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Deriving the Efficiency Function for Type I Censored Sample from Exponential Distribution Using Sup-Entropy

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Abstract

This paper utilizes information theory to quantify Efficiency of Type I Censored sample drawn from Exponential Distribution and the consequent information loss due to Censoring. Based on Awad Sup-Entropy, an Efficiency Function for Censored sample is derived explicitly. The properties of the derived Efficiency Function are explained as a function of the Exponential Parameter and the termination time of the experiment. The estimation for the termination time of the experiment for a given Efficiency is discussed. Furthermore, under certain Efficiency, the Maximum Likelihood and Interval Estimation for the Exponential Parameter are also introduced.

Keywords

Entropy, Type I censoring, Type II censoring, Exponential distribution

1. Introduction

It is naturally anticipated that smaller samples contain less information than larger samples; however, little attempt has been made to quantify the information loss due to the use of subsamples rather than complete samples.

Information is defined in terms of Probability Density Function $f(x)$ of a given random variable X and is measured by the differential Entropy as was suggested by Shannon (1948). The differential Entropy is referred to as the Shannon Entropy for the case of continuous random variables, namely

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$$H(X) = -E(\log f(X)) = -\int f(x)\log f(x)dx. \tag{1.1}$$

A couple of papers have discussed the use of Entropy in analysis of Censored experiments.

Hollander et al. (1987) suggested using Entropy measures for quantifying the information loss in Censored samples from Discrete Distributions. However, for Continuous case, Shannon Entropy could become negative. Therefore, the authors proposed using a variance Entropy measure instead. Ebrahimi and Soofi (1990) discussed the use of Shannon Entropy for measuring information loss for both the Maximum Likelihood Estimation and Bayesian Estimation for type II Censored Exponential data. Awad (1987) suggested a modification for Shannon Entropy (4.1), namely

$$A(X) = -E(\log \frac{f(X)}{\delta}), \text{ where } \delta = \sup_x f(X) \tag{1.2}$$

As opposed to Shannon Entropy (1.1), Awad Sup-Entropy (1.2) or Simply Sup-Entropy is always non-negative since $\frac{f(X)}{\delta} \le 1$; also it attains zero value if and only if X is a uniform random variable (which is a non-informative distribution). Moreover, for the Awad Sup-Entropy, we can also find a complete analogy between Sup-Entropy for Discrete case and that of Continuous case. For more information about Awad Sup-Entropy (see Awad, 1987).

Awad and Alawneh (1987) calculated the loss of Entropy when the life time is assumed to be Truncated Exponentially on [0,t) for the Shannon and the Sup-Entropy cases. They observed numerically that the loss of Sup-Entropy is always between zero and one, and also it decreases with an increase in truncation time; on the other hand, the loss of Shannon Entropy could be negative and could be more than one.

Some other researchers like Abo-Eleneen (2008), Ng et al. (2004), and Zeng and Park (2004) used maximum expected Fisher information measure as an optimality criterion in progressively Censored experiments. Balakrishnan et al. (2008), basing on Fisher information measure, found that the Optimal Censoring scheme for some Distributions was that of one-step Censoring. Haj Ahmed and Awad (2010) based on Sup-Entropy, found that the Optimal Censoring scheme for Pareto Distribution is also one-step Censoring scheme.

As continuum of the exploration of Awad Sup-Entropy usage, the work presented here investigates quantifying Efficiency of type I Censored samples from Exponential Distribution. The rest of this paper is organized as follows: Section 2 suggests the Efficiency Function based on Sup-Entropy for measuring Efficiency of type I Censored sample from Exponential Distribution and its corresponding properties. This section also contains an explicit formula for the termination time of the experiment as function of both the Exponential Parameter and the Efficiency. Also the expected number of measurable observations (Censored sample size) corresponding to a given Efficiency is proposed here. Section 3 discusses the Maximum Likelihood Estimator (MLE) and Confidence Interval for the Exponential Parameter as functions of the Efficiency based on Sup-Entropy. Simulation study is presented in Section 4. Finally, the conclusions are drawn in Section 5.

2. Awad Sup-Entropy and Efficiency of Type I Censored Sample

Let us construct the Efficiency Function of the type I Censored sample from Exponential Distribution using Sup-Entropy (1.2).

We know that in certain types of problems such as life-testing experiments, the ordered observations may occur naturally. In such cases, a great savings in time and cost could be realized by terminating the experiment as soon as the first ‘r’ ordered observations have occurred rather than waiting for all ‘n’ failures to occur. If one terminates the experiment after a fixed time ‘t’, this procedure is referred to as Type I Censored sampling. In this case the number of observations, R is a random variable. The probability that a failure occurs before time ‘t’ for any given trial is p=F(t), where F is the Distribution Function of the assumed life-time model, so for a random sample of size ’n’ the random variable R follows a Binomial Distribution.
Formally, *R~Bin(n,F(t))* with Probability Density Function

$$b(r) = = \frac{n!}{r!(n-r)!}\left[F(t)\right]^r\left[1-F(t)\right]^{n-r} .$$

Type I Censored sampling is related to the concept of Truncated Distributions. Consider an ordered random sample $X_1, X_2, ..., X_n$ from a distribution with Probability Density Function *f(x)* and Cumulative Distribution Function *F(x).* And out of ‘n’ observations in total, we suppose that only ‘r’ observations occur before time ‘t’; then given R=r, the Joint Conditional Density Function of these values say $x_1, x_2, ..., x_r$ is given by

$$h(x_1, x_2, ..., x_r \mid r) = r! \prod_{i=1}^{r} f(x_i \mid x_i < t) = \frac{r!}{[F(t)]^r} \prod_{i=1}^{r} f(x_i).$$

Accordingly, the Joint Probability Density Function of $X_1, X_2, ..., X_r$ is given by

$$f_{cenI}(x_1, x_2, ..., x_r) = \frac{n!}{(n-r)!}[1 - F(t)]^{n-r} \prod_{i=1}^{r} f(x_i), \qquad (2.1)$$

where the subscript cen1 denotes the type I Censoring.

2.1 Awad Sup-Entropy for Type I Censored Sample: The Probability Density Function of Exponential Distribution with Parameter (θ) is given by

$f(x) = \theta e^{-\theta x}, \ x > 0, \theta > 0,$ and zero elsewhere.

The Sup-Entropy for both complete and type I Censored samples from $exp(\theta)$ are derived through the following theorem.

Theorem 2.1: If $X_1, X_2, ..., X_r$ denote the first r ordered statistics of a random sample of size n from Exponential Distribution with Parameter θ that is type I Censored on the right at ‘t’, then

$$A_{com} = n \qquad (2.2)$$

and

$$A_{cen}(\theta) = n\left(1 - e^{-\theta t} - \theta t e^{-\theta t}\right) \qquad (2.3)$$

where A_{com}, and A_{cen} denote the Sup-Entropy of the complete and the Censored samples, respectively.

2.2 Efficiency of Type I Censored Sample: Efficiency of a Censoring scheme based on a given Entropy measure is the ratio of the value of that Entropy in the Censored scheme to its value in the complete scheme. Accordingly, by theorem 2.1 the Efficiency of type I Censored sample from $exp(\theta)$ based on Awad Sup-Entropy is given by

$$C(t, \theta) = \left(1 - e^{-\theta t} - \theta t e^{-\theta t}\right) \qquad (2.4)$$

Also, the relative loss of information due to type I Censoring is given by

$$L(t) = \frac{A_{com} - A_{cen}}{A_{com}} = 1 - C(t, \theta) = e^{-\theta t}(1 + \theta t) \qquad (2.5)$$

It can be seen that the efficiency $C(t,\theta)$ is non-negative strictly increasing function of 't' on $(0,\infty)$ with $C(0,\theta)=0$ and $\lim_{t\to\infty} C(t,\theta)=1$, and hence, $C(t,\theta)$ itself is a Distribution Function defined on the same support of $F(t)$. Also it is interesting to note that $C(t,\theta)$ is just the Distribution Function of the first record value from *exp(θ)*; moreover, $L(t)$ is the Survival Function of that Distribution; but this could not be the general case for all Distributions.

Since $0 \le C(t,\theta) \le 1$, thus $A_{cen} \le A_{com}$; where equality holds if and only if $t \to \infty$. That is, basing on Sup-Entropy, the amount of information in the Censored sample is less than in the complete sample. Therefore, $C(t,\theta)$ might be called 'the expected percentage of information in Censored data with respect to the complete data'.

To determine the termination time that leads to a given Efficiency ε where $0 < \varepsilon < 1$, the solvability of the equation should be discussed.

$$1 - xe^{-x} - e^{-x} = \varepsilon \text{, where } x = \theta t \tag{2.6}$$

The properties of the Efficiency Function (2.4) that are mentioned before guarantee the uniqueness of the solution of (2.6); moreover, many numerical techniques such as Newton-Raphson's method will converge to this unique solution.

Let the unique solution of (2.6) be denoted by x_ε, then the termination time corresponding to ε is given by

$$t_\varepsilon = \frac{x_\varepsilon}{\theta} \tag{2.7}$$

For estimation purposes, since the Parameter θ would be unknown, then unfortunately the termination time in (2.7) will be so. However, the number of failures r_s before t_s can be estimated by the expected value of R at $t = t_\varepsilon$, thus

$$r_\varepsilon = E(R) = nF(t_\varepsilon) = n(1 - e^{-x_\varepsilon}) \tag{2.8}$$

From equation (2.8), the estimated number of failures r_s corresponding to efficiency ε could not be an integer; hence it is reasonable to take the floor values instead.

3. Estimation and Efficiency

In this section, under certain Efficiency, Point and Interval Estimations for θ are provided. Also an approximation to the termination time of the experiment corresponding to this Efficiency is given.

For a given Efficiency ε, equation (2.8) provides approximated number of failures before the termination time t_s, while as mentioned before, the suitable termination time for the experiment to attain this Efficiency is missing. The previous argument suggests estimating θ from the MLE of the type II Censored sample Likelihood instead.

The Likelihood Function for the type II Censored sample is given by Bain and Engelhardt (1992).

$$f_{cenII}(x_1, x_2, ..., x_r) = \frac{n!}{(n-r)!}[1-F(x_r)]^{n-r}\prod_{i=1}^{r} f(x_i) = \frac{n!}{(n-r)!} e^{-\theta x_r (n-r)}\theta^r \exp(-\theta \sum_{i=1}^{r} x_i)$$

if $t > x_i > 0, \theta > 0$, and zero elsewhere.

Thus when r = r_s; the MLE for θ from the type II Censored sample is given by

$$\hat{\theta}_\varepsilon = \frac{r_\varepsilon}{\sum_{i=1}^{r_\varepsilon} x_i + (n - r_\varepsilon).x_r} \quad (3.1)$$

The MLE in equation (3.1) might be called the ε efficient MLE for θ.
By substituting this estimate in equation (2.7), the estimated termination time of the experiment subject to Efficiency ε is given by

$$\hat{t}_\varepsilon = \frac{x_\varepsilon}{\hat{\theta}_\varepsilon} \quad (3.2)$$

The termination time in equation (3.1) might be called the ε _efficient termination time of the experiment.

Johnson and Kotz (1970) showed that for a complete sample of size *n*, $2n\theta/\hat{\theta}$ is distributed as Chi-Square with $2n$ degrees of freedom, where $\hat{\theta}$ is the MLE for θ from the Complete Likelihood Function and thus a $100\alpha\%$ Confidence Interval for θ is given by

$$\frac{\hat{\theta}\,\chi^2_{2n,\alpha/2}}{2n} < \theta < \frac{\hat{\theta}\,\chi^2_{2n,1-\alpha/2}}{2n}$$

where $\chi^2_{n,\alpha}$ is the $100\alpha th$ Percentile of the Chi-Square Distribution with degrees of freedom 'n'.

This result should also hold for Censored samples if the degree of Censoring is not excessive. Accordingly, from the type II Censored sample, a $100\alpha\%$ Confidence Interval for θ is given by

$$\frac{\hat{\theta}_\varepsilon\,\chi^2_{2r,\alpha/2}}{2r} < \theta < \frac{\hat{\theta}_\varepsilon\,\chi^2_{2r,1-\alpha/2}}{2r} \tag{3.3}$$

where $\chi^2_{r,\alpha}$ is the $100\alpha th$ Percentile of the Chi-Square Distribution with degrees of freedom 'r'.
When r = r_s, the interval (3.3) might be called the ε -efficient Confidence Interval for θ.

4. Simulation Study

In this section, the performances of the proposed estimators of 'θ', 't' and 'r' from Censored samples are investigated through a Simulation study under different given values of the Efficiency. The Simulation study is carried out for different values of the combination (θ, n, ε). In this study the Efficiency ε is assumed to be 10%, 50% and 95%. In all these cases we have generated 500 samples of size n(=10, 30 and 50) from an Exponential Distribution with Parameter θ(=0.5, 1 and 10) by using Mathematica 6, and then the average values of the ε-efficient estimators of $\theta(=\hat{\theta}_\varepsilon)$, t $(=\hat{t}_\varepsilon)$and $r(=r_\varepsilon)$ are calculated and reported in Tables 1, 2 and 3 corresponding to ε=(10%, 50% and 95%) respectively, along with the ratio (r_ε / n) are calculated. For the purpose of comparison, the MLE of θ is evaluated from each complete sample $(=\hat{\theta}_{COM})$.

It is straightforward to get from Tables 1, 2 and 3 that the expected number of failures (r_ε) corresponding to a given Efficiency ε is free of 'n' and 'θ', while the ratio $\frac{r_\varepsilon}{n}$ is largely fixed. The variation we find for the values of $\frac{r_\varepsilon}{n}$ in Table 3 is due to rounding. In fact, from equations (2.6) and (2.8) we observe that the ratio $\frac{r_\varepsilon}{n}$ depends solely on the efficiency, but neither on the complete sample size 'n' nor on the parameter 'θ'.

In general, this ratio also depends on the type of the life-time model under consideration. Apart from this, we notice that as the Efficiency ε increases to 100%, r_ε converges to 'n', and $\hat{\theta}_\varepsilon$ converges to $\hat{\theta}_{COM}$, accordingly, in this case the termination time of the experiment ($\hat{t}_\varepsilon$) tends to infinity, at least theoretically. Besides, the proposed estimator of the termination time of the experiment $\hat{t}_\varepsilon$ is free of 'n'. Moreover, it decreases as $\hat{\theta}_\varepsilon$ increases and, for a fixed value of 'n', it increases as ε increases.

It can be noticed from the Simulation studies that the proposed estimator $\hat{\theta}_\varepsilon$ gives reasonably close estimate for $\hat{\theta}_{COM}$ with moderate and large values of ε. This means that the Exponential Parameter can be estimated by $\hat{\theta}_\varepsilon$, with reasonable Efficiency, instead of $\hat{\theta}_{COM}$, if one wants to save time and cost.

5. Conclusion

In this paper, a procedure for quantifying the Efficiency of type I Censored sample from Exponential Distribution has been presented. Criteria for estimating the termination time and the number of failures corresponding to a given Efficiency have been proposed. Moreover, estimations under certain Efficiency have also been introduced.

The work presented here has proved that Sup-Entropy for Censored sample (the amount of information in Censored data) is less than Sup-Entropy for complete sample (the amount of information in complete data), accordingly, Sup-Entropy is a suitable and convenient tool for measuring the Efficiency of type I Censored sample from Exponential Distribution.

Table 1: Expected values of $\hat{\theta}_\varepsilon$, $\hat{t}_\varepsilon$, r_ε , $\hat{\theta}_{COM}$ for various values of 'n' and θ with ε =10%

n	θ	$\hat{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}$	$\hat{\boldsymbol{t}}_{\boldsymbol{s}}$	r_s	$\hat{\boldsymbol{\theta}}_{\boldsymbol{COM}}$
10	0.5	0.661139	1.06362	4(40%)	0.549583
	1	1.41033	0.531812	4	1.14126
	10	12.9098	0.0531812	4	11.113
30	0.5	0.540222	1.06362	12(40%)	0.511134
	1	1.11408	0.531812	12	1.04358
	10	10.9496	0.0531812	12	10.3759
50	0.5	0.523125	1.06362	20(40%)	0.507946
	1	1.0483	0.531812	20	1.01725
	10	10.5416	0.0531812	20	10.1956

Table 2: Expected values of $\hat{\theta}_\varepsilon$, $\hat{t}_\varepsilon$, r_ε , $\hat{\theta}_{COM}$ for various values of 'n' and θ with ε =50%

n	θ	$\hat{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}$	$\hat{\boldsymbol{t}}_{\boldsymbol{s}}$	r_s	$\hat{\boldsymbol{\theta}}_{\boldsymbol{COM}}$
10	0.5	0.573475	3.35669	8(80%)	0.559833
	1	1.16058	1.67835	8	1.11479
	10	11.457	0.167835	8	11.2766
30	0.5	0.51882	3.35669	24(80%)	0.514965
	1	1.04977	1.67835	24	1.03006
	10	10.542	0.167835	24	10.4255
50	0.5	0.518479	3.35669	40(80%)	0.514062
	1	1.02486	1.67835	40	1.02222
	10	10.3394	0.167835	40	10.2366

Table 3: Expected values of $\hat{\theta}_{\varepsilon}$, $\hat{t}_{\varepsilon}$, r_{ε} , $\hat{\theta}_{COM}$ for various values of 'n' and θ with ε =95%

n	θ	$\hat{\boldsymbol{\theta}}_{\boldsymbol{\varepsilon}}$	$\hat{\boldsymbol{t}}_{\boldsymbol{s}}$	r_s (r_s/n)	$\hat{\boldsymbol{\theta}}_{\boldsymbol{COM}}$
10	0.5	0.579318	9.48773	9(90%)	0.569396
	1	1.11625	4.74386	9	1.10103
	10	11.5283	0.474386	9	11.4003
30	0.5	0.513505	9.48773	29(96%)	0.51309
	1	1.02955	4.74386	29	1.02995
	10	10.4495	0.474386	29	10.4452
50	0.5	0.507761	9.48773	49(98%)	0.508264
	1	1.02376	4.74386	49	1.02328
	10	10.2526	0.474386	49	10.2651

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