991).

3.1 Result: The design points $[1-(v,b,r,k,\lambda)]2^{t(k)} \bigcup n_a$ ($^1 \bigcup$ (n_0) will give a v-dimensional SOSRD in $N = b2^{t(k)} + 2vn_a + n_0$ design points, where a^2 is positive real root of the fourth degree polynomial equation

$$(8vn_a^3-4Nn_a^2)a^8+8vr2^{t(k)}a^6n_a^2+$$
$$\left[2vr^2 2^{2t(k)}n_a+\left\{\left(\left(12n_a-2vn_a\right)\lambda-4r\right)N+\left(16\lambda n_a^2-20v\lambda n_a^2+4vrn_a^2\right)\right\}2^{t(k)}\right]a^4+$$
$$\left[4vr^2+\left(16n_a-20vn_a\right)r\lambda\right]2^{2t(k)}a^2+\left[\left(5v-9\right)\lambda^2+\left(6-v\right)r\lambda-r^2\right]N2^{2t(k)}+$$
$$\left(vr+4\lambda-5v\lambda\right)r^2 2^{3t(k)}=0$$

Note: Values of SOSRD using BIBD can be obtained by solving the above equation.

4. Conditions of Measure of Second Order Slope Rotatable Designs

Following Hader and Park (1978), Park and Kim (1992), Victorbabu and Narasimham (1991), equations 1, 2, 3, 4, 5 of (2.2), and (2.3) give the necessary and sufficient conditions for a measure of slope rotatability for any general Second Order Response Surface Designs Further we have

$V(b_i)$ are equal for i,

$V(b_{ii})$ are equal for i,

$V(b_{ij})$ are equal for i, j, where $i \neq j$,

$Cov(b_i, b_{ii})=Cov(b_i, b_{ij})=Cov(b_{ii}, b_{ij})=Cov(b_{ij}, b_{il})=0$ for all $i \neq j \neq l$. (4.1)

Park and Kim (1992) proposed that if the conditions in (2.2) together with (2.3) and (4.1) are met, then the following measure ($Q_v(D)$) given below assess the degree of slope rotatability for any general Second Order Response Surface Design 'D' with v-independent variables.

$$Q_v(D) = \frac{1}{2(v-1)\sigma^4}\left\{(v+2)(v+4)\sum_{i=1}^{v}\left[\sum_{i=1}^{v} v(b_i)) + \frac{(4v(b_{ii}) + \sum_{\substack{j=1 \\ j\neq i}}^{v} v(b_{ij})) - \frac{1}{v}\sum_{i=1}^{v}(4v(b_{ii}) + \sum_{\substack{j=1 \\ j\neq i}}^{v} v(b_{ij}))}{v+2}\right]^2\right\}$$

$$+\frac{4}{v(v+2)}\sum_{i=1}^{v}((4v(b_{ii}) + \sum_{\substack{j=1 \\ j\neq i}}^{v} v(b_{ij})) - \frac{1}{v}\sum_{i=1}^{v}(4v(b_{ii}) + \sum_{\substack{j=1 \\ j\neq i}}^{v} v(b_{ij})))^2$$

$$+2\sum_{i=1}^{v}\left[\left[4v(b_{ii}) - \frac{4v(b_{ii}) + \sum_{\substack{j=1 \\ j\neq i}}^{v} v(b_{ij}))}{v}\right]^2 + \sum_{\substack{i=1 \\ j\neq i}}^{v} v(b_{ij}) - \left[\left[\frac{4v(b_{ii}) + \sum_{\substack{j=1 \\ j\neq i}}^{v} v(b_{ij}))}{v}\right]^2\right]\right]$$

$$+4(v+4)\left[4cov^2(b_i, b_{ii}) + \sum_{\substack{i-1 \\ j\neq i}}^{v} cov^2(b_i, b_{ij})\right] + 4\sum_{i=1}^{v}\left[4\sum_{\substack{i=1 \\ j\neq i}}^{v} cov^2(b_{ii}, b_{ij}) + \sum_{\substack{j<1 \\ j,1\neq i}}^{v}\sum cov^2(b_{ii}, b_{i1})\right]$$

Further, it is simplified to $Q_v(D)=\frac{1}{\sigma^4}\left[4V(b_{ii})-V(b_{ij})\right]^2$.

5. Construction of Measure of Second Order Slope Rotatable Design (SOSRD) Using Balance Incomplete Block Design (BIBD)

In this section, the proposed new method of construction of measure of SOSRD using BIBD is given below.

Let (v, b, r, k, λ) denote a BIBD. For the design points, $[1-(v,b,r,k,\lambda)]2^{t(k)} \cup n_a$ $(a,0,0,\ldots,0)2^1 \cup (n_0)$ generated from BIBD in $N = b2^{t(k)} + 2vn_a + n_0$ design points, simple symmetry conditions 1, 2, 3 of (2.2) are true. Condition 1 of (2.2) is true obviously. Conditions 2 and 3 of (2.2) are true as follows

2. (i) $\sum x_{iu}^2=r2^{t(k)}+2n_a a^2=N\lambda_2$

(ii) $\sum x_{iu}^4 = r2^{t(k)} + 2n_a a^4 = cN\lambda_4$

3. $\sum x_{iu}^2 x_{ju}^2 = \lambda 2^{t(k)} = N\lambda_4$

Measure of SOSRD using BIBD can be obtained by

$$Q_v(D) = \left[\frac{\sum x_{iu}^2}{N}\right]^4 \left[4e - V(b_{ij})\right]^2$$

where,

$$e = V(b_{ii})$$

$$= \frac{(v-1)\left[2^{t(k)}\lambda n_0 + 2^{t(k)+1}\lambda v n_a - 2^{t(k)+2} r n_a a^2 - r^2 2^{2t(k)} + b\lambda 2^{2t(k)}\right] + \left[2^{t(k)+1} b n_a + 2n_a n_0 + 4n_a^2\right]a^4 + (r-\lambda)\left[2^{t(k)+1} v n_a + 2^{t(k)} n_0 + b2^{2t(k)}\right]}{\left[2^{t(k)}(r-k) + 2n_a a^4\right]\left[\begin{array}{c} v2^{t(k)}(\lambda n_0 - 4 r n_a a^2) + 2^{t(k)+1} n_a (v^2\lambda + b a^4) \\ +2n_a n_0 a^4 + (r-k)(b2^{2t(k)} + 2^{t(k)+1} v n_a + 2^{t(k)} n_0) + 2^{2t(k)} v (b\lambda - r^2) \end{array}\right]}$$

Table 1 gives the values of $Q_v(D)$ for SOSRD using various Parameters of BIBD, n_0 and the value of 'a' which make SOSRD using BIBD. It can be verified that $Q_v(D)$ is zero, if and only if, a design 'D' is slope-rotatable. $Q_v(D)$ becomes larger as 'D' deviates from a Slope Rotatable Design.

6. Conclusion

In this paper, general measure has been proposed which enables us to assess the degree of slope rotatability for a given Second Order Response Surface Design using BIBD. This measure, $Q_v(D)$ has the value zero, if and only if, the design 'D' is SOSRD, and $Q_v(D)$ becomes larger as 'D' deviates from a Slope Rotatable Design. It may be used to compare the degree of slope rotatability for the same 'v'. It can be generally used to increase the degree of slope rotatability of SOSRD by the addition of experimental runs. We also point out here that this measure of SOSRD using BIBD has 71 design points for 7-factors, whereas the corresponding measure of SRCCD obtained by Park and Kim (1992) needs 79 design points. Thus the new method leads to a 7-factor measure of SOSRD in less number of design points than the corresponding measure of Slope Rotatable Central Composite Designs (SRCCD). Further, it is pointed out that replication of axial points (n_a) rather than replication of central points provide appreciable advantage in terms of efficiency of the estimates of the Parameters of the model.

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Table 1: Values of measure of SOSRD using BIBD ($n_a = 1$)

(3,3,2,2,1)					
a	**n_0=1,N=19**	**n_0=2,N=20**	**n_0=3,N=21**	**n_0=4,N=22**	**n_0=5,N=23**
1.0	4.8818×10^{-2}	2.5511×10^{-2}	1.5748×10^{-2}	1.0672×10^{-2}	7.6753×10^{-3}
1.3	1.4991×10^{-1}	3.8709×10^{-2}	1.6675×10^{-2}	8.9508×10^{-3}	5.4362×10^{-3}
1.6	8.1146×10^{-2}	1.7971×10^{-2}	6.1608×10^{-3}	2.5693×10^{-3}	1.1949×10^{-3}
1.9	2.7449×10^{-3}	1.4684×10^{-4}	4.4386×10^{-5}	2.6517×10^{-4}	4.5620×10^{-4}
2.2	7.3135×10^{-3}	7.9415×10^{-3}	7.7606×10^{-3}	7.2278×10^{-3}	6.5697×10^{-3}
2.5	3.8942×10^{-2}	3.3665×10^{-2}	2.8907×10^{-2}	2.4785×10^{-2}	2.1276×10^{-2}
2.8	9.8970×10^{-2}	8.2321×10^{-2}	6.8826×10^{-2}	5.7874×10^{-2}	4.8951×10^{-2}
3.1	2.0378×10^{-1}	1.6755×10^{-1}	1.3888×10^{-1}	1.1601×10^{-1}	9.7604×10^{-2}
*	2.0000	1.9330	1.8764	1.8290	1.7894
(4,6,3,2,1)					
a	**n_0=1,N=33**	**n_0=2,N=34**	**n_0=3,N=35**	**n_0=4,N=36**	**n_0=5,N=37**
1.0	7.9676×10^{-3}	4.3725×10^{-3}	2.7504×10^{-3}	1.8819×10^{-3}	1.3630×10^{-3}
1.3	4.0823×10^{-2}	1.0456×10^{-2}	4.3642×10^{-3}	2.2584×10^{-3}	1.3209×10^{-3}
1.6	1.9950×10^{-2}	4.7914×10^{-3}	1.6367×10^{-3}	6.4547×10^{-4}	2.6887×10^{-4}
1.9	8.5114×10^{-5}	1.1294×10^{-5}	1.0811×10^{-4}	2.1175×10^{-4}	2.9080×10^{-4}
2.2	2.8300×10^{-3}	2.8377×10^{-3}	2.7609×10^{-3}	2.6380×10^{-3}	2.4921×10^{-3}
2.5	9.9660×10^{-3}	9.0715×10^{-3}	8.2475×10^{-3}	7.4967×10^{-3}	6.8171×10^{-3}
2.8	2.1792×10^{-2}	1.9277×10^{-2}	1.7292×10^{-2}	1.5546×10^{-2}	1.4010×10^{-2}
3.1	3.9976×10^{-2}	3.5606×10^{-2}	3.1810×10^{-2}	2.8500×10^{-2}	2.5606×10^{-2}
*	1.9348	1.8833	1.8352	1.7909	1.7504
(5,10,4,2,1)					
a	**n_0=1,N=51**	**n_0=2,N=52**	**n_0=3,N=53**	**n_0=4,N=54**	**n_0=5,N=55**
1.0	1.7567×10^{-3}	9.6797×10^{-4}	5.9534×10^{-4}	3.9368×10^{-4}	2.7409×10^{-4}
1.3	1.4342×10^{-2}	3.5871×10^{-3}	1.4176×10^{-3}	6.8471×10^{-4}	3.6975×10^{-4}
1.6	6.1878×10^{-3}	1.5446×10^{-3}	5.0685×10^{-4}	2.4158×10^{-4}	6.1112×10^{-5}
1.9	6.4166×10^{-6}	2.1626×10^{-5}	1.1067×10^{-4}	1.5654×10^{-4}	1.9092×10^{-4}
2.2	1.2470×10^{-3}	1.2302×10^{-3}	1.1998×10^{-3}	1.1609×10^{-3}	1.1169×10^{-3}
2.5	3.4563×10^{-3}	3.2407×10^{-3}	3.0379×10^{-3}	2.8478×10^{-3}	2.6702×10^{-3}
2.8	6.7001×10^{-3}	6.2262×10^{-3}	5.7919×10^{-3}	5.3936×10^{-3}	5.0280×10^{-3}
3.1	1.1547×10^{-2}	1.0703×10^{-2}	9.9344×10^{-3}	9.2327×10^{-3}	8.5913×10^{-3}
*	1.8836	1.8393	1.7955	1.7525	1.7104
(6,15,5,2,1)					
a	**n_0=1,N=73**	**n_0=2,N=74**	**n_0=3,N=75**	**n_0=4,N=76**	**n_0=5,N=77**
1.0	4.5787×10^{-4}	2.4479×10^{-4}	1.4189×10^{-4}	8.6686×10^{-5}	5.4870×10^{-5}

(6,15,5,2,1)					
a	**n_0=1,N=73**	**n_0=2,N=74**	**n_0=3,N=75**	**n_0=4,N=76**	**n_0=5,N=77**
1.3	5.9767×10^{-3}	1.4593×10^{-3}	5.4323×10^{-4}	2.4142×10^{-4}	1.1701×10^{-4}
1.6	2.2530×10^{-3}	5.7160×10^{-4}	1.7637×10^{-4}	5.3795×10^{-5}	1.2973×10^{-5}
1.9	3.3626×10^{-5}	6.4681×10^{-5}	9.1377×10^{-5}	1.1251×10^{-4}	1.2853×10^{-4}
2.2	6.2158×10^{-4}	6.1172×10^{-4}	5.9885×10^{-4}	5.8391×10^{-4}	5.6761×10^{-4}
2.5	1.4678×10^{-3}	1.4012×10^{-3}	1.3375×10^{-3}	1.2767×10^{-3}	1.2189×10^{-3}
2.8	2.6276×10^{-3}	2.4947×10^{-3}	2.3698×10^{-3}	2.2524×10^{-3}	2.1420×10^{-3}
3.1	4.2785×10^{-3}	4.0560×10^{-3}	3.8477×10^{-3}	3.6524×10^{-3}	3.4693×10^{-3}
*	1.8419	1.8015	1.7599	1.7170	1.6728
(7,7,3,3,1)					
a	**n_0=1,N=71**	**n_0=2,N=72**	**n_0=3,N=73**	**n_0=4,N=74**	**n_0=5,N=75**
1.0	3.6876×10^{-4}	3.0765×10^{-4}	2.6256×10^{-4}	2.2798×10^{-4}	2.0065×10^{-4}
1.3	7.4243×10^{-4}	4.7849×10^{-4}	3.3952×10^{-4}	2.5641×10^{-4}	2.0223×10^{-4}
1.6	3.9307×10^{-3}	1.1486×10^{-3}	5.2055×10^{-4}	2.8787×10^{-4}	1.7848×10^{-4}
1.9	2.2780×10^{-3}	5.9303×10^{-4}	2.1790×10^{-4}	9.2232×10^{-5}	4.1254×10^{-5}
2.2	6.1415×10^{-6}	6.7524×10^{-7}	8.5695×10^{-6}	1.8867×10^{-5}	2.8393×10^{-5}
2.5	2.3610×10^{-4}	2.4443×10^{-4}	2.4834×10^{-4}	2.4904×10^{-4}	2.4740×10^{-4}
2.8	7.2031×10^{-4}	6.9292×10^{-4}	6.6562×10^{-4}	6.3872×10^{-4}	6.1243×10^{-4}
3.1	1.3742×10^{-3}	1.3064×10^{-3}	1.2423×10^{-3}	1.1817×10^{-3}	1.1245×10^{-3}
*	2.2305	2.1872	2.1449	2.1039	2.0647
(8,28,7,2,1)					
a	**n_0=1,N=129**	**n_0=2,N=130**	**n_0=3,N=131**	**n_0=4,N=132**	**n_0=5,N=133**
1.0	3.7574×10^{-5}	1.6291×10^{-5}	6.6546×10^{-6}	2.2986×10^{-6}	5.0686×10^{-7}
1.3	1.4462×10^{-3}	3.3949×10^{-4}	1.1237×10^{-4}	4.1305×10^{-5}	1.4865×10^{-5}
1.6	4.0998×10^{-4}	1.0270×10^{-4}	2.6345×10^{-5}	4.7726×10^{-6}	6.9811×10^{-8}
1.9	3.8057×10^{-5}	4.6409×10^{-5}	5.3162×10^{-5}	5.8542×10^{-5}	6.2773×10^{-5}
2.2	2.0258×10^{-4}	1.9991×10^{-4}	1.9697×10^{-4}	1.9380×10^{-4}	1.9047×10^{-4}
2.5	3.8978×10^{-4}	3.7929×10^{-4}	3.6910×10^{-4}	3.5919×10^{-4}	3.4956×10^{-4}
2.8	6.2432×10^{-4}	6.0601×10^{-4}	5.8834×10^{-4}	5.7130×10^{-4}	5.5485×10^{-4}
3.1	9.3614×10^{-4}	9.0806×10^{-4}	8.8103×10^{-4}	8.5498×10^{-4}	8.2988×10^{-4}
*	1.7782	1.7417	1.7023	1.6587	1.6093

* Values of SOSRD using BIBD.

References

1. Box, G. E. P. and Hunter, J. S. (1957). Multifactor experimental designs for exploring response surfaces. *The Annals of Mathematical Statistics*, **28(1)**, 195-241.
2. Hader, R. J. and Park, S. H. (1978). Slope rotatable central composite designs. *Technometrics*, **20(4)**, 413-417.
3. Jang, D. H. and Park, S. H. (1993). A measure and graphical method for evaluating slope rotatability in response surface designs. *Communications in Statistics - Theory and Methods*, **22(7)**, 1849-1863.
4. Park, S. H. (1987). A class of multi-factor designs for estimating the slope of response surface. *Technometrics*, **29(4)**, 449-453.
5. Park, S. H. and Kim, H. J. (1992). A measure of slope rotatability for second order response surface experimental designs. *Journal of Applied Statistics*, **19(3)**, 391 404.
6. Raghavarao, D. (1971). *Constructions and Combinatorial Problems in Design of Experiments*. John Wiley and Sons, New York.
7. Victorbabu, B. Re. (2007). On second order slope rotatable designs A Review. *Journal of the Korean Statistical Society*, **33(3)**, 373-386.
8. Victorbabu, B. Re. and Narasimham, V. L. (1991). Construction of second order slope rotatable designs through balanced incomplete block designs. *Communications in Statistics -Theory and Methods*, **20(8)**, 2467-2478.

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Some Powerful Simple and Composite Goodness of Fit Test Based on UIT and related Bahadur Efficiency

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Abstract

In this paper, we have established powerful Goodness of Fit Tests for the basic situation in which the Hypothesized Distribution is known. A new approach of Parameterization is proposed which is a useful approach to construct a Goodness of Fit Test based on Parametric approaches. Various Goodness of Fit Test procedures have been used in literature. We consider the Union-Intersection approach to make some powerful tests to Goodness of Fit Test problem. We simulate the percentage points of introduced statistics. Also, we study the Bahadur Efficiency of the proposed test.

Keywords

Anderson-Darling test, Berk-Jones statistics, Cumulative distribution function, Goodness of fit test, Likelihood ratio, Power of test, Union-Intersection test

1. Introduction

An essential problem in statistics is whether or not a set of measurements is compatible with the assumption that the measurements are an independently identically distributed sample from a known distribution. A difficulty in testing such a statistical Hypothesis is that the Alternatives (Rival Models) are enormously large and could not be described clearly. As the purpose of a Goodness of Fit Test, these tests are intended as tests for distributional form, not as tests of Parametric values. These kind of problems may be called testing Goodness of Fit. They have some strengths and weaknesses. Goodness of Fit Tests are Hypothesis testing problems. But there are some differences.

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Hypothesis testing is formulated in terms of Null and Alternative Hypotheses, type one and type two Errors and Power of Tests. In search of the best decision, we turn to the search of a test with acceptable Power. On the other hand, there is no specific Alternative Hypothesis for Goodness of Fit Test, so it is impossible to define the Power of Test simply. Traditionally, Goodness of Fit Tests formulated based on the Cumulative Distribution Functions (c.d.f) of a random variable Y, denoted by $F(y)$, is defined by

$$F(y) = P(Y \leq y), \quad \text{for all } y \in \Psi$$

We consider two directions as Simple and Composite Goodness of Fit Tests.
The Simple Hypotheses are given as

$$\mathrm{H}_0\colon F(y) = F_0(y) \ \ \forall y \in \Psi \text{ against } \mathrm{H}_1 : F(y) \neq F_0(y) \text{ for some } y \in \Psi \qquad (1.1)$$

where $F_0(.)$ is a known Distribution Function. We may consider $F_0(.)$ as $F(.,\theta_0)$ where $\theta_0 \in \Theta$ is a specified value of the Vector Parameter θ.
In the Composite situation, we wish to test

$$\mathrm{H}_{0c}\colon F(y) = F(y,\theta) \ \ \forall y \in \Psi \text{ against } \mathrm{H}_{1c} : F(y) \neq F(y,\theta) \text{ for some } y \in \Psi \qquad (1.2)$$

where $\theta \in \Theta$ is an unknown vector of Parameters.

This kind of Hypothesis testing is the problem of whether the underling Distribution Function belongs to a given family of Distribution Functions as $\Phi = \{F(.;\theta), \quad \theta \in \Theta\}$. To test this Composite Hypothesis, we have to estimate θ by $\hat{\theta}_n$ which is a regular estimate of θ. We denote the true value of θ by θ_0. A natural approach to this testing problem is to use the Empirical Distribution Function as an approximation to the true underling Distribution, where the Empirical Distribution Function $F_n(.)$ is defined by

$$F_n(y) = \frac{1}{n}\sum_{i=1}^{n}(Y_i \leq y)$$

Most notable Goodness of Fit Test based on Empirical Distribution Function and $d(F_n(.),F_0(.))$ are the Anderson-Darling A_n^2 Test (1952), Cramer (1928), Kolmogorov (1933), Kuiper V Test (1960), Smirnov (1939, 1941), Von-Mises(1931) and Watson's U^2 Test (1958). The first approach to the problem of testing fit to a fixed distribution is Pearson's (1930) Chi-Squared Test. A way to improve Pearson's statistics consists of employing a functional distance as $d(F_n(.),F_0(.))$. Possibly the best known test statistics based on the Empirical

Distribution Function are the Cramer (1928) and in a more general form Von-Mises(1931) statistics. They proposed

$$\omega_n^2 = n\int_{-\infty}^{\infty}(F_n(y) - F_0(y))^2 \zeta(y)dy$$

for some Weight Function ∂ as an adequate measure of discrepancy. The Kolmogorov Test (1933) is the easiest and also most natural Non-Parametric Test. It is based on the Lnorm and computes the distance between an Empirical and the Hypothesized (theoretical) Distribution Function under the Null Hypothesis. Under Alternative Hypothesis, the difference between the Empirical and Theoretical Distribution Functions will be noticeable. This statistic is given by

$$D_n = \sqrt{n}\sup_{y\in P} | F_n(y) - F_0(y) | .$$

A problem mathematically similar to Kolmogorov's statistic was studied by Smirnov (1939,1941). He has considered D_n^+ and D_n^- where

$$D_n^+ = \sqrt{n}\sup_{y\in P}(F_n(y) - F_0(y))$$

$$D_n^- = \sqrt{n}\sup_{y\in P}(F_0(y) - F_n(y)).$$

The statistics D_n, D_n^+ and D_n^- are known as Kolmogorov-Smirnov statistics. They have the advantage of being distribution free. Thus the same p-values can be used to obtain the significance level when testing it to any Continuous Distribution. In search of this property for ω_n^2 Cramer-Von-Mises have introduced a simple modification. A modification for Cramer-Von-Mises distance is

$$W_n^2(\psi) = n\int_{-\infty}^{\infty}\psi(F_0(y))\{(F_n(y) - F_0(y))^2\}dF_0(y)$$

which was proposed by Smirnov (1936, 1937).

The Parametric version of this statistics when related Parameter is estimated by $\hat{\theta}_n$ is given by

$$\hat{W}_n^2(\psi) = n\int_{-\infty}^{\infty}\psi(F(y;\hat{\theta}_n))\{(F_n(y) - F(y;\hat{\theta}_n))^2\}dF(y;\hat{\theta}_n).$$

The property exhibited by D_n and $W_n^2(\psi)$ of being distribution free does not carry over to the Parametric cases. However, in some cases the Distribution of $F(Y_i;\hat{\theta}_n)$; $i = 1,2,...,n$ does not depend on θ, but only on Φ, the family of underline densities. In those cases, the Distributions of Parametric Goodness of

FitTests are Parameter free. This happens if Φ is a location scale family and $\hat{\theta}_n$ is an equivariant estimator(see, David and Johnson, 1948). All the statistics which can be obtained by varying ψ are usually refereed to as statistics of Cramer-Von-Mises type, two of them are as follows. The Cramer-Von-Mises's statistic obtained by W_n^2 for $\psi(.)=1$, $W_n^2 = n\int_{-\infty}^{\infty}(F_n(y)-F_0(y))^2 dF_0(y)$

and the Anderson-Darling's statistic (1952) for $\psi(t)=(t(1-t))^{-1}$

$$A_n^2 = n\int_{-\infty}^{\infty}\frac{(F_n(y)-F_0(y))^2}{F_0(y)(1-F_0(y))}dF_0(y)$$

with Parametric version as

$$\hat{A}_n^2 = n\int_{-\infty}^{\infty}\frac{(F_n(y)-F(y;\hat{\theta}_n))^2}{F(y;\hat{\theta}_n)(1-F(y;\hat{\theta}_n))}dF(y;\hat{\theta}_n).$$

Consideration of different Weight Functions ψ allows the statistician to put special emphasis on the detection of particular sets of Alternatives. Some people prefer employing Cramer-Von-Mises statistics instead of Kolmogorov-Smirnov statistics. It is because Kolmogorov-Smirnov statistics accounts only for the largest deviation between $F_n(t)$ and $F(t)$, while the other one is a weighted average of all the deviations between $F_n(t)$ and $F(t)$. Anyway we reject H_0 if in each case the value of the statistic is large. The supremum version of the Anderson-Darling statistics is given by

$$B_n^2 = \sup_{-\infty\le y\le\infty}\frac{|(F_n(y)-F_0(y))|}{\sqrt{F_0(y)(1-F_0(y))}}.$$

Eicker (1979) considered $\psi(t)=(t(1-t))^{-1}=\{F_n(y)(1-F_n(y))\}^{-1}$, rather than the Hypothesized variance. Berk and Jones (1979) used the Divergence Function which prepares an approach which give us a test statistic using known Likelihood Ratio Test. More precisely the Berk-Jones statistics as the supremum of the Kullback-Leibler (KL) discrepancy between Hypothesized and Empirical Distribution Functions could be defined as a supreme of $K(F_n(y),F_0(y))$ as

$$K(F_n(y),F_0(y)) = \begin{cases} F_n(y)(\log(\frac{F_n(y)}{F_0(y)})) + (1-F_n(y))\log\frac{1-F_n(y)}{1-F_0(y)} & \text{if} 0 \le F_0(y) < F_n(y) \le 1 \\ 0 & \text{if} 0 \le F_n(y) \le F_0(y) \le 1 \\ \infty & \text{otherwise.} \end{cases}$$

where $K(F_n(y),F_0(y)))$ is the Kullback-Leibler (KL) discrepancy between two Distributions.

It is known that $K(F_n(y),F_0(y))$ behaves as $\frac{1}{2}\frac{(F_n(y)-F_0(y))^2}{F_0(y)(1-F_0(y))}$.

This last term is half of the Pearson statistics for $F_n(y)$. When we consider the Goodness of Fit Test for Multinomial Distribution, the Pearson χ^2 statistic is asymptotically equivalent to the Likelihood Ratio statistic. Berk-Jones proposed that we can fix 'y' and construct a test statistic by Likelihood Ratio Test for Goodness of Fit Test problem. Then, we turn to $F_n(y)$. For each fixed sample $\underline{Y} = (Y_1,\dots,Y_n)$, $F_n(y)$ is a Distribution Function as a function of $y \in \mathrm{P}$. On the other hand, for each fixed value of 'y', $F_n(y)$ is a random variable as a function of the sample and also it is known that $F_n(y)$ is a Unbiased Maximum Likelihood Estimator for $F(y)$. The variance of the Empirical Distribution converges to zero as 'n' goes to infinity. These indicate that $F_n(.)$ is weakly and strongly consistent for estimation of $F(y)$. As we know, $nF_n(y)$: $Bin(n,F(y))$ for a fixed 'y', then under H_0; $nF_n(y)$: $Bin(n,F_0(y))$. We concluded that the Likelihood Ratio statistic for testing H_0 against H_1 in fixed $y \in \mathrm{P}$ is given by

$$\lambda_n(y) = \frac{\sup_{F(y)} \Lambda_n(F(y))}{\Lambda_n(F_0(y))}.$$

where $\Lambda_n(F(y))$ and $\Lambda_n(F_0(y))$ are Likelihood Functions evaluated at $F(y)$ and $F_0(y)$ respectively. A suitable relation between Berk-Jones statistics and Likelihood Ratio statistic is as follows

$$\sup_{y\in[0,1]} K(F_n(y),F_0(y)) = \sup_{y\in[0,1]} n^{-1}\log\lambda_n(y).$$

Einmahl and McKeague (2003) introduced an integral form of Berk-Jones statistic. They also considered testing for symmetry, a change point, independence and for exponentiality. Wellner and Koltchinskii (2002) have given proofs of the Limiting Null Distribution of the Berk and Jones (1979) statistic.

In section 2, we bring our objective to using Berk and Jones idea and using this idea with Union-Intersection Test (UIT). This section is a sketch of UIT. Section 3 shows how we make our test. The Power comparisons are given in section 4. After constructing test using UIT, we will search some good Weight Functions as means of a powerful test in two directions as Simple and Composite Hypothesis. In fact, we develop an approach for Simple Hypothesis and then we will use the results for simulation study in both situations. In section 5, we study the Bahadur Efficiency of the proposed test.

2. Motivation

A large family of statistic which embeds χ^2 and Likelihood Ratio Test statistics are obtained by Cressie and Read (1984) family of divergence statistics defined as following

$$2nI_y^{\kappa} = \frac{2n}{\kappa(\kappa+1)}\left\{F_n(y)\left[\frac{F_n(y)}{F_0(y)}\right]^{\kappa} + \{1-F_n(y)\}\left[\frac{1-F_n(y)}{1-F_0(y)}\right]^{\kappa} - 1\right\}$$

for testing the Goodness of Fit of a Multinomial Distribution for the binary sample $X_{1y},\ldots,X_{ny}$ with 'y' fixed. Our goal is using Berk-Jones idea in fixing 'y' and Likelihood Ratio statistic in fixed 'y' in search of a Goodness of Fit Test for two situations, the simple case where $F_0(.)$ is a known Distribution Function and Composite case where as the common approach for Goodness of Fit Test. We have to estimate the unknown Parameter(s) at first and then apply the test. Parametric case will change our situation for model selection from testing for a specified distribution which belongs to the model in more general situation is actually testing for a Family of Distributions (Models). Einmahl and McKeague (2003) have considered the localized Empirical Likelihood Ratio with Likelihood Function as

$$\Lambda(\bar{F}) = \prod_{i=1}^{n}[\Lambda(\bar{F}(Y_i) - \Lambda(\bar{F}(Y_i -)].$$

We would propose the other approach different than Einmahl and McKeague (2003), using some Weight Functions based on classical approach to Hypothesis testing as Union-Intersection approach. This approach has the same advantage as the well known Likelihood Ratio Test approach. The resulting statistics is the same as the Einmahl and McKeague (2003) statistics but our approach makes theUnion-Intersection Test (UIT) suitable for dealing with the Goodness of Fit problem. UIT introduced by Roy (1953) motivated through the multiple comparisons and simultaneous statistical inference. On the other hand, it is easier to illustrate UIT with a Composite Hypothesis testing problem that leads to simultaneous statistical inference. Generally, we use two types of test statistics for testing H_0 against H_1 that can be defined by

$$T = \int_{-\infty}^{\infty} T_z \, dw(z)$$

or

$$T_{max} = \sup_{z\in(-\infty,\infty)} \{T_z w(z)\}.$$

To construct the global test statistic T (or T_{max}) we need some local test statistics as T_z. In this work, we focus on the test statistic as T only, and illustrate our approach to construct the local test statistic.

Consider a random sample as $\underline{Y} = (Y_1, Y_2, ..., Y_n)$ and a Goodness of Fit Test procedure which introduces a Likelihood Ratio Test for each fixed 'z' which could be between any of two Y_i's. Here we must emphases that $F(y)$ is an unknown Distribution Function, whereas $F(z)$ with fixed 'z' is an unknown Parameter. As we saw $nF_n(y): \ Bin(n, F(y))$, the Likelihood Ratio statistics for testing H_0 against H_1 is given by

$$\lambda_n(z) = \frac{\sup_{F(z)} \Lambda_n(F(z))}{\Lambda_n(F_0(z))} = \frac{\Lambda_n(F_n(z))}{\Lambda_n(F_0(z))} = \left(\frac{F_n(z)}{F_0(z)}\right)^{nF_n(z)} \left(\frac{1-F_n(z)}{1-F_0(z)}\right)^{n(1-F_n(z))}.$$

If we separate the Null Hypothesis $\mathrm{H}_0 : F(y) = F_0(y) \quad \forall y \in \Psi$ (related to a local test) to several Null Hypotheses as $\mathrm{H}_{0z} : F(z) = F_0(z) \quad \forall z \in \mathrm{Z}$, we can construct a Likelihood Ratio for each one of the H_{0z}'s for each fixed 'z', and then construct a test for our essential Hypothesis testing problem. Fortunately, this concept is known in statistics. The Union-Intersection test (UIT), see Casella and

Berger (2002) andSayyareh (2011), is our proposal to solve this problem. The UIT method is a natural solution to this kind of problem. It is because the overall Hypothesis could be rejected if each local Null Hypotheses could be rejected. As a test statistic, we generalized the logic of the Likelihood Ratio Test. On the other hand, we defined•a Weight Function as $w(z)$. This Weight Function permits us to construct different tests. As a choice we consider $w(z)=\cup(F_n(z),F_0(z))$. In the following, after a brief review of UIT, we will construct the Likelihood Ratio Test statistic by UIT, and then we will propose our statistics to model selection. The Likelihood Ratio Test method is a commonly used method of Hypothesis test construction. Another method, which is appropriate when the Null Hypothesis is expressed as an intersection, is the Union-Intersection Test (UIT). In classical statistics we may write

$$\mathrm{H}_0:\theta\in\bigcap_{\gamma\in\Gamma}\Theta_\gamma$$

where Γ is an arbitrary index set that may be finite or infinite, depending on the problem. By this notation we have

$$\mathrm{H}_1:\theta\in\bigcup_{\gamma\in\Gamma}\Theta_\gamma^c.$$

Suppose that for each of the testing $\mathrm{H}_{0\gamma}:\theta\in\Theta_\gamma$ against the AlternativeHypothesis $\mathrm{H}_{1\gamma}:\theta\in\Theta_\gamma^c$, we know that the rejection region for the test of $\mathrm{H}_{0\geq}$ is $\{y:T_\gamma(y)\in R_\gamma\}$ where $T_\gamma(.)$ is the test statistic. Thus, if any of the $\mathrm{H}_{0\geq}$ is rejected, then H_0 must also be rejected, it offers a rejection region for UIT as

$$\bigcup_{\gamma\in\Gamma}\{y:T_\gamma(y)\in R_\gamma\}.$$

As a simple example for UIT, we consider a known Hypothesis test in elementary statistics.

Example: Let $Y_1,Y_2,...,Y_n$ be a independently identically distributed (i.i.d.) random sample from $\mathrm{N}(\mu,\sigma^2)$, where μ and σ^2 are unknown Parameters. We want to test that $\mathrm{H}_0:\mu=\mu_0$ against $\mathrm{H}_1:\mu\neq\mu_0$, where μ_0 is a specified number. As a UIT, we can write

$$\mathrm{H}_0:\{\mu:\mu\leq\mu_0\}\cap\{\mu:\mu\geq\mu_0\}$$

This Null Hypothesis could be written as intersection of two new Null Hypotheses as $\mathrm{H}_{0Lower}:\{\mu:\mu\le\mu_0\}$ and $\mathrm{H}_{0Upper}:\{\mu:\mu\ge\mu_0\}$. Now as the classical approach we will test

$\mathrm{H}_{0Lower}:\mu\le\mu_0$ against $\mathrm{H}_{1Lower}:\mu>\mu_0$

with rejection region $\dfrac{1/n\sum_{i=1}^{n}Y_i-\mu_0}{S/\sqrt{n}}\ge t_{Lower}$ and

$\mathrm{H}_{0Upper}:\mu\ge\mu_0$ against $\mathrm{H}_{1Upper}:\mu<\mu_0$

with rejection region $\dfrac{1/n\sum_{i=1}^{n}Y_i-\mu_0}{S/\sqrt{n}}\le t_{Upper}$.

Then the rejection region of the UIT of

$\mathrm{H}_0:\{\mu:\mu\le\mu_0\}\cap\{\mu:\mu\ge\mu_0\}$

against

$\mathrm{H}_1:\{\mu:\mu\ge\mu_0\}\cup\{\mu:\mu\le\mu_0\}$ for $t_{Lower}=-t_{Upper}$ will be express as $\left|\dfrac{1/n\sum_{i=1}^{n}Y_i-\mu_0}{S/\sqrt{n}}\right|\ge t_{Lower}$ which is the two sided test.

3. Proposed Approach to Construct New Tests

Consider $\underline{Y}=(Y_1,Y_2,\dots,Y_n)$ as anindependently identically distributed (i.i.d.) random sample with unknown Distribution Function $F(.)$. We set $F_0(.)$ as a known Distribution Function. The official Goodness of Fit Test contains testing

$\mathrm{H}_0:F(y)=F_0(y)\quad\forall y\in\Psi$

against

H_1: $F(y)\neq F_0(y)$ for some $y\in\Psi$

A key for proposing a Goodness of Fit Test is that the Distribution Function $F(z)$ for a fixed 'z' is an unknown Parameter. It reduces the Goodness of Fit Test to a Likelihood Ratio Test as

$\mathrm{H}_{0z}:F(z)=F_0(z)\quad\forall z\in Z$

against

H_{1z}: $F(z)\neq F_0(z)$ for some $z\in Z$

As it is a case with Composite Hypotheses testing problems, there may not be, in general, an optimal test for testing H_0 against H_1. However, for a general class of testing problems our idea is to rewrite this Hypothesis testing as the UIT, thus we have

$$H_0 : \bigcap_{z \in Z} H_{0z}$$

against

$$H_1 : \bigcup_{z \in Z} H_{1z}.$$

In this way, there is flexibility in the decomposition of the Hypotheses and choice of appropriate test statistics. To do this, for each 'z' we can define a new random variable (see, Berk and Jones 1979), thus we have

$$Y_{iz} = 1\{Y_i \le z\} = \begin{cases} 1 & \text{if } Y_i \le z \\ 0 & \text{if } Y_i > z \end{cases}$$

for $i = 1,2,\ldots,n.$

Now, we have a Parametric test with a binary variable with value in $\{0,1\}^n$, i.e.

$$Y_{iz} : \; Bin(1, F(z))$$

and

$$\sum_{i=1}^{n} Y_{iz} = nF_n(z) : \; Bin(n, F(z)).$$

The Likelihood Function is

$$\Lambda_n(F(z)) = \Lambda_n(F(z); \underline{Y}_{iz}) = (F(z))^{nF_n(z)} (1 - F(z))^{n(1-F_n(z))}$$

and the Likelihood Ratio Test is given by

$$\approx_n(z) = \frac{\sup_{F(z)} \Lambda_n(F(z))}{\Lambda_n(F_0(z))}$$

$$\approx_n(z) = \Lambda^{F_n(z)/F_0(z)} \; \alpha \; \frac{\Lambda_n(F_n(z))}{\Lambda_n(F_0(z))}$$

for the large value of $\lambda_n(z)$ we reject the Null Hypothesis. The log-Likelihood Function is given by

$$\log \lambda_n(z) = \log \Lambda^{F_n(z)/F_0(z)} = nF_n(z) \log(\frac{F_n(z)}{F_0(z)}) + n(1 - F_n(z)) \log(\frac{1 - F_n(z)}{1 - F_0(z)}).$$

The proposed test statistics for testing H_0 against H_1 is

$$U_n = \int_{P} \log \Lambda^{F_n(z)/F_0(z)} d(w(z)) = \int_{P} \log \Lambda^{F_n(z)/F_0(z)} d(\psi(F_n(z), F_0(z)))$$

Note that the decision rule is built from the logical equivalence that H_0 is wrong, if and only if, any of its components H_{0z} is wrong or equivalently H_0 is true if all the H_{0z}'s are individually true. Also assume that we can test H_{0z} using a statistic $T_z(y)$ such that for any Hypothesis included in H_{0z}, $p(\{y \in \Psi; T_z(y) \geq c\})$ is known, for all $c \in P$ and z. Using this idea in the search of the more powerful test we will consider the different $\psi(F_n(z), F_0(z))$'s for U_n in the next section.

4. Investigation of the Power of Some New Tests by Simulation

To generate new tests we have to choose appropriate Weight Function. Then we need to modify $F_n(z)$ at its discontinuity points $Y_{(i)}$ for $i = 1,2,...,n$. It is trivial that for a point $Y_{(i)} = y$ there are $\frac{i-0.5}{2}$ observations among $Y_1,...,Y_n$ which are less than y. This lead us to consider $F_n(Y_{(i)})$ as $\frac{i-0.5}{2}$.

Selected Weight Function generates a test like a member of the class of the Cramer-Von-Mises. In this section, we propose some Weight Function and simulate their Power against several Alternatives. For each Alternative, the Power result was derived from 10^4 samples of size $n = 50,70,100,120,150,200,250$ depending on choosing κ (related to the Weight Function) and $\psi(F_n(z), F_0(z))$ for Simple and Composite situations.

4.1 Simple Hypothesis: The reasonable choosing of the Weight Function will give us a reasonable test statistic. At the first we consider

$$d\psi(F_n(z), F_0(z)) = 2\{F_n(z)(1 - F_n(z))\}^{-\sqrt{\kappa}} dF_n(z) \quad \text{for} \quad \kappa \geq 0,$$

which is an empiric version of the Weight Function $\psi(F_n(z), F_0(z))$.

By this choice we have

$$T_n = \int_{\mathrm{P}} \log \Lambda^{F_n(z)/F_0(z)} d(w(z)) = \int_{\mathrm{P}} \log \Lambda^{F_n(z)/F_0(z)} 2\{F_n(z)(1-F_n(z))\}^{-\sqrt{\kappa}} d(F_n(z)).$$

Then T_n is given by

$$T_n = 2\frac{1}{n}\sum_{i=1}^{n}\{\log \Lambda^{F_n(X_{(i)})/F_0(X_{(i)})}\{F_n(X_{(i)})(1-F_n(X_{(i)})\}^{-\sqrt{\kappa}}\} =$$

$$2\sum_{i=1}^{n}\{F_n(X_{(i)})\log\frac{F_n(X_{(i)})}{F_0(X_{(i)})}+(1-F_n(X_{(i)})\log\frac{1-F_n(X_{(i)})}{1-F_0(X_{(i)})}\}\{F_n(X_{(i)})(1-F_n(X_{(i)})\}^{-\sqrt{\kappa}}. \qquad (4.1)$$

We generate 10^4 observations from a Beta Distribution, say, $\beta(\eta,\theta)$. Consider the $\beta(1,1)$ as the true (data generate) density. As $F_0(.)$, we consider the Beta Distributions with Parameters as $(\eta,\theta)=(1.5,1.5),(.8,.8),(.6,.6),(1.1,0.8)$ and ≡ =0.1, 0.3, 0.5, 0.7, 1, 1.2, 1.4. For all of tests we set $\alpha = 0.05$ as the level of test. At this given level the critical values of the tests are simulated independently.

Table 1 shows the result of simulations for T_n. For any candidate κ we see that the Power of Test grows when the sample size increases and the Powers converge to 1 very fast. On the other hand, the Power of the new test is always greater than the Power of Anderson-Darling Test. When we set $(\eta,\theta)=(1.5,1.5)$ and $\kappa \le 0.3$ the Power of Anderson-Darling Test is greater than our test.

Table 2 shows the power of the empirical version of the Anderson-Darling test in the same situation as above. In Table 2, the Power of the empirical version of the Anderson-Darling Test in the same situation is stimulated.

Now we consider a simple Weight as $\psi(F_n(z),F_0(z)) = 2dF_n$, which gives T_n as empiric version of the Likelihood Ratio statistic, thus

$$E_n = \int_{\mathrm{P}}[nF_n(z)\log(\frac{F_n(z)}{F_0(z)})+n(1-F_n(z))\log(\frac{1-F_n(z)}{1-F_0(z)})]d(2F_n(z)) =$$

$$\sum_{i=1}^{n}2\{F_n(X_{(i)})\log(\frac{F_n(X_{(i)})}{F_0(X_{(i)})})+(1-F_n(X_{(i)}))\log(\frac{1-F_n(X_{(i)})}{1-F_0(X_{(i)})})\} =$$

$$2\sum_{i=1}^{n}\{F_n(X_{(i)})\log(\frac{F_n(X_{(i)})}{F_0(X_{(i)})})+(1-F_n(X_{(i)}))\log(\frac{1-F_n(X_{(i)})}{1-F_0(X_{(i)})})\}. \qquad (4.2)$$

This test is more powerful than the Anderson-Darling Test when we set $(\eta,\theta)=(0.6,0.6)$, or $(1.1,0.8)$. When we choose $(\eta,\theta)=(0.8,0.8)$ our test is the same as the Anderson-Darling Test. But for $(\eta,\theta)=(1.5,1.5)$ our test has a Power which is a little (about 0.1) lower then the Anderson-Darling Test. see Table 3.

The other test of this type may be construct by the Weight Function as

$$d\psi(F_n(z),F_0(z))=\frac{1}{2}\frac{F_n(z)}{F_0(z)}\frac{1-F_n(z)}{1-F_0(z)}dF_n(z),$$

which introduce a new test, say K_n, where

$$K_n=\int_{\mathrm{P}}\log\Lambda^{F_n(z)/F_0(z)}d(w(z))=\int_{\mathrm{P}}\log\Lambda^{F_n(z)/F_0(z)}\frac{1}{2}\{\frac{F_n(z)}{F_0(z)}\frac{1-F_n(z)}{1-F_0(z)}\}^{-\sqrt{\kappa}}d(F_n(z)).$$

After simplification K_n is given by

$$K_n=\frac{1}{2}\frac{1}{n}\sum_{i=1}^{n}\log\Lambda^{F_n(z)/F_0(z)}\{\frac{F_n(z)}{F_0(z)}\frac{1-F_n(z)}{1-F_0(z)}\}^{-\sqrt{\kappa}}=$$

$$\frac{1}{2}\sum_{i=1}^{n}\{F_n(X_{(i)})\log\frac{F_n(X_{(i)})}{F_0(X_{(i)})}+(1-F_n(X_{(i)})\log\{\frac{1-F_n(X_{(i)})}{1-F_0(X_{(i)})}\}\}\{\frac{F_n(X_{(i)})(1-F_n(X_{(i)})}{F_0(X_{(i)})(1-F_0(X_{(i)})}\}^{-\sqrt{\kappa}}. \qquad (4.3)$$

By this Weight Function, our test has a good Power, but for some $\equiv$ the Power of this test is lower than Anderson-Darling (Table 4). On the other hand, always our test is more powerful than χ^2 test. The Power of χ^2 test for some value of (η,θ) is given in Table 5.

4.2 Composite Hyopthesis: In Composite case, we are testing the Goodness of Fit for a Family of Distributions. To Goodness of Test in (1.2) our test Function will be

$$T_{nc}=2\frac{1}{n}\sum_{i=1}^{n}\{\log\Lambda^{F_n(z)/F(z;\hat{\theta}_n)}\{F_n(z)(1-F_n(z)\}^{-\sqrt{\kappa}}$$

$$2\sum_{i=1}^{n}\{F_n(X_{(i)})\log\frac{F_n(X_{(i)})}{F_0(X_{(i)};\hat{\theta}_n)}+(1-F_n(X_{(i)})\log\frac{1-F_n(X_{(i)})}{1-F_0(X_{(i)};\hat{\theta}_n)}\}\{F_n(X_{(i)})(1-F_n(X_{(i)})\}^{-\sqrt{\kappa}} \qquad (4.4)$$

And

$$K_{nc} = \frac{1}{2}\frac{1}{n}\sum_{i=1}^{n} \log \Lambda^{F_n(z)/F(z;\hat{\theta}_n)}\{\frac{F_n(z)}{F(z;\hat{\theta}_n)}\frac{1-F_n(z)}{1-F(z;\hat{\theta}_n)}\}^{-\sqrt{\kappa}} =$$

$$\frac{1}{2}\sum_{i=1}^{n}\{F_n(X_{(i)})\log\frac{F_n(X_{(i)})}{F_0(X_{(i)};\hat{\theta}_n)} + (1-F_n(X_{(i)})\log\{\frac{1-F_n(X_{(i)})}{1-F_0(X_{(i)};\hat{\theta}_n)}\}\{\frac{F_n(X_{(i)})(1-F_n(X_{(i)})}{F_0(X_{(i)};\hat{\theta}_n)(1-F_0(X_{(i)};\hat{\theta}_n)}\}^{-\sqrt{\kappa}} \quad (4.5)$$

similar to 4.1 and 4.3.

We assume that Y has a Normal Distribution, say $\mathrm{N}(\mu,\sigma^2)$ with $\theta=(\mu,\sigma^2)$ unknown. We can estimate θ by $\hat{\theta}_n = (\overline{Y}_n, S_n^2)$, the mean and sample variance. Then T_{nc} and K_{nc} will be applied to test the Goodness of Fit Test for normality. For Power study, we verify the Power for

$\mathrm{H}_{0c}: Y: \ \mathrm{N}(\mu,\sigma^2)$

against

$\mathrm{H}_{1c}: Y: \ \mathrm{N}(a+\hat{\mu}, b\hat{\sigma}^2)$

where $a \in \mathrm{P}$ and $b \in \mathrm{P}^+$. The Power of this test for $\equiv = 0.1, 0.5, 1, 1.4$, $a=0.3$ and $b=1$ using T_{nc} and K_{nc} are given in Tables 6 and 7 respectively. As we see these tests have a good Power to choose a Model to describe the data at hand.

5. Bahadur Efficiency of Proposed Test

Considering a case when $\psi(F_n(z), F_0(z)) = F_n(z)$ we define a test statistic as

$$S_n = \frac{1}{n}\sum_{z_i \in A_i} \log \Lambda^{F_n(z_i)/F_0(z_i)}$$

then

$$P_{\mathrm{H}_0}(\frac{1}{n}S_n \geq t) = P_{\cap H_{0z}}\{\frac{1}{n}(\frac{1}{n}\sum_{z}\log \Lambda^{F_n(z)/F_0(z)}) \geq t\}$$

$$\leq P_{\mathrm{H}_{0z_{max}}}\{\frac{1}{n}\frac{1}{n}n(\max_{z}\log \Lambda^{F_n(z)/F_0(z)}) \geq t\} =$$

$$P_{\mathrm{H}_{0z_{max}}}\{\frac{1}{n}\log \Lambda_{z_{max}}^{F_n(z)/F_0(z)} \geq t\} \leq \sum_{\mathrm{H}_{1z_{max}}} P_{\mathrm{H}_{0z_{max}}}\{\frac{1}{n}\sum_{i=1}^{n}\log\frac{Bin(1,F(z_{max}))}{Bin(1,F_0(z_{max}))} \geq t\} \quad (*)$$

$$\leq |\mathrm{H}_{1z_{max}}| \max_{F(z_{max})} P_{\mathrm{H}_{0z_{max}}}\{\exp\sum_{i=1}^{n}\log\frac{Bin(1,F(z_{max}))}{Bin(1,F_0(z_{max}))} \geq \exp(nt)\} =$$

$$|\mathrm{H}_1| \max_{F(z_{max})} P_{\mathrm{H}_{0z_{max}}} \{\exp \sum_{i=1}^{n} \log \frac{Bin(1, F(z_{max}))}{Bin(1, F_0(z_{max}))} \geq \exp(nt)\} \leq$$

$$|\mathrm{H}_1| \max_{F(z_{max})} \mathrm{E}_{\mathrm{H}_{0z_{max}}} \{\frac{\exp \sum_{i=1}^{n} \log \frac{Bin(1, F(z_{max}))}{Bin(1, F_0(z_{max}))}}{\exp(nt)}\} = \quad \text{(by Markovin equality)}$$

$$|\mathrm{H}_1| \max_{F(z_{max})} \exp(-nt) \mathrm{E}_{\mathrm{H}_{0z_{max}}} \{\prod_{i=1}^{n} \frac{Bin(1, F(z_{max}))}{Bin(1, F_0(z_{max}))}\} =$$

$$|\mathrm{H}_1| \max_{F(z_{max})} \exp(-nt) \{\mathrm{E}_{\mathrm{H}_{0z_{max}}} \frac{Bin(1, F(z_{max}))}{Bin(1, F_0(z_{max}))}\}^n =$$

$$|\mathrm{H}_1| \max_{F(z_{max})} \exp(-nt) \{\sum Bin(1, F_0(z_{max}))\}^n \leq |\mathrm{H}_1| \exp(-nt)$$

so

$$-\frac{2}{n} \log P_{\mathrm{H}_0}(\frac{1}{n} S_n \geq t) \geq 2t - \frac{2 \log |\mathrm{H}_1|}{n}$$

we know that

$$1/n \log \Lambda^{F_n(z)/F_0(z)} = \frac{1}{n}(\log \Lambda^{F_n(z)} - \log \Lambda^{F_0(z)}) =$$

$$\frac{1}{n}(\log \Lambda^{F(z)} - \log \Lambda^{F_0(z)}) + \frac{1}{n}(\log \Lambda^{F_n(z)} - \log \Lambda^{F(z)}) \qquad \text{(iff)} \quad F(z) \neq F_0(z) \quad \text{(under} \quad H_{1z})$$

$$= \frac{1}{n} \sum_{i=1}^{n} \log \frac{Bin(1, F(z))}{Bin(1, F_0(z))} + \frac{1}{n}(\log \Lambda^{F_n(z)} - (\log \Lambda^{F(z)}) \xrightarrow{\Pi} \mathrm{E}_{H_{1z}} \log \frac{Bin(1, F(z))}{Bin(1, F_0(z))} + (0)\chi^2 =$$

$KL\{Bin(1, F(z)), Bin(1, F_0(z))\}$ a.s under H_{1z}.

Thus

$$\frac{1}{n} \sum_{z} \log \Lambda^{F_n(z)/F_0(z)} \xrightarrow{\Pi} \mathrm{E}_{H1} KL(Bin(1, F(Y)), Bin(1, F_0(Y)))$$

$$-\frac{2}{n} \log P_{\mathrm{H}_0}(\frac{1}{n} S_n \geq t \geq \inf_{\mathrm{H}_0} \mathrm{E}_{\mathrm{H}1} KL(Bin(1, F(Y)), Bin(1, F_0(Y)))$$

Bahadur (1967) showed that the other part of inequality for all of tests is right, then

$$-\frac{2}{n} \log P_{\mathrm{H}_0}(\frac{1}{n} S_n \geq t) = 2 \inf_{\mathrm{H}_0} \mathrm{E}_{\mathrm{H}1} KL(Bin(1, F(Y)), Bin(1, F_0(Y))).$$

The inequality which namely (*) is correct because

$$\frac{1}{n}\log \Lambda_{z_{max}}^{F_n(z)/F_0(z)} \le \sup_{F\in H_1}\frac{1}{n}\sum_{i=1}^{n}\log\frac{Bin(1,F(z_{max}))}{Bin(1,F_0(z_{max}))}.$$

So the Bahadur Efficiency is achieved.

6. Conclusion

The Goodness of Fit Tests are used for verifying whether or not the experimental data come from the postulated model. In this direction,one must decide if theoretical and experimental Distributions are the same. Then,Goodness of Fit is a Hypothesis testing problem and the problem is concerned with the choice of one of the Alternative Hypothesis. This problem contained the Parameters or not. In this work, we consider a simple situation where the Distribution Function is completely known, and also the Composite case. We have introduced an approach which is known to all statisticians, the Likelihood Ratio approach to Hypothesis testing problem. For simple situation, the family which we consider to simulation study is a simple family, but sensitive to choice of Parameter. This family is U-shaped if both of its Parameters (η,θ) are less than one, is J-shaped if $(\eta-1)(\theta-1)<0$, and is otherwise Unimodal. In the case (η=1, θ=1), this distribution is Uniform Distribution on $(0,1)$. This sensibility to Parameters lets us verify our test to various situations. For Composite situation, we consider Location-Scale Family as Normal Family. Development of this approach to a Weight Function which could be morepowerful than Anderson-Darling Test for any κ is our idea. On the other hand, we showed that a member of this kind of tests is efficient in Bahadur sense.

Table 1: Power computations of H_0: $F(.) = \beta(1,1)$ against H_1: $F(.) = \beta(\eta,\theta)$ at level $\alpha = 0.05$ for $(\eta,\theta) = (1.5,1.5)$ using test function T_n

κ	n=50	n=70	n=100	n=120	n=150	n=200	n=250
0.1	0.137	0.251	0.485	0.557	0.702	0.890	0.957
0.3	0.257	0.446	0.597	0.706	0.825	0.947	0.984
0.5	0.322	0.475	0.664	0.776	0.888	0.972	0.993
0.7	0.333	0.568	0.684	0.780	0.911	0.973	0.994
1.0	0.398	0.610	0.729	0.827	0.915	0.970	0.995
1.2	0.413	0.570	0.742	0.824	0.912	0.975	0.991
1.4	0.438	0.606	0.785	0.829	0.910	0.955	0.987

Table 2: Power computation of H_0: $F(.) = \beta(1,1)$ against H_1: $F(.) = \beta(\eta,\theta)$ at level $\alpha = 0.05$ for $(\eta,\theta) = (1.5,1.5)$, $(0.8,0.8)$, $(0.6,0.6)$, $(1.1,0.8)$ based onAnderson-Darling Test

(η,θ)	n=50	n=70	n=100	n=120	n=150	n=200	n=250
(1.5,1.5)	0.302	0.395	0.622	0.640	0.829	0.933	0.975
(0.8,0.8)	0.041	0.082	0.104	0.109	0.129	0.212	0.297
(0.6,0.6)	0.261	0.393	0.570	0.690	0.889	0.955	0.992
(1.1,0.8)	0.495	0.638	0.782	0.849	0.906	0.964	0.980

Table 3: Power computation of H_0: $F(.) = \beta(1,1)$ against H_1: $F(.) = \beta(\eta,\theta)$ at level $\alpha = 0.05$ for $(\eta,\theta) = (1.5,1.5)$, $(0.8,0.8)$, $(0.6,0.6)$, $(1.1,0.8)$ using test function E_n

(η,θ)	n=50	n=70	n=100	n=120	n=150	n=200	n=250
(1.5,1.5)	0.083	0.166	0.254	0.420	0.566	0.747	0.894
(0.8,0.8)	0.146	0.176	0.210	0.241	0.256	0.339	0.447
(0.6,0.6)	0.580	0.725	0.876	0.922	0.973	0.995	0.999
(1.1,0.8)	0.506	0.646	0.704	0.844	0.923	0.978	0.993

Table 4: Power computation of H_0: $F(.) = \beta(1,1)$ against H_1: $F(.) = \beta(\eta,\theta)$ at level $\alpha = 0.05$ for $(\eta,\theta) = (1.5,1.5)$ using test function K_n

κ	n=50	n=70	n=100	n=120	n=150	n=200	n=250
0.1	0.196	0.300	0.423	0.496	0.712	0.887	0.945
0.3	0.247	0.355	0.526	0.640	0.727	0.896	0.976
0.5	0.275	0.428	0.590	0.686	0.806	0.910	0.962
0.7	0.342	0.441	0.633	0.696	0.827	0.938	0.982
1.0	0.352	0.502	0.674	0.755	0.870	0.945	0.983

κ	n=50	n=70	n=100	n=120	n=150	n=200	n=250
1.2	0.373	0.529	0.676	0.765	0.871	0.947	0.985
1.4	0.407	0.552	0.704	0.793	0.900	0.959	0.986

Table 5: Power computation of H_0: $F(.) = \beta(1,1)$ against H_1: $F(.) = \beta(\eta,\theta)$ at level $\alpha = 0.05$ for $(\eta,\theta) = (1.5,1.5), (0.8,0.8), (0.6,0.6), (1.1,0.8)$ based on χ^2 test

(η,θ)	n=50	n=70	n=100	n=120	n=150	n=200	n=250
(1.5,1.5)	0.210	0.400	0.500	0.591	0.630	0.785	0.810
(0.8,0.8)	0.102	0.170	0.175	0.185	0.200	0.280	0.320
(0.6,0.6)	0.382	0.575	0.711	0.823	0.900	0.984	0.999
(1.1,0.8)	0.085	0.145	0.155	0.165	0.183	0.212	0.264

Table 6: Power computation of H_{0c}: $Y: N(\mu, \sigma^2)$ against H_{1c}: $Y: N(0.3 + \hat{\mu}, \hat{\sigma}^2)$ at level $\alpha = 0.05$ using test function T_{nc}

κ	n=50	n=70	n=100	n=120	n=150	n=200	n=250
0.1	0.654	0.739	0.880	0.918	0.989	1.0	1.0
0.5	0.718	0.765	0.871	0.880	0.900	0.999	1.0
1.0	0.509	0.674	0.792	0.912	0.907	0.951	0.994
1.4	0.460	0.637	0.659	0.718	0.850	0.908	0.991

Table 7: Power computation of H_{0c}: $Y: N(\mu, \sigma^2)$ against H_{1c}: $Y: N(0.3 + \hat{\mu}, \hat{\sigma}^2)$ at level $\alpha = 0.05$ using test function K_{nc}

κ	n=50	n=70	n=100	n=120	n=150	n=200	n=250
0.1	0.660	0.829	0.939	0.949	0.963	0.980	0.999
0.5	0.453	0.657	0.790	0.949	0.941	0.920	0.990
1.0	0.436	0.676	0.957	0.960	0.972	0.987	0.992
1.4	0.320	0.505	0.638	0.870	0.880	0.947	0.980

References

1. Anderson, T. W. and Darling, D. A. (1952). Asymptotic theory of certain goodness of fit criteria based on stochastic processes.*The Annals of Mathematical Statistics*, **23(2)**, 193-212.
2. Bahadur, R. R. (1967). Rates of convergence of estimates and test statistics. *The Annals of Mathematical Statistics*, **38**, 303–324.
3. Berk, R. H. and Jones, D. H. (1979). Goodness of fit test that dominate the Kolmogorov statistics. *Zeitschrift fur Wahrscheinlichkeitstheorie und Verwandte Gebiete*, **47**, 47-59.
4. Cassella, G. and Berger, R. L. (2002). *Statistical Inference*, 2nd edition, Pacific Grove CA: Duxbury Press.
5. Cramer, H. (1928). On the composition of elementary errors. Second paper: Statistical application. *Skand. Atuar tidskr*, **11**, 141-180.
6. Cressie, N. and Read, T. R. C. (1984). Multivariate goodness-of-fit tests. *Journal of the Royal Statistical Society, Series B*, **46(3)**, 440-464.
7. David, F.N. and Johnson, N.L. (1948). The probability integral transformation of when parameters are estimated from the sample. *Biometrika*, **35(1-2)**, 182-190
8. Eicker, F. (1979). The asymptotic distribution of the suprema of the standardized empirical process. *The Annals of Statistics*, **7(1)**, 116-138.
9. Einmahl, J. H. J. and McKeague, I. (2003). Empirical likelihood based hypothesis testing. *Bernoulli*, **9(2)**, 267-290.
10. Kolmogorov, A. (1933). Sulla determinazione empirica di una legge di distribuzione. *Giorn. Ist. Ital. Attuari*, **4**, 83-91.
11. Kuiper, N. H. (1960). Tests concerning random points on a circle. *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen, Series A*, **63**, 38–47.
12. Pearson, E. S. (1930). A further development of test for normality. *Biometrika*, **22**, 239-249.
13. Roy, S. N. (1953). On a heuristic method of test construction and its use in multivariate analysis. *The Annals of Mathematical Statistics*, **24**, 220-238.
14. Sayyareh, A. (2011). Limiting distribution and simulated power of some proposed Cramer-Von-Mises type statistic. *International Mathematical Forum*, **6(49)**, 2427-2446.
15. Smirnov, N. V. (1936). Sur la distribution de W_n^2 . *C. R. Acad. Sci. Paris*, **202**, 449-452.

16. Smirnov, N. V. (1937). On the distribution of Mises's W_n^2 criterion. *Mat. Sb.*, Nov., Series 2, 973-993.
17. Smirnov, N. V. (1939). Sur les ecarts de la courbe de distribution empirique. *Matematiceskii Sbornik N.S.*, **6**, 3-26.
18. Smirnov, N. V. (1941). Approximation laws of distribution of random variables from empirical data. *Uspekhi Matematicheskikh Nauk*, **10**, 179-206.
19. Von- Mises, R. (1931). *Wahrscheinlichkeitsrechnung*. Wein, Leipzig.
20. Watson, G.S. (1958). On chi-square goodness-of-fit tests for continuous distributions. *Journal of the Royal Statistical Society, series B*, **20**, 44-61.
21. Wellner, J. A. and Koltchinskii, V. (2002). A note on the asymptotic distribution of Berk-Jones type statistics under the null hypothesis. (http://www.stat.washington.edu/jaw/JAW-papers/NR/HDP2002-rev%20copy.pdf)

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A Hybrid Group Acceptance Sampling Plan for Lifetimes Having Generalized Pareto Distribution

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Abstract

In this paper, a Hybrid Group Acceptance Sampling Plan (HGASP) is proposed for a Truncated Life Test if lifetimes of the items follow the Generalized Pareto Distribution. The Designed Parameters such as minimum number of testers and acceptance number are found when the Consumer's Risk, test termination time and Group Size are pre-specified. The Operating Characteristic values, minimum ratios of the true average (mean) life for the given Producer's Risk are also determined. A comparative study of conventional plan and existing plan is elaborated.

Keywords

Group sampling plan, Generalized pareto distribution, Consumer's risk, Producer's risk. Operating characteristics

1. Introduction

An Acceptance Sampling is a scheme, which consists of sampling, observation and inference in determining the acceptance or rejection of a lot of items submitted by the vendor. It is very important and useful tool if the quality of an item is explained by its lifetime. It has been an important decision to choose an appropriate Probability Distribution in describing the lifetime of the testing items.

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The selection of a Moderate Life Test Sampling Plan is a crucial decision because a good plan not only can help producer but is also very necessary for the consumer. Acceptance Sampling Plan is used when testing is destructive; examining every item is not possible and large number of items are inspected in a short interval of time. In an ordinary Acceptance Sampling Plan, it is assumed that a single item is observed in a tester but in practice more than one item can be examined by the availability of the testers. The items put in a tester can be considered as a group and the number of items in a group is called Group Size. Any Acceptance Sampling Plan which follows such type of pattern is called Group Acceptance Sampling Plan (GASP). The technique of obtaining minimum number of testers for a specified number of groups is called Hybrid Group Acceptance Sampling Plan (HGASP).

An ordinary Acceptance Sampling Plan based on Truncated Life Test for a variety of Lifetime Distributions were developed by Baklizi (2003), Balakrishnan et al. (2007), Epstein (1954), Kantam et al. (2006), Rosaiah et al. (2008) and Tsai and Wu (2006). More recently, Aslam et al. (2010b), Aslam et al. (2010c), Aslam and Jun (2009), Mughal (2011a), Mughal et al. (2010), Mughal et al. (2011b), Mughal et al. (2011c), and Rao (2011) have developed the Group Acceptance Sampling Plan based on Truncated Life Test. The objective of this study is to develop a Hybrid Group Acceptance Sampling Plan (HGASP) based on Truncated Life Test when the lifetime of an item follows the Generalized Pareto Distribution (GPD) introduced by Abd Elfattah et al. (2007). The Probability Density Function (p.d.f.) and Cumulative Distribution Function (c.d.f.) of GPD are given respectively

$$f(t;\alpha,\beta,\lambda,\delta)=\frac{\delta\alpha}{\beta}\left(\frac{t-\lambda}{\beta}\right)^{\delta-1}\left[1+\left(\frac{t-\lambda}{\beta}\right)^{\delta}\right]^{-(\alpha+1)} \tag{1.1}$$

where $\lambda<t<\infty$, $\beta>0$, $\alpha>0$, $\delta>0$, λ is the Location Parameter, β is the Scale Parameter and (α , δ) are Shape Parameters.

$$F(t;\alpha,\beta,\lambda,\delta)=1-\left[1+\left(\frac{t-\lambda}{\beta}\right)^{\delta}\right]^{-\alpha} \tag{1.2}$$

Aslam et al. (2010a) studied Group Acceptance Sampling Plan based on Generalized Pareto Distribution (GPD). The mean and variance of GPD are

$$\mu = \beta \frac{\Gamma\left(\alpha - \frac{1}{\delta}\right)\Gamma\left(1 + \frac{1}{\delta}\right)}{\Gamma(\alpha)} + \lambda \tag{1.3}$$

$$\sigma^2 = \beta^2 \left[\frac{\Gamma\left(1 + \frac{2}{\delta}\right)\Gamma\left(\alpha - \frac{2}{\delta}\right)}{\Gamma(\alpha)} - \left(\frac{\Gamma\left(1 + \frac{1}{\delta}\right)\Gamma\left(\alpha - \frac{1}{\delta}\right)}{\Gamma(\alpha)} \right)^2 \right] \tag{1.4}$$

For the existence of mean and variance, we assume the following conditions namely, $\alpha > 1/\delta$ and $\alpha > 2/\delta$ respectively.

2. Hybrid Group Acceptance Sampling Plans (HGASP)

Consider μ denotes the true mean life and μ_0 represents the specified mean life of an item which follows Generalized Pareto Distribution (GPD). It is wished to propose a HGASP if the true mean life is higher than the specified mean life i.e. $H_1 = \mu > \mu_0$. The HGASP also develop the mutual agreement of the both Producer's and the Consumer's Risk. The chance of rejecting a good item is the Producer's risk whereas the chance of accepting a bad item is called the Consumer's risk denoting by γ and β respectively. Now, we propose the HGASP followed by Truncated Life Tests:

Step 1: Find the number of testers 'r' and allocate the 'r' items to each pre-specified 'g' groups. The required sample size for a lot in the Truncated Life Test is n = gr.

Step 2: Specify the acceptance number 'c' for every group and the termination time t_0.

Step 3: Terminate the experiment and reject the lot if more than 'c' failures are found in each group.

If $r = 1$, the proposed HGASP convert to the ordinary Acceptance Sampling Plan and we can say that the proposed HGASP is an extension of the ordinary plans available in the literature. Our concern is to find the number of testers 'r', required for Generalized Pareto Distribution and different values of acceptance number 'c' whereas the number of groups 'g' and the test truncation time t_0 are considered to be pre-specified. For convenience, we will consider that the test termination time

Table 2 indicates the minimum number of testers for the proposed HGASP for the Shape Parameters $(\alpha,\delta)=2$. The frequently used values of test termination time a=0.7, 0.8, 1.0, 1.2, 1.5, 2.0, number of groups $g=2(1)9$, acceptance number $c=0(1)7$ and Consumer's Risk β^*=0.25, 0.10, 0.05, 0.01 are given in Table 1. The selections of the shape of Parameters $(\alpha,\delta)=2$ are used for the comparison purpose.

On the other hand, Operating Characteristic (OC) involves a system of principles, techniques and their purpose is to construct decision rules to accept or reject the lot through numerically or graphically based on the sample information. The Operating Characteristic (OC) curve indicates the Probability of Acceptance for various levels of submitted lot quality. If the minimum number of testers is found, one can be delighted to obtain the lot Acceptance Probability when the quality of an item is highly good. As discussed earlier, an item is assumed to be bad or poor quality if $\mu < \mu_0$. For $(\alpha,\delta)=2$, the probabilities of acceptance are showed in Table 3 based on Equation 3.4 for given Design Parameters. From Table 3, we observe that OC values decreases as quickly as the mean ratio decreases. For example, when β^*=0.05, g=4, c=2 and a=1.0, the number of testers required is r=3. If the true mean lifetime is twice the specified mean lifetime $(\mu/\mu_0=2)$, the Producer's Risk is approximately γ=1-0.6796=0.3204 and γ=0.0003 when the true value of mean is 8 times the specified value.

The producer can be concerned in enlarging the quality level of an item so that the Acceptance Probability should be higher than the pre-assumed level. When the Producer's Risk is given, the minimum ratio (μ/μ_0) can be found by solving the following inequality (2.4),

$$L(p_1) \geq 1-\gamma \tag{2.4}$$

Table 4 shows the minimum ratio for Generalized Pareto Distribution with $(\alpha,\delta)=2$, at the Producer's Risks of γ=0.05 based on the Designed Parameters given in Table 1 and one may see that the mean ratio decreased as the Test Termination Ratio decreased. For example, when β^*=0.25, g=4, c=2, r=3 and a=0.8 for determining a Producer's Risk γ=0.05 will be increased the true value of mean μ to 2.51 times the specified value of μ_0.

Table 2 indicates the minimum number of testers for the proposed HGASP for the Shape Parameters $(\alpha,\delta)=2$. The frequently used values of test termination time a=0.7, 0.8, 1.0, 1.2, 1.5, 2.0, number of groups $g=2(1)9$, acceptance number $c=0(1)7$ and Consumer's Risk β^{*}=0.25, 0.10, 0.05, 0.01 are given in Table 1. The selections of the shape of Parameters $(\alpha,\delta)=2$ are used for the comparison purpose.

On the other hand, Operating Characteristic (OC) involves a system of principles, techniques and their purpose is to construct decision rules to accept or reject the lot through numerically or graphically based on the sample information. The Operating Characteristic (OC) curve indicates the Probability of Acceptance for various levels of submitted lot quality. If the minimum number of testers is found, one can be delighted to obtain the lot Acceptance Probability when the quality of an item is highly good. As discussed earlier, an item is assumed to be bad or poor quality if $\mu<\mu_0$. For $(\alpha,\delta)=2$, the probabilities of acceptance are showed in Table 3 based on Equation 3.4 for given Design Parameters. From Table 3, we observe that OC values decreases as quickly as the mean ratio decreases. For example, when β^{*}=0.05, g=4, c=2 and a=1.0, the number of testers required is r=3. If the true mean lifetime is twice the specified mean lifetime $(\mu/\mu_0=2)$, the Producer's Risk is approximately γ=1-0.6796=0.3204 and γ=0.0003 when the true value of mean is 8 times the specified value.

The producer can be concerned in enlarging the quality level of an item so that the Acceptance Probability should be higher than the pre-assumed level. When the Producer's Risk is given, the minimum ratio (μ/μ_0) can be found by solving the following inequality (2.4),

$$L(p_1)\geq 1-\gamma \tag{2.4}$$

Table 4 shows the minimum ratio for Generalized Pareto Distribution with $(\alpha,\delta)=2$, at the Producer's Risks of γ=0.05 based on the Designed Parameters given in Table 1 and one may see that the mean ratio decreased as the Test Termination Ratio decreased. For example, when β^{*}=0.25, g=4, c=2, r=3 and a=0.8 for determining a Producer's Risk γ=0.05 will be increased the true value of mean μ to 2.51 times the specified value of μ_0.

It is concluded that the proposed HGASP is more economical and beneficial than the existing HGASP in terms of minimum sample size, cost, test truncation time and labor.

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The authors are grateful to the Assistant Editor Ms. Maliha Butt for her suggestions and guidance during the preparation of the paper and to the referees whose fruitful comments led to an improvement in the paper.

Table 1: Group size (g), test termination time (a), Consumer's Risk (β^*) and acceptance number (c)

β*	g	c	a
0.25	2	0	0.5
0.10	3	1	0.7
0.05	4	2	0.8
0.01	5	3	1.0
	6	4	1.2
	7	5	1.5
	8	6	
	9	7	

It is concluded that the proposed HGASP is more economical and beneficial than the existing HGASP in terms of minimum sample size, cost, test truncation time and labor.

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0.01	5	3	1.0
	6	4	1.2
	7	5	1.5
	8	6	
	9	7	

Table 3: Operating Characteristics values of the group sampling plan for Generalized Pareto Distribution α=2, δ=2, g=4 and c=2

β	r	a	2	4	6	8	10	12
0.25	5	0.5	0.8894	0.9971	0.9997	0.9999	1.0000	1.0000
	4	0.7	0.7683	0.9922	0.9992	0.9999	1.0000	1.0000
	3	0.8	0.8606	0.9956	0.9996	0.9999	1.0000	1.0000
	3	1.0	0.6796	0.9853	0.9984	0.9997	0.9999	1.0000
	3	1.2	0.4657	0.9627	0.9956	0.9991	0.9998	0.9999
	3	1.5	0.2081	0.8942	0.9853	0.9969	0.9991	0.9997
0.10	6	0.5	0.8104	0.9945	0.9995	0.9999	1.0000	1.0000
	4	0.7	0.7683	0.9922	0.9992	0.9999	1.0000	1.0000
	4	0.8	0.6258	0.9838	0.9983	0.9997	0.9999	1.0000
	3	1.0	0.6796	0.9853	0.9984	0.9997	0.9999	1.0000
	3	1.2	0.4657	0.9627	0.9956	0.9991	0.9998	0.9999
	3	1.5	0.2081	0.8942	0.9853	0.9969	0.9991	0.9997
0.05	6	0.5	0.8104	0.9945	0.9995	0.9999	1.0000	1.0000
	4	0.7	0.7683	0.9922	0.9992	0.9999	1.0000	1.0000
	4	0.8	0.6258	0.9838	0.9983	0.9997	0.9999	1.0000
	3	1.0	0.6796	0.9853	0.9984	0.9997	0.9999	1.0000
	3	1.2	0.4657	0.9627	0.9956	0.9991	0.9998	0.9999
	3	1.5	0.2081	0.8942	0.9853	0.9969	0.9991	0.9997
	7	0.5	0.7180	0.9907	0.9991	0.9998	1.0000	1.0000
	5	0.7	0.5795	0.9816	0.9981	0.9996	0.9999	1.0000
	4	0.8	0.6258	0.9838	0.9983	0.9997	0.9999	1.0000
	4	1.0	0.3275	0.9488	0.9940	0.9988	0.9997	0.9999
	3	1.2	0.4657	0.9627	0.9956	0.9991	0.9998	0.9999
	3	1.5	0.2081	0.8942	0.9853	0.9969	0.9991	0.9997

Table 4: Minimum ratio of true average life to specified life for the Producer's Risk of 0.05, for Generalized Pareto Distribution $\alpha = 2, \delta = 2$

			a					
β	g	c	0.5	0.7	0.8	1.0	1.2	1.5
0.25	2	0	7.35	7.26	8.3	10.38	12.46	15.57
	3	1	2.92	3.07	3.51	4.4	5.29	6.6
	4	2	2.35	2.81	2.51	3.13	3.76	4.69
	5	3	1.86	2.25	2.06	2.57	3.08	3.86

Table 3: Operating Characteristics values of the group sampling plan for Generalized Pareto Distribution α=2, δ=2, g=4 and c=2

β	r	a	2	4	6	8	10	12
0.25	5	0.5	0.8894	0.9971	0.9997	0.9999	1.0000	1.0000
	4	0.7	0.7683	0.9922	0.9992	0.9999	1.0000	1.0000
	3	0.8	0.8606	0.9956	0.9996	0.9999	1.0000	1.0000
	3	1.0	0.6796	0.9853	0.9984	0.9997	0.9999	1.0000
	3	1.2	0.4657	0.9627	0.9956	0.9991	0.9998	0.9999
	3	1.5	0.2081	0.8942	0.9853	0.9969	0.9991	0.9997
0.10	6	0.5	0.8104	0.9945	0.9995	0.9999	1.0000	1.0000
	4	0.7	0.7683	0.9922	0.9992	0.9999	1.0000	1.0000
	4	0.8	0.6258	0.9838	0.9983	0.9997	0.9999	1.0000
	3	1.0	0.6796	0.9853	0.9984	0.9997	0.9999	1.0000
	3	1.2	0.4657	0.9627	0.9956	0.9991	0.9998	0.9999
	3	1.5	0.2081	0.8942	0.9853	0.9969	0.9991	0.9997
0.05	6	0.5	0.8104	0.9945	0.9995	0.9999	1.0000	1.0000
	4	0.7	0.7683	0.9922	0.9992	0.9999	1.0000	1.0000
	4	0.8	0.6258	0.9838	0.9983	0.9997	0.9999	1.0000
	3	1.0	0.6796	0.9853	0.9984	0.9997	0.9999	1.0000
	3	1.2	0.4657	0.9627	0.9956	0.9991	0.9998	0.9999
	3	1.5	0.2081	0.8942	0.9853	0.9969	0.9991	0.9997
	7	0.5	0.7180	0.9907	0.9991	0.9998	1.0000	1.0000
	5	0.7	0.5795	0.9816	0.9981	0.9996	0.9999	1.0000
	4	0.8	0.6258	0.9838	0.9983	0.9997	0.9999	1.0000
	4	1.0	0.3275	0.9488	0.9940	0.9988	0.9997	0.9999
	3	1.2	0.4657	0.9627	0.9956	0.9991	0.9998	0.9999
	3	1.5	0.2081	0.8942	0.9853	0.9969	0.9991	0.9997

Table 4: Minimum ratio of true average life to specified life for the Producer's Risk of 0.05, for Generalized Pareto Distribution $\alpha = 2, \delta = 2$

			a					
β	g	c	0.5	0.7	0.8	1.0	1.2	1.5
0.25	2	0	7.35	7.26	8.3	10.38	12.46	15.57
	3	1	2.92	3.07	3.51	4.4	5.29	6.6
	4	2	2.35	2.81	2.51	3.13	3.76	4.69
	5	3	1.86	2.25	2.06	2.57	3.08	3.86

			a					
β	**g**	**c**	**0.5**	**0.7**	**0.8**	**1.0**	**1.2**	**1.5**
0.25	**6**	**4**	1.76	1.93	2.21	2.25	2.7	3.37
	7	**5**	1.69	1.73	1.98	2.03	2.44	3.05
	8	**6**	1.53	1.58	1.81	1.88	2.26	2.82
	9	**7**	1.5	1.66	1.89	1.76	2.11	2.64
0.10	**2**	**0**	7.35	10.3	8.3	10.38	12.46	15.57
	3	**1**	3.49	4.08	4.67	4.4	5.29	6.6
	4	**2**	2.64	2.81	3.2	3.13	3.76	4.69
	5	**3**	2.08	2.25	2.57	2.57	3.08	3.86
	6	**4**	1.92	2.22	2.21	2.25	2.7	3.37
	7	**5**	1.81	1.97	1.98	2.03	2.44	3.05
	8	**6**	1.63	1.79	1.81	1.88	2.26	2.82
	9	**7**	1.59	1.66	1.68	2.1	2.11	2.64
0.05	**2**	**0**	9.01	10.3	11.76	10.38	12.46	15.57
	3	**1**	3.96	4.08	4.67	4.4	5.29	6.6
	4	**2**	2.64	2.81	3.2	3.13	3.76	4.69
	5	**3**	2.28	2.61	2.57	2.57	3.08	3.86
	6	**4**	2.07	2.22	2.21	2.76	2.7	3.37
	7	**5**	1.81	1.97	1.98	2.47	2.44	3.05
	8	**6**	1.73	1.79	2.05	2.26	2.26	2.82
	9	**7**	1.67	1.82	1.89	2.1	2.11	2.64
0.01	**2**	**0**	10.4	12.61	11.76	14.71	17.66	15.57
	3	**1**	3.96	4.87	4.67	5.83	5.29	6.6
	4	**2**	2.92	3.3	3.2	4.01	3.76	4.69
	5	**3**	2.46	2.61	2.98	3.21	3.08	3.86
	6	**4**	2.2	2.47	2.54	2.76	2.7	3.37
	7	**5**	2.03	2.18	2.25	2.47	2.44	3.05
	8	**6**	1.83	1.97	2.05	2.26	2.26	2.82
	9	**7**	1.75	1.82	1.89	2.1	2.52	2.64

Table 5: Comparisons of sample size (n) when g=7 and c=5.

a	β*	Existing HGASP	Proposed HGASP
0.7	**0.05**	105	56
0.8		91	49
1.0		70	49
1.2		63	42
1.5		56	42

References

1. Abd Elfattah, A. M., Elsherpieny, E. A. and Hussein, E. A. (2007). A new generlized pareto distribution. *Interstat (electronic journal)*. **12**, 1-6.
2. Aslam, M. and Jun, C. H. (2009). A group acceptance sampling plan for truncated life tests based the inverse rayleigh distribution and log-logistics distribution. *Pakistan Journal of Statistics*, **25(2)**, 107-119.
3. Aslam, M., Mughal, A. R., and Ahmed, M. (2010a). Group acceptance sampling plan for lifetimes having generalized pareto distribution. *Pakistan Journal of Commerce and Social Sciences*, **4(2)**, 185-193.
4. Aslam, M., Mughal, A. R., Ahmed, M. and Yab, Z. (2010b). Group acceptance sampling plan for pareto distribution of the second kind. *Journal of Testing and Evaluation*, **38(2)**,1-8.
5. Aslam, M., Mughal, A. R., Hanif, M. and Ahmed, M. (2010c). Economic reliability group acceptance sampling based on truncated life tests using pareto distribution of the second kind. *Communications of the Korean Statistical Society*, **17(5)**, 725–732.
6. Baklizi, A. (2003). Acceptance sampling based on truncated life tests in the Pareto distribution of the second kind. *Advances and Applications in Statistics*, **3(1)**, 33-48.
7. Balakrishnan, N., Leiva, V. and Lopez, J. (2007). Acceptance sampling plans from truncated life tests based on the generalized Birnbaum-Saunders distribution. *Communication in Statistics - Simulation and Computation*, **36**, 643-656.

8. Epstein, B. (1954). Truncated life tests in the exponential case. *The Annals of Mathematical Statistics*, **25**, 555-564.
9. Kantam, R. R. L., Rao, G. S. and Sriram, G. (2006). An economic reliability test plan: log-logistic distribution. *Journal of Applied Statistics*, **33(3)**, 291-296.
10. Mughal, A. R. (2011a). A hybrid economic group acceptance sampling plan for Exponential lifetime distribution. *Economic Quality Control*, **26(2)**, 163-171.
11. Mughal, A. R. and Aslam, M. (2011b).Efficient group acceptance sampling plan for family pareto distributions. *Continental Journal of Applied Sciences,* **6(3)**, 40-52.
12. Mughal, A. R., Aslam, M., Hussain, J. and Rehman, A., (2010). Economic Reliability group acceptance sampling plans for lifetimes following a Marshall-Olkin extended distribution. *Middle Eastern Finance and Economics,* **7**, 87-93.
13. Mughal, A. R., Hanif, M., Imran, A. A., Rafi, M. and Ahmed, M. (2011c). Economic reliability two-stage group sampling plan for truncated life test having Weibull distribution. *European Journal of Scientific Research,* **54(4)**, 593-599.
14. Rao, G. S. (2011). A hybrid group acceptance sampling plan for lifetimes based on generlized exponential distribution. *Journal of Applied Sciences,* **11(12)**, 2232-2237
15. Rosaiah, K., Kantam, R. R. L. and Reddy, J. R. (2008). Economic reliability test plan with inverse Rayleigh variate. *Pakistan Journal of Statistics*, **24**, 57-65.
16. Tsai, T. R. and Wu, S. J. (2006). Acceptance sampling based on truncated life tests for generalized Rayleigh distribution. *Journal of Applied Statistics,* **33(6)**, 595-600.

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Deriving the Efficiency Function for Type I Censored Sample from Exponential Distribution Using Sup-Entropy

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Abstract

This paper utilizes information theory to quantify Efficiency of Type I Censored sample drawn from Exponential Distribution and the consequent information loss due to Censoring. Based on Awad Sup-Entropy, an Efficiency Function for Censored sample is derived explicitly. The properties of the derived Efficiency Function are explained as a function of the Exponential Parameter and the termination time of the experiment. The estimation for the termination time of the experiment for a given Efficiency is discussed. Furthermore, under certain Efficiency, the Maximum Likelihood and Interval Estimation for the Exponential Parameter are also introduced.

Keywords

Entropy, Type I censoring, Type II censoring, Exponential distribution

1. Introduction

It is naturally anticipated that smaller samples contain less information than larger samples; however, little attempt has been made to quantify the information loss due to the use of subsamples rather than complete samples.

Information is defined in terms of Probability Density Function $f(x)$ of a given random variable X and is measured by the differential Entropy as was suggested by Shannon (1948). The differential Entropy is referred to as the Shannon Entropy for the case of continuous random variables, namely

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$$H(X) = -E(\log f(X)) = -\int f(x)\log f(x)dx. \tag{1.1}$$

A couple of papers have discussed the use of Entropy in analysis of Censored experiments.

Hollander et al. (1987) suggested using Entropy measures for quantifying the information loss in Censored samples from Discrete Distributions. However, for Continuous case, Shannon Entropy could become negative. Therefore, the authors proposed using a variance Entropy measure instead. Ebrahimi and Soofi (1990) discussed the use of Shannon Entropy for measuring information loss for both the Maximum Likelihood Estimation and Bayesian Estimation for type II Censored Exponential data. Awad (1987) suggested a modification for Shannon Entropy (4.1), namely

$$A(X) = -E(\log\frac{f(X)}{\delta}), \text{ where } \delta = \sup_x f(X) \tag{1.2}$$

As opposed to Shannon Entropy (1.1), Awad Sup-Entropy (1.2) or Simply Sup-Entropy is always non-negative since $\frac{f(X)}{\delta} \leq 1$; also it attains zero value if and only if X is a uniform random variable (which is a non-informative distribution). Moreover, for the Awad Sup-Entropy, we can also find a complete analogy between Sup-Entropy for Discrete case and that of Continuous case. For more information about Awad Sup-Entropy (see Awad, 1987).

Awad and Alawneh (1987) calculated the loss of Entropy when the life time is assumed to be Truncated Exponentially on [0,t) for the Shannon and the Sup-Entropy cases. They observed numerically that the loss of Sup-Entropy is always between zero and one, and also it decreases with an increase in truncation time; on the other hand, the loss of Shannon Entropy could be negative and could be more than one.

Some other researchers like Abo-Eleneen (2008), Ng et al. (2004), and Zeng and Park (2004) used maximum expected Fisher information measure as an optimality criterion in progressively Censored experiments. Balakrishnan et al. (2008), basing on Fisher information measure, found that the Optimal Censoring scheme for some Distributions was that of one-step Censoring. Haj Ahmed and Awad (2010) based on Sup-Entropy, found that the Optimal Censoring scheme for Pareto Distribution is also one-step Censoring scheme.

As continuum of the exploration of Awad Sup-Entropy usage, the work presented here investigates quantifying Efficiency of type I Censored samples from Exponential Distribution. The rest of this paper is organized as follows: Section 2 suggests the Efficiency Function based on Sup-Entropy for measuring Efficiency of type I Censored sample from Exponential Distribution and its corresponding properties. This section also contains an explicit formula for the termination time of the experiment as function of both the Exponential Parameter and the Efficiency. Also the expected number of measurable observations (Censored sample size) corresponding to a given Efficiency is proposed here. Section 3 discusses the Maximum Likelihood Estimator (MLE) and Confidence Interval for the Exponential Parameter as functions of the Efficiency based on Sup-Entropy. Simulation study is presented in Section 4. Finally, the conclusions are drawn in Section 5.

2. Awad Sup-Entropy and Efficiency of Type I Censored Sample

Let us construct the Efficiency Function of the type I Censored sample from Exponential Distribution using Sup-Entropy (1.2).

We know that in certain types of problems such as life-testing experiments, the ordered observations may occur naturally. In such cases, a great savings in time and cost could be realized by terminating the experiment as soon as the first 'r' ordered observations have occurred rather than waiting for all 'n' failures to occur. If one terminates the experiment after a fixed time 't', this procedure is referred to as Type I Censored sampling. In this case the number of observations, R is a random variable. The probability that a failure occurs before time 't' for any given trial is p=F(t), where F is the Distribution Function of the assumed life-time model, so for a random sample of size 'n' the random variable R follows a Binomial Distribution.
Formally, $R \sim Bin(n,F(t))$ with Probability Density Function

$$b(r) = = \frac{n!}{r!(n-r)!}\left[F(t)\right]^{r}\left[1-F(t)\right]^{n-r}$$

Type I Censored sampling is related to the concept of Truncated Distributions. Consider an ordered random sample $X_1, X_2, \ldots, X_n$ from a distribution with Probability Density Function $f(x)$ and Cumulative Distribution Function $F(x)$. And out of 'n' observations in total, we suppose that only 'r' observations occur before time 't'; then given R=r, the Joint Conditional Density Function of these values say $x_1, x_2, \ldots, x_r$ is given by

$$h(x_1, x_2, \ldots, x_r \mid r) = r! \prod_{i=1}^{r} f(x_i \mid x_i < t) = \frac{r!}{[F(t)]^r} \prod_{i=1}^{r} f(x_i).$$

Accordingly, the Joint Probability Density Function of $X_1, X_2, \ldots, X_r$ is given by

$$f_{cenI}(x_1, x_2, \ldots, x_r) = \frac{n!}{(n-r)!}[1 - F(t)]^{n-r} \prod_{i=1}^{r} f(x_i), \qquad (2.1)$$

where the subscript cen1 denotes the type I Censoring.

2.1 Awad Sup-Entropy for Type I Censored Sample: The Probability Density Function of Exponential Distribution with Parameter (θ) is given by

$f(x) = \theta e^{-\theta x},\ x > 0, \theta > 0,$ and zero elsewhere.

The Sup-Entropy for both complete and type I Censored samples from $exp(\theta)$ are derived through the following theorem.

Theorem 2.1: If $X_1, X_2, \ldots, X_r$ denote the first r ordered statistics of a random sample of size n from Exponential Distribution with Parameter θ that is type I Censored on the right at 't', then

$$A_{com} = n \qquad (2.2)$$

and

$$A_{cen}(\theta) = n\left(1 - e^{-\theta t} - \theta t e^{-\theta t}\right) \qquad (2.3)$$

where A_{com}, and A_{cen} denote the Sup-Entropy of the complete and the Censored samples, respectively.

2.2 Efficiency of Type I Censored Sample: Efficiency of a Censoring scheme based on a given Entropy measure is the ratio of the value of that Entropy in the Censored scheme to its value in the complete scheme. Accordingly, by theorem 2.1 the Efficiency of type I Censored sample from $exp(\theta)$ based on Awad Sup-Entropy is given by

$$C(t,\theta) = \left(1 - e^{-\theta t} - \theta t e^{-\theta t}\right) \qquad (2.4)$$

Also, the relative loss of information due to type I Censoring is given by

$$L(t) = \frac{A_{com} - A_{cen}}{A_{com}} = 1 - C(t,\theta) = e^{-\theta t}(1 + \theta t) \qquad (2.5)$$

It can be seen that the efficiency $C(t,\theta)$ is non-negative strictly increasing function of 't' on $(0,\infty)$ with $C(0,\theta)=0$ and $\lim_{t\to\infty} C(t,\theta)=1$, and hence, $C(t,\theta)$ itself is a Distribution Function defined on the same support of $F(t)$. Also it is interesting to note that $C(t,\theta)$ is just the Distribution Function of the first record value from *exp(θ)*; moreover, $L(t)$ is the Survival Function of that Distribution; but this could not be the general case for all Distributions.

Since $0 \le C(t,\theta) \le 1$, thus $A_{cen} \le A_{com}$; where equality holds if and only if $t \to \infty$. That is, basing on Sup-Entropy, the amount of information in the Censored sample is less than in the complete sample. Therefore, $C(t,\theta)$ might be called 'the expected percentage of information in Censored data with respect to the complete data'.

To determine the termination time that leads to a given Efficiency ε where $0 < \varepsilon < 1$, the solvability of the equation should be discussed.

$$1 - xe^{-x} - e^{-x} = \varepsilon \text{, where } x = \theta t \tag{2.6}$$

The properties of the Efficiency Function (2.4) that are mentioned before guarantee the uniqueness of the solution of (2.6); moreover, many numerical techniques such as Newton-Raphson's method will converge to this unique solution.

Let the unique solution of (2.6) be denoted by x_ε, then the termination time corresponding to ε is given by

$$t_\varepsilon = \frac{x_\varepsilon}{\theta} \tag{2.7}$$

For estimation purposes, since the Parameter θ would be unknown, then unfortunately the termination time in (2.7) will be so. However, the number of failures r_ε before t_ε can be estimated by the expected value of R at $t = t_\varepsilon$, thus

$$r_\varepsilon = E(R) = nF(t_\varepsilon) = n(1 - e^{-x_\varepsilon}) \tag{2.8}$$

From equation (2.8), the estimated number of failures r_ε corresponding to efficiency ε could not be an integer; hence it is reasonable to take the floor values instead.

3. Estimation and Efficiency

In this section, under certain Efficiency, Point and Interval Estimations for θ are provided. Also an approximation to the termination time of the experiment corresponding to this Efficiency is given.

For a given Efficiency ε, equation (2.8) provides approximated number of failures before the termination time t_s, while as mentioned before, the suitable termination time for the experiment to attain this Efficiency is missing. The previous argument suggests estimating θ from the MLE of the type II Censored sample Likelihood instead.

The Likelihood Function for the type II Censored sample is given by Bain and Engelhardt (1992).

$$f_{cenII}(x_1,x_2,\ldots,x_r)=\frac{n!}{(n-r)!}[1-F(x_r)]^{n-r}\prod_{i=1}^{r}f(x_i)=\frac{n!}{(n-r)!}e^{-\theta x_r(n-r)}\theta^r\exp(-\theta\sum_{i=1}^{r}x_i)$$

if $t>x_i>0,\theta>0,$ and zero elsewhere.

Thus when r = r_ε; the MLE for θ from the type II Censored sample is given by

$$\hat{\theta}_\varepsilon=\frac{r_\varepsilon}{\sum_{i=1}^{r_\varepsilon}x_i+(n-r_\varepsilon)x_r} \quad (3.1)$$

The MLE in equation (3.1) might be called the ε efficient MLE for θ.
By substituting this estimate in equation (2.7), the estimated termination time of the experiment subject to Efficiency ε is given by

$$\hat{t}_\varepsilon=\frac{x_\varepsilon}{\hat{\theta}_\varepsilon} \quad (3.2)$$

The termination time in equation (3.1) might be called the ε -efficient termination time of the experiment.

Johnson and Kotz (1970) showed that for a complete sample of size n, $2n\theta/\theta$ is distributed as Chi-Square with $2n$ degrees of freedom, where θ is the MLE for θ from the Complete Likelihood Function and thus a $100\alpha\%$ Confidence Interval for θ is given by

$$\frac{\hat{\theta}\,\chi^2_{2n,\alpha/2}}{2n} < \theta < \frac{\hat{\theta}\,\chi^2_{2n,1-\alpha/2}}{2n}$$

where $\chi^2_{n,\alpha}$ is the $100\alpha th$ Percentile of the Chi-Square Distribution with degrees of freedom 'n'.

This result should also hold for Censored samples if the degree of Censoring is not excessive. Accordingly, from the type II Censored sample, a $100\alpha\%$ Confidence Interval for θ is given by

$$\frac{\hat{\theta}_\varepsilon\,\chi^2_{2r,\alpha/2}}{2r} < \theta < \frac{\hat{\theta}_\varepsilon\,\chi^2_{2r,1-\alpha/2}}{2r} \tag{3.3}$$

where $\chi^2_{r,\alpha}$ is the $100\alpha th$ Percentile of the Chi-Square Distribution with degrees of freedom 'r'.

When r = r_s, the interval (3.3) might be called the ε_efficient Confidence Interval for θ.

4. Simulation Study

In this section, the performances of the proposed estimators of 'θ', 't' and 'r' from Censored samples are investigated through a Simulation study under different given values of the Efficiency. The Simulation study is carried out for different values of the combination (θ, n, ε). In this study the Efficiency ε is assumed to be 10%, 50% and 95%. In all these cases we have generated 500 samples of size n(=10, 30 and 50) from an Exponential Distribution with Parameter θ(=0.5, 1 and 10) by using Mathematica 6, and then the average values of the ε-efficient estimators of $\theta(=\hat{\theta}_\varepsilon)$, t $(=\hat{t}_\varepsilon)$and $r(=r_\varepsilon)$ are calculated and reported in Tables 1, 2 and 3 corresponding to ε=(10%, 50% and 95%) respectively, along with the ratio (r_ε/n) are calculated. For the purpose of comparison, the MLE of θ is evaluated from each complete sample $(=\hat{\theta}_{COM})$.

It is straightforward to get from Tables 1, 2 and 3 that the expected number of failures (r_ε) corresponding to a given Efficiency ε is free of 'n' and 'θ', while the ratio $\frac{r_\varepsilon}{n}$ is largely fixed. The variation we find for the values of $\frac{r_\varepsilon}{n}$ in Table 3 is due to rounding. In fact, from equations (2.6) and (2.8) we observe that the ratio $\frac{r_\varepsilon}{n}$ depends solely on the efficiency, but neither on the complete sample size 'n' nor on the parameter 'θ'.

In general, this ratio also depends on the type of the life-time model under consideration. Apart from this, we notice that as the Efficiency ε increases to 100%, r_ε converges to 'n', and $\hat{\theta}_\varepsilon$ converges to $\hat{\theta}_{COM}$, accordingly, in this case the termination time of the experiment ($\hat{t}_\varepsilon$) tends to infinity, at least theoretically. Besides, the proposed estimator of the termination time of the experiment $\hat{t}_\varepsilon$ is free of 'n'. Moreover, it decreases as $\hat{\theta}_\varepsilon$ increases and, for a fixed value of 'n', it increases as ε increases.

It can be noticed from the Simulation studies that the proposed estimator θ_ε gives reasonably close estimate for θ_{COM} with moderate and large values of ε. This means that the Exponential Parameter can be estimated by $\hat{\theta}_\varepsilon$, with reasonable Efficiency, instead of $\hat{\theta}_{COM}$, if one wants to save time and cost.

5. Conclusion

In this paper, a procedure for quantifying the Efficiency of type I Censored sample from Exponential Distribution has been presented. Criteria for estimating the termination time and the number of failures corresponding to a given Efficiency have been proposed. Moreover, estjmations under certain Efficiency have also been introduced.

The work presented here has proved that Sup-Entropy for Censored sample (the amount of information in Censored data) is less than Sup-Entropy for complete sample (the amount of information in complete data), accordingly, Sup-Entropy is a suitable and convenient tool for measuring the Efficiency of type I Censored sample from Exponential Distribution.

Table 1: Expected values of $\hat{\theta}_\varepsilon$, $\hat{t}_\varepsilon$, r_ε , $\hat{\theta}_{COM}$ for various values of 'n' and θ with ε =10%

n	θ	$\hat{\theta}_\varepsilon$	$\hat{t}_s$	r_s	$\hat{\theta}_{COM}$
10	0.5	0.661139	1.06362	4(40%)	0.549583
	1	1.41033	0.531812	4	1.14126
	10	12.9098	0.0531812	4	11.113
30	0.5	0.540222	1.06362	12(40%)	0.511134
	1	1.11408	0.531812	12	1.04358
	10	10.9496	0.0531812	12	10.3759
50	0.5	0.523125	1.06362	20(40%)	0.507946
	1	1.0483	0.531812	20	1.01725
	10	10.5416	0.0531812	20	10.1956

Table 2: Expected values of $\hat{\theta}_\varepsilon$, $\hat{t}_\varepsilon$, r_ε , $\hat{\theta}_{COM}$ for various values of 'n' and θ with ε =50%

n	θ	$\hat{\theta}_\varepsilon$	$\hat{t}_s$	r_s	$\hat{\theta}_{COM}$
10	0.5	0.573475	3.35669	8(80%)	0.559833
	1	1.16058	1.67835	8	1.11479
	10	11.457	0.167835	8	11.2766
30	0.5	0.51882	3.35669	24(80%)	0.514965
	1	1.04977	1.67835	24	1.03006
	10	10.542	0.167835	24	10.4255
50	0.5	0.518479	3.35669	40(80%)	0.514062
	1	1.02486	1.67835	40	1.02222
	10	10.3394	0.167835	40	10.2366

Table 3: Expected values of $\hat{\theta}_\varepsilon$, $\hat{t}_\varepsilon$, r_ε , $\hat{\theta}_{COM}$ for various values of 'n' and θ with ε =95%

n	θ	$\hat{\theta}_\varepsilon$	$\hat{t}_\varepsilon$	r_ε (r_ε/n)	$\hat{\theta}_{COM}$
10	0.5	0.579318	9.48773	9(90%)	0.569396
	1	1.11625	4.74386	9	1.10103
	10	11.5283	0.474386	9	11.4003
30	0.5	0.513505	9.48773	29(96%)	0.51309
	1	1.02955	4.74386	29	1.02995
	10	10.4495	0.474386	29	10.4452
50	0.5	0.507761	9.48773	49(98%)	0.508264
	1	1.02376	4.74386	49	1.02328
	10	10.2526	0.474386	49	10.2651

References

1. Abo-Eleneen, Z. A. (2008). Fisher information in type II progressive censored samples. *Communications in Statistics - Theory Methods*, **37**, 682-691.
2. Awad, A. M. (1987). A statistical information measure. *Dirasat (Science) XIV*, **12**, 7-20.
3. Awad, A. M. and Alawneh, A. (1987). Application of entropy of a life time model. *IMA Journal of Mathematical Control and Information*, **4**, 143-147.
4. Bain, J. and Engelhardt, M. (1992). *Introduction to Probability and Mathematical Statistics*. California: Duxbury Press.
5. Balakrishnan, N., Burkschat, M., Cramer, E. and Hofman, G. (2008). Fisher information based progressive censoring planes. COMSTA 4123, doi: 10.1016/j.csda.07.038.
6. Ebrahimi, N. and Soofi, E. (1990). Relative information loss under type-II censored exponential data. *Biometrica,* **77(2)**, 429-435.
7. Haj Ahmed, H. A. and Awad, A. M. (2010). Optimality criterion for progressive type-II right censoring based on Awad sup-entropy measures. *Journal of Statistics,* **16**, 12-27.
8. Hollander, M., Proshcan, F. and Sconing, J. (1987). Measuring information in right-censored models. *Naval Rcs. Logist*, **34**, 669-681.

9. Johnson, N. L. and Kotz, S. (1970). *Continuous Univariate Distributions*, Vol. I. Boston: Houghton Mifflin.
10. Ng, H.K.T., Chan, P. S. and Balakrishnan, N. (2004). Optimal progressive censoring plans for the Weibull distribution. *Technometrics*, **46**, 470-481.
11. Shannon, C. E. (1948). A mathematical theory of communication. *Bell System Technical Journal*, **27**, 379-423.
12. Zeng, G. and Park, S. (2004). On the Fisher information in multiply censored and progressively censored data. *Communication in Statistics - Theory Methods*, **33(8)**, 1821-1835.

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Determinants of Higher Order Birth Intervals in Pakistan

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Abstract

Birth interval pattern can be used to draw attention on the significant characteristics of reproduction and dynamics of fertility transition. The focus of current paper is to study the effect of socioeconomic, demographic and proximate determinants on the length of birth intervals for Pakistani women. Cox Regression Model is used for modeling the birth intervals. It is evident from higher order birth interval models that age of women, preceding birth interval, education of women and survival status of preceding child are major determinants of all birth intervals. In the Proximate Determinant Models, period of breastfeeding and age of women have played significant role in the determination of all birth intervals. Enhancement in women's education, discouraging gender biasness, improvement in health facilities and promotion of long breastfeeding period can be helpful in expanding the birth spacing.

Keywords

Abstinence, Amenorrhea, Breastfeeding, Cox regression model, Parity, Proximate determinants of fertility

1. Introduction

Fertility analysis is very important for policy makers to get guidance for population control and also for the evaluation of family planning programs. Knodel (1987) had presented the idea of three fertility inhibiting behaviors during early transitional period of fertility. These are starting, spacing and stopping behavior of fertility. Intentional long birth spacing limits child bearing and is known as 'spacing behavior' of fertility.

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Birth interval analysis is more susceptible technique for measuring fertility than other conservative methods of measuring fertility (Rodriguez and Hobcraft as cited in Nath et al., 2000). Pattern of birth intervals not only provides pace of child bearing but also chances of transition to higher parity. In developing countries if urgent results are required for fertility consideration then birth interval analysis is preferred over total children ever born to women (completed parity).

Singh et al. (2011) had found that infant mortality, period of breastfeeding, use of contraceptives, women's age at marriage, birth order and gender of preceding child were major birth interval dynamics in Manipur, India. Eini-Zinab and Agha (2005) explored that current age of women, education of women, survival status of preceding child and maternal age at the time of delivery were responsible for the postponement of second child in Iran. Ramesh (2006) had used both open and closed birth intervals for understanding the dynamics of fertility in Orissa, India and concluded that effect of various factors on the determination of birth interval length varies with parity. Kiani and Nazli (1988) had concluded that spacing behavior of fertility had not shown any change in the marital fertility of Pakistani women.

Pakistan is confronted with the problem of rapid population growth which is a great hindrance to the economic growth. Increase in adolescent population and reduction in dependency ratio exhibit that phase of population transition has got started. Pakistan has entered in the early stage of fertility transition from the past two decades (Ali and Buriro, 2008). The average of more than six children per women has started to turn down in late 1980's (Arnold and Sultan, 1992; Feeney and Alam, 2003). The total fertility rate declined from 6.0 to 5.4 children in 1992-96. In the last decade, this decline became more rapid and reached 4.1 children per women in 2006-07 (Ali and Buriro, 2008). But still it is far away from replacement level of fertility. In Pakistan, approximately 33% of women had birth interval less than two years. In spite of Governmental campaigns on family planning issues for past few years, there is little contraction in the length of closed birth interval. Desire for long birth interval is now increasing in Pakistan (Catalyst Consortium, 2003). In the past, more attention was paid to study the stopping behavior of fertility while spacing behavior of Pakistani women was ignored. The decline in fertility could very well be due to spacing behavior rather than only due to stopping behavior of fertility. There is need to study the factors which affect birth intervals. It will help in understanding the spacing behavior of fertility of Pakistani women and identifying the factors which are useful in declining the fertility through long birth intervals. It will also identify the factors which are creating hindrance in long birth intervals. Potential factors and covariates which can affect birth interval length are illustrated under the two broad classes of socio-economic/ demographical factors and biological factors (Proximate Determinants). Both of these are elaborated under separate headings as follows.

1.2 Socio-economic and Demographic Determinants of Birth Interval: Women's age, education and length of previous interval had great effect on the subsequent birth spacing (Rodriguez et al., 1983). Age at first birth, urban residence and sex of previous child were taken as predictor for birth interval by Rindfuss et al, (1983). Consequences of marital age on fertility are influenced by biological factors and maturity of couple's behavior towards reproductive decisions (Kallan and Udry, 1986).

Urban and rural attitudes about reproductive decisions may also differ. If social and cultural norms are one reason, the others may be awareness and access to health facilities. Certain trends are expected, for instance, long exclusive breastfeeding in rural area so it widens the interval. Characteristics of women which are potential candidates for modeling the birth intervals are maternal age at first birth, parity, education of women, work status of women and place of residence (Birth Spacing three to five saves lives, 2002).

Education is linked with awareness of an individual regarding health and reproduction. Effect of both education and age at marriage was found significant on birth spacing (Hirschman and Rindfuss, 1980; Rindfuss et al., 1983). Educated women may have long birth interval than uneducated women due to delay in marriage, employment status, use of contraceptives and awareness about reproductive health. But Bumpass et al. (1986) had reported short second birth interval for women with higher education. Ramarao et al. (2006) had named the reason of short interval for highly educated women as 'compressing the child bearing'. Education of husband is important factor particularly in those societies where woman takes her reproductive decision with the consent of her husband. Gender composition also influences birth interval (Maitra and Pal, 2004).

Birth spacing and child survival are correlated to each other. Death of previous child shortens birth interval. Maitra and Pal (2004) named it the phenomenon of 'the child replacement effect'. There is also a biological reason of short interval. Death of preceding child disrupts breastfeeding. Duration of amenorrhea is also reduced in this case. Both of these can result in short interval (Santow, 1987).

Work status of women had shown short interval in some of the countries. On the other side, Mturi (1997) and Setty-Venugopal and Upadhyay (2002) had reported long interval for employed women. According to the theory of opportunity cost it would be long. But if child rearing is not conflicting with work then it would not be long.

Quantity/ Quality theory of fertility may also affect spacing behavior similarly as it affects stopping behavior. Usually birth intervals are expected to be short for lower income group than higher income group. Relationship between birth spacing and women's occupation is not clear and quite uncertain (Bavel and Kok, 2004).

Inclusion of preceding previous interval in birth interval model is criticized because if its impact is meaningful then what about marriage to first birth interval. Purpose of its inclusion is, in fact, to measure the indirect effect of breastfeeding and contraception on birth spacing. Trussell et al. (1985) had also included this variable even in the presence of some Proximate Determinants i.e. breastfeeding and contraceptive use. Demographers did not think the necessity of previous birth interval in the model if data on both contraceptive usage and breastfeeding are available in surveys (Richards, 1982 as cited in Trussell et al., 1985). But information on all Proximate Determinants is not available so their effect can be captured through previous birth interval.

1.2 Proximate Determinants (Biological Factors) of Fertility for Birth Interval: Biological factors which contribute towards fertility are breastfeeding practice, deliberate fertility control through contraception, coital frequency, abortions and reproductiveness (Baschieri and Hinde, 2007). Davis and Blake (1956) have defined some biological and behavioral factors through which social, economic and cultural factors influence fertility. Davis and Blake (1956) had defined relationship between socio-economic and biological or intermediate fertility variables. This relationship is shown in the diagram (Figure 1).

Breastfeeding is widespread in developing countries. Diversification is found in its period and regularity pattern due to cultural norms. Ramarao et al. (2006) had enlisted many studies which had shown wider birth interval due to breastfeeding. Postpartum amenorrhea (PPA) is also a biological factor which affects birth spacing. Its minimum period may be of month and maximum may exceed over a year (Singh et al., 2007).

Postpartum abstinence is frequently exercised in many societies. People also believe that sexual relationship during breastfeeding period pollutes the milk and is harmful for child health (Regassa, 2007). Setty-Venugopal and Upadhyay (2002) reported that postpartum amenorrhea and abstinence both lead to birth spacing of up to two years.

Objective:

- The focus of current paper is to study the effect of socioeconomic and demographic factors on the length of higher order birth intervals for Pakistani women. The analyses provide insight into spacing behavior of fertility through parity specific birth interval analyses.
- Effect of biological factors is investigated by fitting separate model on these factors. It will helpful in understanding whether the effect of these factors varies across parities.

2. Data and Methodology

2.1 Data: Source of data is Pakistan Demographic and Health Survey which was conducted in 2005-06. Two Stage Stratified Sample Design was used for selection of sample. One thousand sample points were chosen using Probability Proportional to Size Sampling from rural and urban stratum. Distribution of 1000 sample points was given as: Punjab (440), Sindh (260), NWFP (180), Baluchistan (100), and Federally Administered Tribal Areas (20). From these 1000 sample points, 105 households were chosen with the help of Systematic Random Sampling from each sample point. Ten households were chosen from these 105 households using Systematic Random Sampling for women reproductive history related questionnaire. Information about reproductive health, including birth histories data was collected from 10023 women of age 15-49. Birth history data include twenty entries, one for each birth. Preceding birth interval is calculated as the difference in months between the current birth and the previous birth.

Birth interval for higher order births is defined as time since previous birth (Woldemicael, 2008). All birth intervals are closed. Birth interval models are fitted parity-wise i.e. models for second, third, fourth and fifth births. It is assumed that pattern of transition above parity five is same. For socio-economic model, data is not truncated. Factors and covariates used for analysis of socioeconomic model are current age of women, age of women at marriage, region (Punjab, Sindh, NWFP and Baluchistan), education of both spouses, wealth index, occupation of both spouses, preceding birth interval, gender of preceding child, and survival status of preceding child. Proximate Determinant Model is fitted on age of women, age of women at marriage, preceding interval, period of breastfeeding for preceding child, period of amenorrhea for preceding child and period of abstinence for preceding child.

It is preferred in this paper to report the results of only higher order birth intervals (2^{nd}, 3^{rd}, 4^{th} and 5^{th}) to understand the spacing behavior of fertility. Because first birth interval is most of the times inconsistent and irregular due to cultural norms and bans of society (Singh et al., 1993 as cited in Nath et al., 2000). In Pakistan there is evidence of not using contraceptives at the start of marriage (PSLM, 2005-06). A woman has to prove her fecundability so couples start planning the child soon after marriage irrespective of their education, work status and wealth index status. Birth interval analysis is carried up to 5^{th} birth because it would capture most of the birth transition.

Information about the period of breastfeeding, amenorrhea and abstinence is available only for births from January 2001 till the date of survey. That is why Proximate Determinant Model is fitted only for births after January 2001.

2.2 Methods: Detail about methods is given as follows;

2.2.1 Product Limit Survivorship Function: Chakraborty et al. (1996) had used Product Limit Survivorship Function to study the differential pattern of birth intervals in Bangladesh. The Product Limit Estimates provide better estimates than life table analysis. The Product Limit Estimate of the Survival Function (Kaplan and Meier, 1958) is defined as

$$S(t_i) = \prod_{j=1} (1 - \frac{d_j}{n_j})$$

Chakraborty et al. (1996) had defined the terms in Product Limit Survivorship according to birth interval analysis as follows.

d_j = number of women having births at time t_j.

n_j = number of women just prior to time t_j exposed to the risk of having birth.

t_j = time since the previous birth of a child to that woman.

2.2.2 Log-Rank Test: It is used to compare Survival Distribution of various categories of factors (Nathan, 1966). For the two groups, Hypotheses are given as

$$h(t,X) = h_0(t).e^{b'X} \tag{2.1}$$

where $b = (b_1, b_2, ...b_p)$ are regression coefficients. Simplifying (2.1) generates a Regression Model

$$\log \frac{h(t)}{h_0(t)} = b_1 x_{1i} + b_2 x_{2i} + ... + b_p x_{pi} \tag{2.2}$$

Proportionality assumption is essential to be verified prior to the application of Cox Regression Model. Non-Proportional Hazard Model is recommended in case of violation of assumption. Proportionality assumption is also related to the nature of covariates used in the Hazard Models.

3. Results

In descriptive analysis average length for all birth intervals is computed. Second birth interval is the shortest among all birth intervals (Table 1). Length of second birth interval is approximately 28 months. Marginal difference is observed in the length of third, fourth and fifth birth interval length. The average length of these birth intervals is one month more than average length of second birth interval. It means birth spacing behavior of Pakistani women is almost same for different parities.

3.1 Non-Parametric Analysis (Kaplan-Meier Product Limit Estimate of Survival Time): Kaplan-Meier average survival time is given in Table 2. This will help in understanding the average length of birth intervals among various categories of factors and covariates. Average birth interval increases consistently with the age of women for all parties. The average birth interval length for Bangali women of more than 35 years of age was approximately 18 months for higher order birth intervals (Chakraborty et al., 1996).

Average birth interval is short for the women getting married after the age of twenty five. For second birth interval, there is absolute decline with the increase in age at marriage. Consistent increase in the length of birth interval for age and marital duration was also observed for Jordan (Youssef, 2005). Marginal difference in the length of birth interval is observed for the women belonging to different regions of Pakistan. Women belonging to Punjab and NWFP have short second and third birth intervals than Balochi and Sindhi women. Minimum average birth interval for fourth birth is for the women belonging to NWFP. Up to parity four, Balochi women have longest birth interval as compared to other provinces. For fifth birth, Balochi women have minimum birth interval as

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By comparing difference in average birth interval length, it is found that for Pakistani women, major contribution in the length of birth intervals is due to biological factors and child mortality (death of preceding child).

3.2 Log-Rank Test: There is significant variations in the failure time (occurrence of birth) of different categories for age of woman, education of woman, education of husband, wealth index, preceding birth interval, survival status of preceding child and period of breastfeeding for all parities (Table 3). It means significant variability exists between birth intervals among various categories of these factors. Gender of preceding child and period of amenorrhea have significant variation among their categories up to fourth birth whereas region, age at marriage, occupation of woman, occupation of husband and period of abstinence do not have significant variation among their categories for all parities. Log-Rank Test for second and third birth interval had shown significant difference among various categories of gender of preceding consecutive births and household income (Nath et al., 2000). Khan and Raeside (1998) had concluded that long rank test for higher order births had shown consistently significant difference among various categories of factors in both urban and rural areas like age at first birth, education of both spouses and death of preceding child in Bangladesh. Women's work status, region of residence and sex of preceding child had shown significant variation among different categories in rural areas (Khan and Raeside, 1998).

Direction and magnitude of individual factor on birth interval length is studied through Multivariate analysis. Some factors are observed to behave differently in the Multivariate analysis. In Multivariate analyses, when the remaining factors are controlled some factors which were significant in Univariate analyses, became insignificant.

fourteen months was observed in average birth interval due to death of preceding child in Bangladesh (Chakraborty et al., 1996). In Jordan, decline of five months was found (Youssef, 2005). Increasing trend is observed in birth interval length with an increase in the preceding birth interval, duration of breastfeeding, amenorrhea or abstinence. The maximum difference in average birth interval length is observed for the breastfeeding factor. Women who breastfeed their child have at least five months longer next birth interval than those who do not. In Jordan those who breastfeed the child had four months longer birth interval than those who did not (Youssef, 2005).

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Direction and magnitude of individual factor on birth interval length is studied through Multivariate analysis. Some factors are observed to behave differently in the Multivariate analysis. In Multivariate analyses, when the remaining factors are controlled some factors which were significant in Univariate analyses, became insignificant.

Region has significant effect in the determination of second birth interval. Punjab and NWFP have shorter birth interval length while Balochistan has long birth interval up to parity four. Shortest second birth interval is for women who belong to NWFP. Third birth interval is significantly short for Punjabi women as compared to Balochi women. Women belonging to NWFP have significantly long fifth birth interval as compared to Balochi women. The difference in birth interval might be due to difference in breastfeeding period. The pattern of birth interval among provinces is found similar as reported by median duration of breastfeeding among children born in the past three years. Ali and Sultan (2008) have found that Punjabi women have shorter median breastfeeding period (17 months) as compared to Balochi women (20.7 months).

Urban women have shown significantly shorter second interval as compared to rural women. Only fifth birth interval is insignificantly longer for urban women than for the rural women. In univariate analyses (Kaplan-Meier) for parity above two, urban women have shown longer interval than rural women. This difference is marginal. But nature of relationship is changed when rest of variables are controlled in the multivariate analyses i.e. short birth interval for urban women. The reason of long interval for rural women is actually not related to rural urban differentials (Trussell et al., 1985 and Woldemicael, 2008). The reason given in support of result is prevalence of romantic/love marriages in urban areas (Suwal, 2001). Sleeping arrangement in rural areas also causes delays in birth interval (Suwal, 2001). Moreover, in current analyses birth interval of only live births is included. Unavailability of health facilities may cause miscarriages in rural areas which results in long birth interval for live birth. Ali and Sultan (2008) have reported that median breastfeeding duration is found to be shorter in urban (18 months) areas than rural (19.4 months).

Education of women has shown positive association with birth spacing. The effect of education is significant in all models except for fourth birth interval. The impact of education of husband, like education of woman, has positive but insignificant effect on birth spacing for second and fourth birth intervals. For third and fifth birth intervals effect of husband's education is reversed. Gangadharan and Maitra (2001) argued for this negative effect of husband's education that a man with high education usually marries with highly educated woman. Educated women generally have delayed marriages and older women have short birth interval. So age of woman at marriage may be the reason behind positive effect of husband's education. Suwal (2001) had found short second birth interval in Nepal for the woman whose husband was secondary or post-secondary educated. Khan

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Covariate survival curve for second, third, fourth and fifth birth interval (Figures 2, 3, 4, 5) shows gradual decline which means decline in survival rate (short survival time). The chance of not having a subsequent birth of child decreases with increase in birth interval. The decline is sharper after three years. It means majority of married women have next birth within three-year interval. After eight years it approaches the bottom line and remains constant at zero for all further durations.

3.4 Cox Regression Model for Proximate Determinants (Biological Factors) of Birth Intervals: Analysis of effect of biological factors on birth interval is no doubt very important for understanding the reproductive behavior. Period of breastfeeding, amenorrhea and abstinence are available only for births from January 2001 till the date of survey, for biological model only those cases are selected who had given births after 2001.

Proximate Determinants which play major role in transition to next births are age of women, age at marriage and period of breastfeeding. This is true for all parities. There is increase in birth interval length with an increase in age of women and period of breastfeeding. Increase in the preceding birth interval and period of abstinence contract the birth intervals. Strength and nature of relationship of preceding birth interval is changed in the biological model. It can be concluded from this change that in socioeconomic model this factor has very well captured the influence of biological factors. But when model for biological factors is fitted its effect is diminished due to inclusion of very important biological factor i.e. period of breastfeeding. Apart from parity two, significant negative relationship is found between period of abstinence and birth interval. It means with extended period of abstinence, the possibility of conception for next birth increases. Period of amenorrhea has shown inconsistent influence on birth intervals. If duration of amenorrhea is less than abstinence it may increase birth space (Setty-Venugopal and Upadhyay, 2002). But in the current analysis median duration of amenorrhea is equal to median duration of abstinence only for second birth interval.

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mother regarding the child brought up and also decreases probability of child survival.

Birth interval length is found to be significantly longer in case of increase in preceding breastfeeding period. Government should educate women for exclusive breastfeeding period of six month and at least two years with weaning (WHO, The World Health Report, 2006). It widens not only birth interval but also breast milk increases the chance of child survival by increasing the immunity of child. And in case of child survival birth interval is also expanded (Maitra and Pal, 2004).

5. Recommendations for Future Research and Policy Implication

All these recommendation are given in the context of Pakistan.

- The effect of frequency and use of contraceptives can be studied on birth intervals.
- Government should motivate couples to increase the birth interval length in case of death of preceding child and also strengthen health programs. It is necessary for the maternal and child health. If long birth interval is promoted in case of death of preceding child, it will cause decline in fertility.
- Family planning programs should be made more effective to get the favorable results (longer birth intervals) for age of women at marriage, urbanization and modernization factors.
- Birth interval length is shorter than three years for all higher order births. There is need of effective policy for promotion of long birth space (at least 4 to 5 years) between two consecutive children. Lady health visitors can be used for this purpose.

6. Limitations

Limitations are mentioned as follows;

- Effect of contraceptives is not studied due to unavailability of data within birth interval.
- Proximate Determinants like period of breastfeeding, amenorrhea and abstinence are only for those births which occurred after January 2001.

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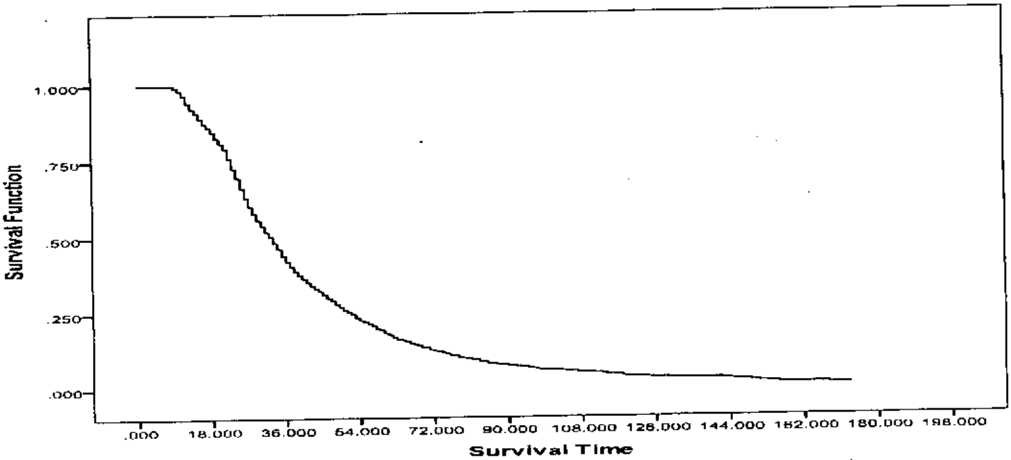


Figure 3: Survival curve for third birth interval

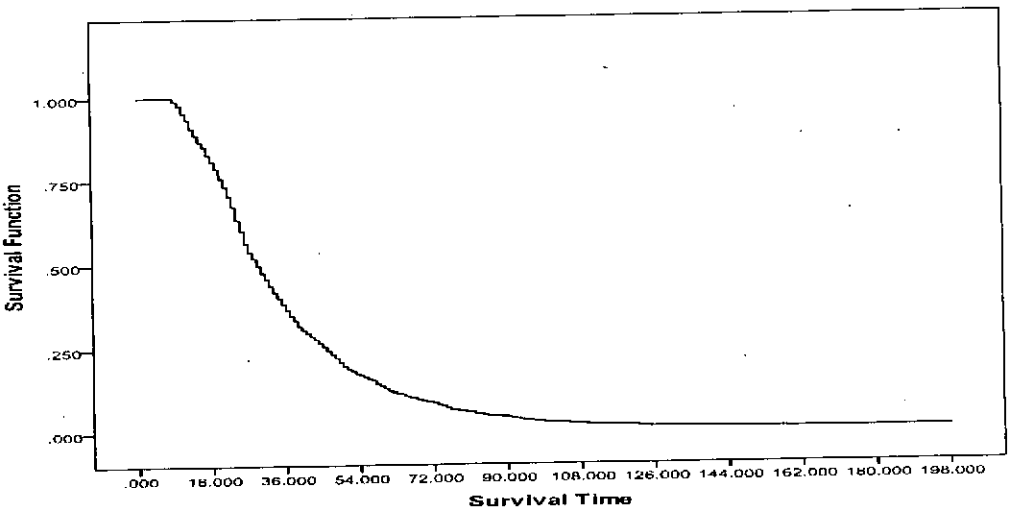
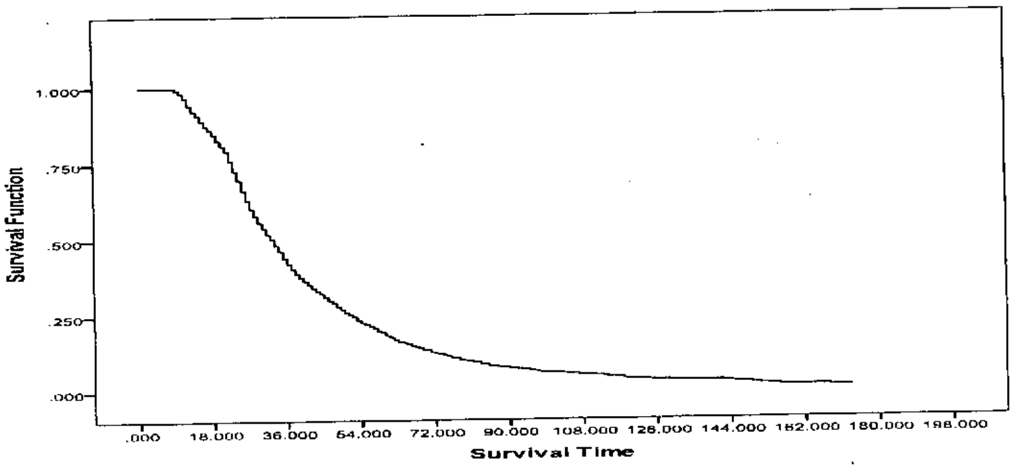


Figure 4: Survival curve for fourth birth interval



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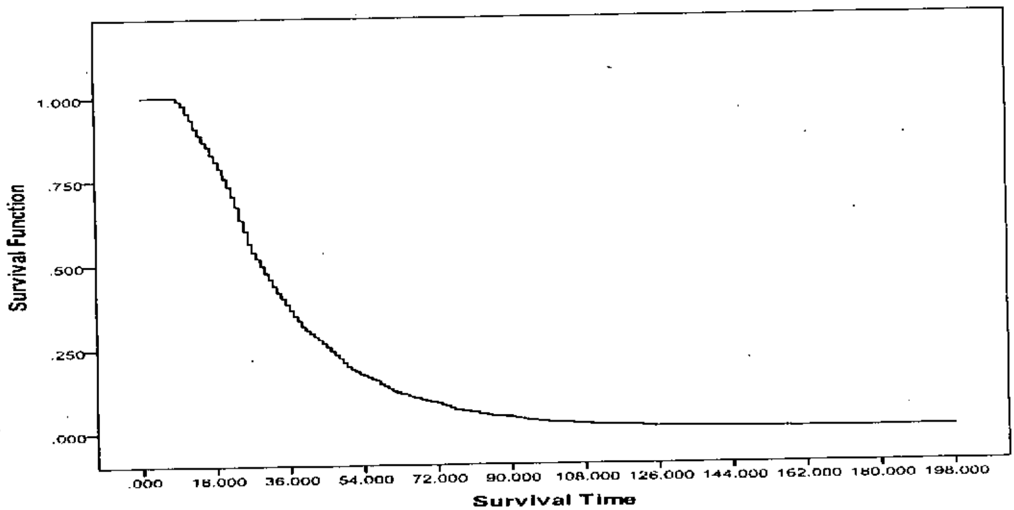

Figure 4: Survival curve for fourth birth interval

Table 2: Kaplan-Meier Estimates of mean survival time by socio-economic/ demographic and Proximate Determinants for higher order birth intervals

Factors/ Covariates	Levels	Second Interval		Third Interval		Fourth Interval		Fifth Interval	
		Estimate	S.E.	Estimate	S.E.	Estimate	S.E.	Estimate	S.E.
age of woman	? 24 months	24.755	.400	25.443	.637	23.750	.950	25.219	1.841
	25-34	27.361	.276	28.420	.314	28.637	.373	28.069	.473
	35+	29.131	.324	30.003	.319	29.435	.317	30.273	.360
Overall		27.998	.201	29.192	.221	29.052	.241	29.638	.290
Age of women at marriage	? 18 months	28.818	.268	29.243	.282	29.001	.299	29.569	.344
	19-21	26.702	.388	29.275	.466	29.210	.505	30.289	.704
	22-24	26.565	.591	29.319	.718	29.383	.868	29.637	1.091
	25+	26.277	.695	27.929	.805	28.488	1.06	27.489	1.138
Overall		27.998	.201	29.192	.221	29.052	.241	29.638	.290
Region	Punjab	27.828	.310	28.589	.322	29.323	.378	29.794	.448
	Sindh	28.196	.405	29.819	.473	28.894	.480	29.833	.634
	NWFP	27.278	.413	29.217	.465	28.432	.522	29.801	.582
	Baloch.	29.363	.615	29.960	.686	29.465	.658	28.385	.755
Overall		27.998	.201	29.192	.221	29.052	.241	29.638	.290
Residence	Urban	27.000	.304	29.667	.377	29.650	.418	30.893	.538
	Rural	28.635	.267	28.895	.272	28.700	.293	28.959	.339
Overall		27.998	.201	29.192	.221	29.052	.241	29.638	.290
Education of woman	No	28.343	.253	28.838	.257	28.365	.268	29.099	.314
	Primary	27.240	.526	27.967	.583	29.957	.744	30.796	.925
	Secon.	26.582	.475	30.475	.676	31.872	.850	32.518	1.232
	Higher	28.478	.783	35.172	1.318	34.540	1.71	35.237	2.869
Overall		27.998	.201	29.192	.221	29.052	.241	29.638	.290
Education of husband	No	28.749	.358	29.012	.358	28.611	.369	29.015	.411
	Primary	27.515	.486	28.175	.501	27.281	.507	28.987	.642
	Secon.	27.331	.342	29.022	.399	29.773	.475	30.737	.576
	Higher	27.866	.463	31.072	.627	31.334	.737	30.824	1.104
Overall		27.987	.202	29.179	.222	29.051	.242	29.653	.291
wealth index	Poorest	29.366	.475	28.515	.476	27.796	.478	28.947	.631
	Poorer	28.374	.458	28.796	.465	28.348	.501	28.500	.548
	Middle	27.431	.463	28.574	.485	29.005	.530	28.656	.563

Table 2: Kaplan-Meier Estimates of mean survival time by socio-economic/ demographic and Proximate Determinants for higher order birth intervals

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Factors/ Covariates	Levels	Second Interval		Third Interval		Forth Interval		Fifth Interval	
		Estimate	S.E.	Estimate	S.E.	Estimate	S.E.	Estimate	S.E.
Period of Breastfeeding for preceding child	0 month	18.652	1.65	21.063	2.883	20.053	2.018	17.917	1.932
	1-6	19.602	.801	20.524	1.039	20.288	1.06	20.816	1.339
	7-12	20.438	.426	23.306	.673	24.140	.811	23.622	.848
	13-18	26.085	.585	25.404	.714	26.254	.933	25.671	.940
	19-24	30.509	.748	30.406	.759	30.772	.831	30.745	.956
	>24	26.750	1.833	26.879	2.164	30.167	2.790	31.696	2.806
Overall		23.939	.338	25.347	.408	25.855	.468	25.979	.519
Period of Amenorrhea for preceding child	0-1 months	22.093	.486	23.143	.868	23.544	.847	24.050	.974
	2-3	22.982	.641	24.433	1.105	24.490	.943	25.143	1.090
	>4	26.792	.544	27.991	.639	28.071	.630	27.994	.713
Overall		23.887	.323	25.851	.479	25.716	.461	26.044	.521
Period of Abstinence for preceding child	0-1 months	23.015	.457	24.516	.556	24.618	.590	24.733	.673
	2-3	24.082	.516	25.868	.654	26.910	.835	26.891	.914
	>4	25.081	1.024	25.760	1.406	28.435	1.449	26.628	1.759
Overall		23.678	.326	25.138	.406	25.785	.464	25.675	.522

Table 3: Comparison of Survival Distribution using Log-Rank Test for higher birth intervals

Factors/Covariates	Second Interval Chi-square	Third Interval Chi-square	Fourth Interval Chi-square	Fifth Interval Chi-square
Age of woman	46.939**	28.980**	18.269**	14.000**
Age at marriage	30.703**	1.465	.485	2.889
Region	8.559**	7.307	2.160	2.680
Residence	15.668**	3.436	3.936*	9.632**
Education of woman	10.671*	34.702**	30.540**	16.339**

Factors/ Covariates	Levels	Second Interval Estimate	Second Interval S.E.	Third Interval Estimate	Third Interval S.E.	Forth Interval Estimate	Forth Interval S.E.	Fifth Interval Estimate	Fifth Interval S.E.
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Table 4: Cox Hazard Model for higher order birth intervals, Pakistan 2006-07 (Odd Ratios)

Factors/ Covariates	Levels	Second Birth Interval	Third Birth Interval	Fourth Birth Interval	Fifth Birth Interval
Age of woman	None	.987**	.989**	.994*	.993*
Age at marriage	None	1.026**	1.015**	1.007	1.006
Preceding birth interval	None	1.000	.993**	.993**	.993**
Region	Punjab	1.117**	1.120*	1.085	.924
	Sindh	1.110*	1.002	1.062	.929
	NWFP	1.140**	1.043	1.103	.888+
	Baloch.	1.000	1.000	1.000	1.000
Residence	Urban	1.107*	1.041	1.076	.979
	Rural	1.000	1.000	1.000	1.000
Education of woman	No	1.257**	1.529**	1.265+	1.644**
	Primary	1.189*	1.536**	1.180	1.529*
	Secondary	1.211**	1.343**	1.103	1.466+
	Higher	1.000	1.000	1.000	1.000
Education of husband	No	1.021	.913	1.006	.960
	Primary	1.043	.942	1.132+	.979
	Secondary	1.020	.966	1.005	.938
	Higher	1.000	1.000	1.000	1.000
Wealth index	Poorest	.932	1.139+	1.089	1.002
	Poorer	.966	1.075	1.029	1.082
	Middle	1.026	1.121*	1.038	1.066
	Richer	.993	1.046	1.091	1.015
	Richest	1.000	1.000	1.000	1.000
Occupation of woman	no work	1.067	1.036	.994	1.028
	profess	1.158	1.128	.948	1.042
	agri	1.132	1.051	1.043	.896
	manual	1.000	1.000	1.000	1.000
Occupation of husband	no work	1.032	1.123	1.205*	1.063
	profess	1.018	1.016	.988	.938

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Residence	Urban	1.107*	1.041	1.076	.979
	Rural	1.000	1.000	1.000	1.000
Education of woman	No	1.257**	1.529**	1.265+	1.644**
	Primary	1.189*	1.536**	1.180	1.529*
	Secondary	1.211**	1.343**	1.103	1.466+
	Higher	1.000	1.000	1.000	1.000
Education of husband	No	1.021	.913	1.006	.960
	Primary	1.043	.942	1.132+	.979
	Secondary	1.020	.966	1.005	.938
	Higher	1.000	1.000	1.000	1.000
Wealth index	Poorest	.932	1.139+	1.089	1.002
	Poorer	.966	1.075	1.029	1.082
	Middle	1.026	1.121*	1.038	1.066
	Richer	.993	1.046	1.091	1.015
	Richest	1.000	1.000	1.000	1.000
Occupation of woman	no work	1.067	1.036	.994	1.028
	profess	1.158	1.128	.948	1.042
	agri	1.132	1.051	1.043	.896
	manual	1.000	1.000	1.000	1.000
Occupation of husband	no work	1.032	1.123	1.205*	1.063
	profess	1.018	1.016	.988	.938

References

1. Ali, M. S. and Buriro, A. A. (2008). Fertility, Pakistan demographic and health survey 2006-07 (41-47). Islamabad, Pakistan: National Institute of Population Studies Islamabad, Pakistan and Macro International Inc. Calverton, Maryland, USA.
2. Ali, M. S. and Sultan, M. (2008). Nutrition, Pakistan demographic and health survey 2006-07(139-144). Islamabad, Pakistan: National Institute of Population Studies Islamabad, Pakistan and Macro International Inc. Calverton, Maryland, USA.
3. Arnold, F. and Sultan, M. (1992). Fertility, in Pakistan demographic and health survey 1990-91 (35-52). Islamabad, Pakistan: National Institute of Population Studies Islamabad, Pakistan and Macro International Inc. Columbia, Maryland, USA.
4. Baschieri, A. and Hinde, A. (2007). The proximate determinants of fertility and birth intervals in Egypt: An application of calendar data. *Demographic Research*, **16**, 59-96.
5. Bavel, J. V. and Kok, J. (2004). Birth spacing in the Netherlands: The effects of family composition, occupation and religion on birth intervals 1820-1885. *European Journal of Population*, **2(2)**, 119-140.
6. Birth Spacing three to five saves lives, (2002). Population reports. The Johns Hopkins University Bloomberg School of Public Health, Serial L (3), **30(3)**, Issues in World Health. Retrieved from info.k4health.org/pr/l13/l13bib.shtml
7. Bumpass, L. L., Rindfuss, R. R. and James, P. (1986). Determinants of Korean birth intervals: The confrontation of theory and data. *Population Studies*, **40**, 403-423.
8. Catalyst Consortium, (2003). Focus group studies: Bolivia, India, Pakistan, and Peru. Washington, D. C.: USAID. Retrieved from http://www.coregroup.org/storage/documents/Workingpapers/smrh_OBSI_Overview.pdf
9. Chakraborty, N., Sharmin, S. and Islam M. A. (1996). Differential pattern of birth interval in Bangladesh. *Asia Pacific Population Journal*, **11(4)**, 73-86.
10. Cox, D. R. (1972). Regression models and life-tables. *Journal of the Royal Statistical Society. Series B (Methodological)*, **34(2)**, 187–220.
11. Davis, K. and Blake, J. (1956). Social structure and fertility: An analytic framework. *Economic Development and Cultural Change*, **4(3)**, 211-235.

References

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22. Knodel, J. (1987). Starting, stopping, and spacing during the early stages of fertility transition: The Experience of German Village Populations in the 18th and 19th Centuries. *Demography*, **24 (2)**, 143-162.
23. Maitra, P. and Pal, S. (2004). Birth spacing and child survival: Comparative evidence from India and Pakistan. *Labor and Demography*, Retrieved from http://129.3.20.41/eps/lab/papers/0403/0403023.pdf
24. Mturi, A. J. (1997). The determinants of birth intervals among non-contracepting Tanzanian women. *African Population Studies*, **12 (2)**. Retrieved from http://www.bioline.org.br/request?ep97011
25. Nath, D. C., Leonetti, D. L. and Steele, M. (2000). Analysis of birth intervals in a non-contracepting Indian population: An evolutionary ecological approach. *Journal of biosocial Science*, **32**, 343-354.
26. Nathan, M. (1966). Evaluation of survival data and two new rank order statistics arising in its consideration. *Cancer Chemotherapy Reports*, **50 (3)**, 163–170.
27. Pakistan Social and Living Standards Measurement Survey (PSLM), 2005-06. *Population Welfare*. Retrieved from http://www.statpak.gov.pk/depts/fbs/statistics/pslm2005_06/population_welfare.pdf
28. Ramarao, S., Townsend, J. and Askew, I. (2006). Correlates of inter-birth intervals: Implications of optimal birth spacing strategies in Mozambique. *Population Council, USAID*, 2-3.
29. Ramesh, P. (2006). Determinants of birth interval dynamics in Orissa, India. Draft of the paper prepared for presentation at the European Population Conference 2006 (EPC–2006) on Population Challenges in Ageing Societies held during 21-24 June 2006, at Liverpool, UK.
30. Regassa, N. (2007). Socio-economic correlates of high fertility among low contraceptive communities of Southern Ethiopia. *The Journal of Human Ecology*, **2(3)**, 203-213.
31. Rindfuss, R., Parnell, A. and Hirschman, C. (1983). The timing of entry into motherhood in Asia: A comparative perspective. *Population Studies*, **37**, 253-272.
32. Rodriguez, G., Hobcraft, J., Mcdonald, J., Menken, J. and Trussell, J. (1983). A comparative analysis of the determinants of birth analysis (Report No. 30). WFS Comparative Studies. Vooburg Netherlands: International Statistical Institute.
33. Santow, G. (1987). Reassessing the contraceptive effect of breastfeeding. *Population Studies*, **41(1)**, 147-160.

22. Knodel, J. (1987). Starting, stopping, and spacing during the early stages of fertility transition: The Experience of German Village Populations in the 18th and 19th Centuries. *Demography*, **24 (2)**, 143-162.
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24. Mturi, A. J. (1997). The determinants of birth intervals among non-contracepting Tanzanian women. *African Population Studies*, **12 (2)**. Retrieved from http://www.bioline.org.br/request?ep97011
25. Nath, D. C., Leonetti, D. L. and Steele, M. (2000). Analysis of birth intervals in a non-contracepting Indian population: An evolutionary ecological approach. *Journal of biosocial Science*, **32**, 343-354.
26. Nathan, M. (1966). Evaluation of survival data and two new rank order statistics arising in its consideration. *Cancer Chemotherapy Reports*, **50 (3)**, 163–170.
27. Pakistan Social and Living Standards Measurement Survey (PSLM), 2005-06. *Population Welfare*. Retrieved from http://www.statpak.gov.pk/depts/fbs/statistics/pslm2005_06/population_welfare.pdf
28. Ramarao, S., Townsend, J. and Askew, I. (2006). Correlates of inter-birth intervals: Implications of optimal birth spacing strategies in Mozambique. *Population Council, USAID*, 2-3.
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Nasir Abbas[1], Syed Mohsin Ali Kazmi[2] and Muhammad Aslam[3]

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Bayesian Inference is a technique of statistical inference that, in addition to using the sample information, utilizes prior information about Parameter(s) to draw results about the Parameters. But the beauty is subdued by huge and cumbersome algebraic calculations necessary to find Posterior estimates. This article suggests numerical methods to derive Posterior distributions under all types of priors – uninformative and informative – and to find Bayes Estimates. We use both numerical differentiation and numerical integration to serve the purpose. The entire estimation procedure is illustrated using real as well as simulated datasets.

Keywords

Numerical differentiation, Numerical integration, Fisher's information, Jeffreys prior, Squared error loss function (SELF), Bayes estimator, Exponential distribution, Normal distribution

1. Introduction

The main difference between the Bayesian and the Frequentistic schools of thoughts is that the former associate randomness with population Parameters and formally incorporate in their analysis any prior information pertaining to Parameters. Prior information about Parameters is updated with current information (data) to yield Posterior Distribution, which is a work-bench for the Bayesians. Adams (2005) throws light on the advantages of using Bayesian approach. But the major problem, which Bayesians face, is the calculation of Posterior Estimates via the Posterior Distribution.

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If the nature of expressions involved in the determinant is complicated, we may use the numerical methods for finding the second partial derivatives to calculate the Jeffreys Prior using the relations

$$f''_{uu} = \frac{\partial^2 f(u,v)}{\partial u^2}\bigg|_{(u_0,v_0)} = \frac{f(u_0-h,v_0)-2f(u_0,v_0)+f(u_0+h,v_0)}{h^2}, \tag{2.2}$$

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where $f(u,v)$ is a bi-variate function and f''_{uu} denotes the partial double derivative with regard to the random variable u. For more details, one can see Bernardo (1979), Berger and Bernardo (1989, 1992a, 1992b), Datta and Ghosh (1995), Jeffreys (1961).

3. The Quadrature Method

We usually need to evaluate multiple integrals to find Bayes Estimates, for example, Posterior means, predictive probabilities, Posterior probabilities for Hypotheses testing etc. based on complicated nature of the Posterior Distribution, particularly when there is a Vector of Parameters and the expressions involve complicated algebraic functions. Considering a one-dimensional case, the Quadrature refers to any method for numerically approximating the value of the definite integral $\int_a^b p(\theta)d\theta$, where $p(\theta)$ may be any proper density. The procedure is to calculate it at a number of points in the range 'a' through 'b' and find the result as a weighted average as

$$\int_a^b p(\theta)d\theta = \sum_{i=0}^{n} \varepsilon_i\, p(\theta_i) \tag{3.1}$$

where $a = \theta_0 < \theta_1 < \theta_2, \dots, \theta_n = b$, $\theta_{i+1} = \theta_1 + \varepsilon_i$, for all $i = 0,1,2,\cdots,n$ and ε_i stands for the size of increment used to approach 'b' from 'a'. Here it is important to note that the accuracy and size of the increment are inversely related to each other. Two-dimensional integrals may be evaluated using the relation

$$\int_a^b \int_c^d p(\theta_i,\theta_j)d\theta_i d\theta_j \cong \sum_{i=0}^{n_i}\sum_{i=0}^{n_i} \varepsilon_i\, \varepsilon_j\, p(\theta_i,\theta_j) \tag{3.2}$$

where $\min(\theta_i) = \theta_0 < \theta_1 < \theta_2, \dots, \theta_{n_i} = \max(\theta_i)$, for all θ_i, $\min(\theta_j) = \theta_0 < \theta_1 < \theta_2, \dots, \theta_{n_j} = \max(\theta_j)$, for all θ_j; ε_i and ε_j respectively denote the size of increments in the Parametric values θ_i and θ_j, and $p(\theta_i,\theta_j)$ symbolizes any Bivariate Density.

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- The entire estimation algorithm may be understood by Figure 1.

5. Illustrations

For the purpose of illustration of the entire estimation procedure, we take two examples: one for single-Parameter Density and the other for two-Parameter Density and consider the Exponential and the Normal Distributions respectively.

5.1 One-Parameter Distribution: For a one-Parameter case, for instance, we consider the Exponential Distribution with density $f(x|\theta) = \theta \exp(-\theta x)$, $\theta > 0$, for $x \geq 0$; and zero elsewhere. Let an observed sample of size 'n' with values $X_1, X_2, \dots, X_n$ be taken from the Exponential Distribution. The Likelihood Function is $L(\boldsymbol{X};\theta) = \theta^n \exp(-\theta \sum_{i=1}^{n} x_i)$ and the logarithm of the Likelihood Function is $l(\boldsymbol{X};\theta) = n \ln \theta - \theta \sum_{i=1}^{n} x_i$, which implies $\frac{\partial^2 l(X;\theta)}{\partial \theta^2} = -\frac{n}{\theta^2}$. Since it does not depend upon **x**, so we get the Fisher's information as $I(\theta) = -E\left\{\frac{\partial^2 l(X;\theta)}{\partial \theta^2}\right\} = \frac{n}{\theta^2}$ and hence the Jeffreys Prior takes the form $J(\theta) \propto \theta^{-1}$. The Posterior Distribution follows the Gamma Distribution as $f(\theta|X) \propto \theta^{n-1}\exp(-\theta \sum_{i=1}^{n} x_i)$, i.e., $\theta|X \sim g(n, \sum_{i=1}^{n} x_i)$ with Posterior mean $E_{\theta|x}(\theta) = \frac{n}{\sum_{i=1}^{n} x_i} = (\bar{x})^{-1}$.

For illustration, let the time in minutes required to serve a customer at certain facility have an Exponential Distribution with unknown Parameter θ. If the average time required to serving a random sample of 20 customers is observed to be 3.8 minutes. Obviously, the Posterior Distribution for the Parameter θ under the Jeffreys Prior, as derived in Section 5.1, is the Gamma with Parameters 20 and $20 \times 3.8 = 76$, i.e., $\theta|X \sim g(20, 76)$ and the Posterior Bayes estimator under the Squared-Error Loss Function is $(\bar{x})^{-1}$, i.e., 0.263158.
Using the numerical estimation criteria explained in Section 4, we run a set of C codes to get $E_{\theta|x}(\theta) \equiv (\bar{x})^{-1} =$ 0.263158. Even for complex Posterior Distributions, the numerical estimation criteria give good results.

5.2 Two-Parameter Distribution: Similarly, if we consider the Normal Distribution with Parameters mean μ and variance δ^2, both unknown, with density $f(x|\mu,\delta^2) = 1/\sqrt{2\pi\delta^2}\,.\exp\{-(x-\mu)^2/(2\delta^2)\}$, $-\infty \leq \mu \leq \infty$, $\delta > 0$, constant to get a proper density. In this scenario, the Posterior Distribution is automatically used to yield the desired Posterior Bayes Estimates.

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6. Concluding Remarks

In this article, an effort is made to elaborate on the numerical methods to find the Jeffreys Prior and the Posterior Bayes Estimates. Numerical differentiation and Quadrature are considered to find the Jeffreys Prior and the Posterior Bayes Estimates. For instance of a one-Parameter case, we used the Exponential Distribution to derive the Jeffreys Prior and the Posterior Estimates, whereas for the two-Parameter case, we used the Normal Distribution. The observed and simulated datasets are studied. It is seen that the theoretical and numerical results fairly agree.

The same procedure can easily be employed for the case of uninformative uniform, informative and conjugate priors too. The method works equally well when priors and datasets are assumed to follow non regular density functions. The complicated Posterior Distributions can also be handled with equal ease and accuracy.

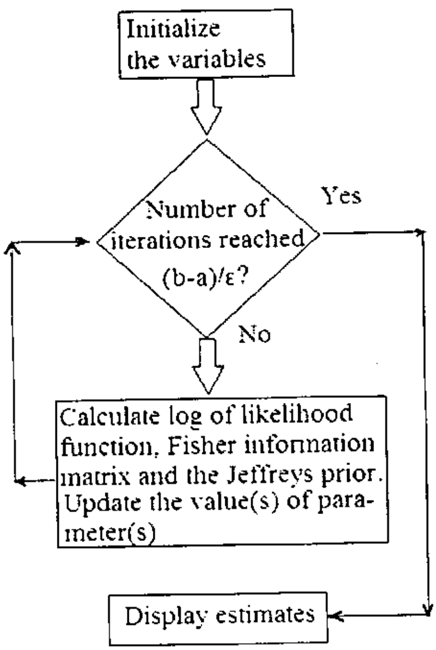

Figure 1: The numerical-estimation procedure

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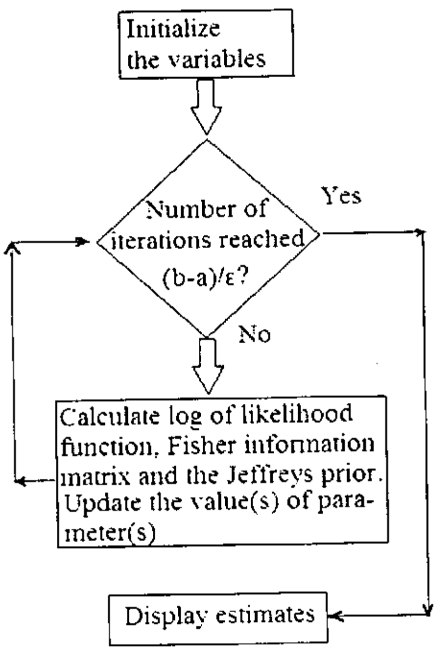

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