5ISSN 1684-8403
Journal of Statistics
Volume 18, 2011, pp. 10-28

A Comparative Study of Ten Asymmetry Tests

H.E.Thomas Holgersson[1]

Abstract

In this paper, we investigate the properties of some common tests for Asymmetry. The tests are based on Moments, Order Statistics and Empirical Characteristic Functions, respectively. These tests have completely different origins, rely on different characterizations of symmetry and have very different size and power properties. It is demonstrated that tests based on Empirical Characteristic Functions are strongly dependent on the choice of working region and that some previously proposed tests may be improved considerably by using Bootstrapped critical values. It is also concluded that no test is uniformly better than the others but that the Empirical Characteristic Function tests have best over-all properties.

Keywords

Asymmetry testing, Skewness, Empirical characteristic function, Order statistics, Multimodality.

1. Introduction

It is often of great importance to know if a variable x is symmetric about its median (M) or not. In some cases, the sole analysis involves the question of whether the probability of events at $M-x$ equals those of $M+x$, while in other analyses, the question of symmetry is mainly used to decide between parametric and non-parametric methods. The issue of symmetry may also concern the choice of model. For example, if Macroeconomic Models are linear then it is not desirable to use them with Asymmetric dependent variables. There are also cases when asymmetry may deteriorate the properties of a point estimator.

[1] Jonkoping University, SE-551 11 Jonkoping, Sweden.

It is well known that point estimators of Regression parameters may be inefficient if the error term is asymmetrically distributed, in which, other measures of central tendency like the median should be considered as a more suitable alternative (for example, Hammarstedt and Shukur, 2006). Asymmetry of data may also lead to erroneous interval estimates (Bakliz and Kibria, 2009). In other words, there is a wide range of statistical analyses that requires an appropriate method for assessing symmetry. However, it is not obvious that which test to employ. The history of assessing symmetry dates back to the early nineteen hundreds, when a number of different skewness coefficients were considered by Fisher (1929), Pearson (1905), Yule (1917) and others. These were formed by different functions of measures of location (mean, mode or median) and scale (standard deviation or inter-percentile distances). Later, on the Standardized third order central moment came to dominates the literature and is sometimes considered to be identical to the term "skewness statistic", even though it is formally only one of several possible measures. The third central moment is well known to have high power against many Skewed Distributions and also uniquely determines the symmetry of distributions within the Pearson family (Ord, 1972), and is a standard option in many computer packages. But, as recognized in the 1940s and later on, this statistic cannot be expected to have power against Asymmetric Distributions with zero third order central moment (Churchill, 1946). Eventually, the words "skewness" and "asymmetry" came to refer to two separate things. While, skewness is usually understood to mean a non-zero third order central moment, Asymmetry refers to the more general property that $P(X < M - x) \neq P(X > M + x)$ for some *x*. Therefore, several approaches have been developed for testing this hypothesis. Some of these assume a known centre, for example, the median of the distribution (Balalakrishnan, 2004; Fuerverger and Mureika, 1977). Such tests are of particular relevance when there is a composite null hypothesis of symmetry about a specified mean versus Asymmetry and/or location shift, but are of less relevance if the sole question involves only the issue of symmetry irrespectively the location parameter. In this paper, we will restrict ourselves to the latter, that is, tests for Asymmetry, when no prior hypothesis of the centre of the distribution is available. Roughly, one may divide the tests into three classes: (i) Moment-based tests, including the Classical Skewness Measure proposed by Pearson and Fisher. The relation between skewness coefficients and Asymmetry for many distributions is zero, skewness implies symmetry. (ii) there are tests based on Order Statistics, which have a strong graphical interpretation in the sense that observations below the centre of the distribution are compared to what they "should be" above the centre if data is symmetric. Such tests are usually constructed by functions of Order Statistics. Some important references include Antille et al. (1982) and David and Johnson (1956). (iii) It involves Empirical Characteristic Function, which have strong potential for detecting Asymmetry since the Characteristic Function is real about its mean, if and only if, the underlying distribution is symmetric (Csorgo and Heathcote, 1987). On the other hand, these tests depend upon a "working region" that needs to be estimated from the sample and this complicates its usage. There are obviously tests not belonging to any of the classes (i) - (iii) though most standard tests met in the literature will fall into some of these categories.

In this paper, we will investigate the properties of some tests that we believe are of particular importance. We will pay attention to the choice of working region and its effect on Empirical Characteristic Functions, the choice of smoothing parameter for Quantile tests and also demonstrate that size distortions may be reduced by using Bootstrapped critical values. This is done through a Monte Carlo Simulation that uses variables with finite and infinite range, Unimodal and Multimodal as well as Asymmetric Distributions with zero skewness coefficients. The paper develops as follows; in section 2 the tests are briefly presented and discussed, the Monte Carlo simulation is presented in Section 3 while a summary is given in Section 4.

2. Asymmetry Tests

A random variable X is said to be symmetric, if and only if, $F_X(a-x)+F_X(a+x)=1$ for all x in $\mathbb{R}$. In this section, we will discuss some different tests for Asymmetry that is common in the literature, restricting ourselves to properties of purely continuous variables $\{x_i\}_{i=1}^n$ which are independently and identically distributed realizations of X .

2.1 The Pearson/Fisher Skewness Test

The skewness coefficient as defined by Pearson and Fisher in the early nineteen hundreds is defined by;

$$\gamma = \mu_3 / \mu_2^{3/2} \text{ where } \mu_r = E\left(X - E[X]\right)^r . \tag{2.1}$$

Its sample counterpart is given by $g = m_3 / m_2^{3/2}$ where $m_r = n^{-1}\sum_{i=1}^n \left(x_i - m_1\right)^r$. When X is symmetrically distributed with kurtosis $\mu_4 / \mu_2^2 = 3$, we have the well-known property;

$(n/6)g^2 \xrightarrow{\ell} \chi^2_{(1)}$

(Bowman and Shenton, 1975; Fisher, 1929; Stuart and Ord, 1993). Hence, $\theta_1 := (n/6)g^2$ can be used to test Asymmetry by rejecting the null hypothesis at the $\alpha-$level when $\theta_1 > q_\alpha$, where q_α is a critical value determined by $P\left(\chi^2_{(1)} > q_\alpha\right) = \alpha$. We will refer to this test as "Test 1" T_1. The Asymptotic Distribution of θ_1 requires a kurtosis equal to the value 3 (Horsewell and Looney, 1993) and homoscedasticity of X. As a remedy to these shortcomings, Holgersson (2006) proposed the use of Bootstrapped critical values: if $\chi_b = \{x_{ib}\}_{i=1}^n$ denotes a Bootstrap Resample taken from the original sample $\{x_i\}_{i=1}^n$, $\theta_j := (n/6)g_j^2$ is calculated from χ_j and $\tilde{q}_\alpha = \theta_{((1-\alpha)B)}$ is the $(1-\alpha)B^{th}$ Ordered Statistic from the B Bootstrap Resamples $\{\chi_j\}_{j=1}^B$, then the null hypothesis of symmetry is rejected when $\theta_1 > \tilde{q}_\alpha$. We will refer to this test as T_2.

2.2 The Empirical Characteristic Function

The Characteristic Function of a random variable defined by;

$$\varphi(t) := E\left[e^{itX}\right] = E\left[Cos(tX)\right] + iE\left[Sin(tX)\right] = U(t) + iV(t) \qquad (2.2)$$

uniquely determines the complete distribution of X. It is well known that $\varphi(t)$ is real, if and only if, X is symmetric about the origin. Hence, symmetry of $(X-\mu)$ is equivalent to $E\left[Sin\left(t(X-\mu)\right)\right] = 0$. An estimate of $\varphi(t)$ is available by;

$$\varphi_n(t) = n^{-1}\sum_{i=1}^n Cos(tx_i) + in^{-1}\sum_{i=1}^n Sin(tx_i) = U_n(t) + iV_n(t)$$

which is a Consistent Estimate of $\varphi(t)$ under some mild conditions (Feuerverger and Mureika, 1977). A specific test was suggested by Csorgo and Heathcote (1987) based on the statistic defined by;

$$\theta_n(t) = t^{-1}arc\,tan\left(V_n(t)/U_n(t)\right), \qquad t \in \ell. \qquad (2.3)$$

This statistic depends on the choice of t because $U_n(t) = 0$ may occur infinitely often in t. The solution to this problem is to determine a "working region" $\ell \in (0, A_0)$, where $A_0 = \inf\{t \geq 0 : U_n(t) = 0\}$ and then evaluating an appropriate test statistic at its maximum in ℓ. Therefore, an estimate of A_0 is required. Welsh (1986) proposed the Iterative Estimate of A_0 by;

$$K_{n,k+1} = K_{n,k} + \left(U_n\left(K_{n,k}\right)/2^{1-\beta} m_\beta\right)^{1/\beta},\ k = 0,\ 1,\ \ldots, \tag{2.4}$$

where $m_\beta = n^{-1}\sum_{i=1}^{n}|x_i|^\beta,\ 0<\beta\le 1,\ K_{n,0} = \left(2/m_2\right)^{1/2}$ and $m_2 = n^{-1}\sum_{i=1}^{n} x_i^2$.

Then, for each fixed n and $K_{n,k} \uparrow A_0$, the empirical working region may be determined by this estimate. An upper bound is available by $A_n < \left(2/m_2\right)^{1/2}$ which is argued to be very conservative (Welsh, 1986) but may never the less be of interest due to its simplicity. Further details concerning the first positive zero are available in Braker and Husler (1991), Heathcote and Husler (1990) and Ushakov (1999). Using an appropriate estimate of A_0 and the corresponding working region, Czorgo and Heathcote (1987) considered the statistic defined by $\sqrt{n}\left|\theta_n(s_n)-\theta_n(t_n)\right|$. The points s_n and t_n are defined as follows;

Let,

$$\sigma_n(t,s) = \frac{h_n(t,s)}{2ts\left\{U_n^2(t)+V_n^2(t)\right\}\left\{U_n^2(s)+V_n^2(s)\right\}}$$

where,

$$h_n(t) = U_n(t-s)\left\{U_n(t)U_n(s)+V_n(t)V_n(s)\right\} + U_n(t+s)\left\{V_n(t)V_n(s)-U_n(t)U_n(s)\right\}$$
$$+V_n(t-s)\left\{V_n(t)U_n(s)-U_n(t)V_n(s)\right\} - V_n(t+s)\left\{U_n(t)V_n(s)+V_n(t)U_n(s)\right\}.$$

Then, $s_n = \min\left\{s : \sigma_n^2(s) = \sup_{t\in\ell}\sigma_n^2(t)\right\}$, $t_n = \max\left\{t : \sigma_n^2(t) = \inf_{s\in\ell}\sigma_n^2(s)\right\}$ and the test statistic for testing Asymmetry is defined as;

$$\theta(\ell) = \frac{\left|\theta_n(s_n)-\theta_n(t_n)\right|}{\left\{\sigma_n^2(s_n)+\sigma_n^2(t_n)-2\sigma_n(s_n,t_n)\right\}}. \tag{2.5}$$

This statistic will depend upon the choice of working region and hence on the choice of β. For the purpose of comparison, we will use two values, $\beta = 1$ and $\beta = 0.4$, and define the corresponding test statistics by $\theta_3 = \theta(1)$ and $\theta_4 = \theta(0.4)$ respectively. When the approximated bound is determined by $A_n < \left(2/m_2\right)^{1/2}$ the corresponding test statistic will be denoted by;

$\theta_5 = \theta\left(\left(2/m_2\right)^{1/2}\right)$

The null hypothesis of symmetry is then rejected at the $\alpha-$level when $\theta_3 > z_{\alpha/2}$, $\theta_4 > z_{\alpha/2}$ or $\theta_5 > z_{\alpha/2}$ and the corresponding tests will be labeled by T_3, T_4 and T_5, respectively.

2.3 The Trimmed Wilcoxon Test

Instead of characterizing symmetry by moments, symmetry about the origin may be defined by $F(x) = 1 - F(-x)$ or equivalently by $f(-x) = f(x)$ for all *x*. The latter definition may be more appealing, since the shape of a distribution can always be presented by the graph of $f_n(x)$ against *x*. Hence, observations below the centre of the distribution may be compared to what they "should be" above the centre. There are several ways of doing this empirically. Antille et al. (1982) considered the following general statistic for testing Asymmetry;

$$U(\eta) = n^{-1/2} \sum_{i=1}^{n} \phi_\eta \left(\frac{R_i(M)}{2(n+1)} \right) sign(X_i - M), \qquad (2.6)$$

where, M is the median of *X*, $R_i(M)$ is the rank of $|X_i - M|$, $\phi_\eta(z) = \min(z,\ 1/2 - \eta)$, $0 \le z \le 1/2$ and $0 \le \eta \le 1/2$. For $\eta = 0$ this statistic is a Wilcoxon Signed Rank Test with ranks centered about the sample median, originally proposed by Gupta (1967). In order to achieve a statistic which is more robust to irregularities in the tails of the distributions, one may use a Trimmed version$(\eta > 0)$. Unfortunately, the sampling variance of $U(\eta)$ depends upon the distribution of X, even for Symmetric Distributions. Antille et al. (1982) argued that, as the sample size *n* increases

$$U(0) \xrightarrow{\ell} N\left(0,\ \sigma_{0,n}^2\right)$$

where,

$$\sigma_{0,n}^2 = 0.03156 \exp(-5.66/n).$$

A Trimmed version will have the Limiting Distribution;

$$U(1/6) \xrightarrow{\ell} N\left(0,\ \sigma_{1/6,n}^2\right)$$

where,

$$\sigma_{1/6,n}^2 = 0.01736 \exp(-5.77/n).$$

These approximations assume an odd sample size (even sample sizes require a trivial modification).

Hence, if $\theta_6 = U(0) \big/ \sqrt{\sigma^2_{0,n}}$ and $\theta_7 = U(1/6) \big/ \sqrt{\sigma^2_{1/6,n}}$, then the null hypothesis of symmetry is rejected when $|\theta_6| > z_{\alpha/2}$ or $|\theta_7| > z_{\alpha/2}$. We will refer to the corresponding tests as T_6 and T_7, respectively.

2.4 Gaps Test

Another family of tests which is also based on Ordered Statistics may be defined as follows;

Let, $x_{(i)}$ be the i^{th} order statistic of an observable variable x and define $R_i = x_{(n-k+i)} - x_{(n-1-k+i)}$, $L_i = x_{(k+2-i)} - x_{(k+1-i)}$ for $i = 1, 2, ..., k$ where $k = [n/2]$.

Then, R_i and L_i may be thought of as gaps on the left and right hand side of the median and may hence be compared to each other. Finch (1977) proposed the statistic;

$$V = \sum_{i=1}^{k} w_i V_i, \tag{2.7}$$

where,

w_i are fixed (non-random) weights and $V_i = (R_i - L_i) / (R_i + L_i)$.

The values of the optimal weights depend on the underlying Asymmetric Distribution. Since, an Asymmetry Test is often conducted against no specific alternative; one single choice of weights cannot be optimal against all Asymmetric Distributions. Moreover, the Asymptotic Standard Error of V depends on the distribution of X even under symmetry. Finch (1977) derived the optimal weights for Tukey Distribution (Karian and Dudewicz, 2000) and the Logistic Distribution, respectively, and argued that the choice of optimal weights is particularly intricate for heavy Tailed Distributions. Antille et al. (1982) considered weights for the Gap Test defined by;

$$w_i = J\left(i/(n+1)\right)$$

where, J is a Smoothing Indicator Function defined by;

$$J = I[0.05, 0.5]$$

And

I is the Indicator Function assigning the value 1 for values in the interval and zero otherwise. These authors, hence, proposed the Normalized Gap Statistic defined by;

$$\theta_8 = n^{-1/2} \sum_{i=1}^{k} J\left(i/(n+1)\right) V_i \tag{2.8}$$

and argued that $\theta_8 \xrightarrow{\ell} N(0,\ 0.1293)$ as n increases. The null hypothesis of Asymmetry is hence rejected at the $\alpha-$level when $|\theta_8| > z_{\alpha/2}$. This test will be labeled as T_8.

2.5 Order Statistics

Yule (1917) proposed a skewness measure defined by;

$$\theta = \frac{\hat{\pi}_{0.75} + \hat{\pi}_{0.25} - 2\hat{\pi}_{0.5}}{\hat{\pi}_{0.75} - \hat{\pi}_{0.25}}$$

where, $\hat{\pi}_p$ is an estimate of the pth population percentile.

Later on more general skewness measures of Order Statistics were considered, defined by;

$$\theta = \frac{\hat{\pi}_p + \hat{\pi}_{1-p} - 2\hat{\pi}_{0.5}}{\hat{\sigma}}$$

where, $\hat{\sigma}$ is some measure of spread. David and Jonson (1956) proposed the specific choice;

$$\theta(p) = \frac{\hat{\pi}_p + \hat{\pi}_{1-p} - 2\hat{\pi}_{0.5}}{\hat{\pi}_p + \hat{\pi}_{1-p}}\ , \tag{2.9}$$

where, $p = 0.9875$.

As with most Order Statistics, the sampling variance of test statistics depends on the density of X. Resek (1974) derived the variance of $\theta(0.9875)$ of a normally distributed X for a single sample size ($n = 100$) but that variance estimate is not expected to be useful in small samples or for symmetric non normal data. But due to this statistics very simple functional form (critical values) may be consistently Bootstrapped. The test is then defined as follows;

Let,

$$\chi_b = \{x_{ib}\}_{i=1}^{n}$$

denote a Bootstrap Resample taken from the original sample $\{x_i\}_{i=1}^{n}$, $\theta_b(p) = \frac{\hat{\pi}_{b,p} + \hat{\pi}_{b,1-p} - 2\hat{\pi}_{b,0.5}}{\hat{\pi}_{b,p} + \hat{\pi}_{b,1-p}}$ be calculated from χ_b and $\theta_i(p)_{((1-\alpha)B)}$ be the $(1-\alpha)B^{th}$ Ordered Statistic from the B Bootstrap Resamples $\{\chi_b\}_{b=1}^{B}$. Two values of the trimming constant p will be used, $p = 0.9875$ and $p = 0.95$ respectively. The null hypothesis of symmetry is then rejected when

$\theta(0.9875) > \theta(0.9875)_{((1-\alpha)B)}$ and $\theta(0.95) > \theta(0.95)_{((1-\alpha)B)}$. The two tests based on $\theta(p)$ with Bootstrapped critical values and different trimming constants will be denoted by T_9 and T_{10}.

The above tests, T_1- T_{10}, all rely on Asymptotical properties and are based on different characterizations of symmetry. Therefore, it will be undue to derive rejection probabilities of T_1- T_{10} analytically. The next section, therefore, involves a Monte Carlo Simulation.

3. Monte Carlo Simulations

Since, all tests presented in the previous section only have asymptotically known null distributions, it is crucial that the rejection frequencies for Symmetric Distributions are investigated carefully. Only tests that have a rejection frequency close to the nominal size should be considered as useful in real applications. Good size properties, however, are not sufficient when the tests lack power to reject false null hypotheses. The power of statistical tests is usually investigated by letting the parameter of interest which increases away from its null value in the parameter space while keeping the sample size fixed. This, however, is not feasible to do when investigating Asymmetry Tests: firstly, there is no unique way to quantify Asymmetry, and secondly, different tests are optimal for different types of Asymmetry. Therefore, we will design the power Simulation by using a Fixed Asymmetric Distribution and then letting the sample size increase. We will consider a somewhat wide variety of distributions reaching outside the Pearson family of distributions. These include Asymmetric Distributions with zero third-order central moment, distributions with heavy tails or extreme values, variables with Finite/Infinite Range, Unimodal, Multimodal Distributions and distributions with only a few existing moments. A summary of the distributions is presented in Tables 1 and 2. The Kintchine Distribution (α, τ) is described in Johnson (1987, page 35) and the Burr XII distribution (c, k) is described in Kotz et al. (2005, page 679). Our "Double Centred Gamma Distribution" is defined by;

$$Z = ((Y_1 - E[Y_1]) - (X_1 - E[X_1]))$$

where,

X and Y are individually and mutually independent gamma variates such that $Y_1 \sim Gamma(rate = 1, shape = 0.125)$ and $X_1 \sim Gamma(rate = 3.41995, shape = 5)$.

This Asymmetric Distribution has its third central moment equal to zero. We will also use the Stieltje Distribution (Churchhill, 1947; Stuart and Ord, 1993) that has all odd central moments equal to zero, a discontinuity point at $x = 0$ and multiple modes. The Trimodal Distribution Z is defined such that;

$$p(Z = X_1) = p(Z = X_2) = p(Z = X_3) = 1/3$$

where,

$X_1 \sim N(-10,4)$, $X_2 \sim N(0,16)$, $X_3 \sim N(10,4)$ are individually and mutually independent.

Details about Standard Distributions (Beta, Students *t* etc.) are available in Johnson, et al. (1994 and 1995) and Stuart and Ord (1993).

The Monte Carlo Simulations of the tests have been conducted by counting rejection frequencies of r = 10,000 replicates for each sample size $n = 10 - 500$. The Bootstrapped critical values were obtained by $B = 99$ nonparametric resamples. The size properties (rejection frequencies for symmetric data) are displayed in Tables 3-10. The Classical Fisher/Pearson Skewness Test T_1 does not have a rejection frequency limiting its nominal size (0.05) for any distribution but the normal. This is due to the fact that its null distribution depends on a kurtosis value of 3. The Bootstrapped version, on the other hand (T_2) rapidly limits its nominal size for all distributions except for the $t_{(2)}$ and Cauchy Distributions where it seems to diverge. This is not surprising since none of these distributions have a finite value of the kurtosis and the Cauchy Distribution even lacks finite variance. The tests based on the Characteristic Function (T_3-T_5) using three different estimates of the working region, also reveal some interesting properties. For the Normal, Uniform and Beta Distribution they limit the nominal size asymptotically and behave similarly to each other (Tables 3-5). The rejection frequencies when applied to the Trimodal Distribution (Table 7), on the other hand, reveal that the choice of working region ℓ does have an impact on the tests. When ℓ is determined by the Iterative Estimator and using Smoothing parameter $\beta = 0$ (=T_3) the rejection frequency of the test actually diverges as *n* increases, whereas, for the other two options (T_4-T_5) it seems to limit the nominal though very slowly. For the Cauchy Distribution, the three tests (T_3-T_5) stay at a rejection frequency of about 0.02 (Table 8). The properties of the tests based on the Characteristic Function when applied to the Khintchine and Laplace Distributions are presented in Tables 9-10. These show that the T_3 test fails to limit the nominal size, whereas, the two others (T_4-T_5) do limit 0.05. The two versions of the Wilcoxon Test (T_6- T_7) perform well in terms of size for the Normal Distribution (Table 3) but diverge for the Uniform and Beta Distributions (Tables 4-5). For the

$t_{(2)}$ and Chauchy Distribution, the rejection frequencies of the T_6-T_7 tests stay between 0.02-0.03 for all sample sizes, while they diverges for the Trimodal Distribution (Table 7). The Gaps Test T_8 strongly over rejects for all Symmetric Distributions except the Normal and Uniform Distribution. Therefore, its power properties are of limited interest. The Order Statistics tests (T_9-T_{10}) differ by their smoothing constants. It is interesting to note that they behave rather differently for Heavy-tailed Distributions (Students $t_{(2)}$, Cauchy, Laplace and Kinthine) in that the T_{10} over rejects for these distributions, whereas, the size properties of the T_{10} remain well behaved when applied to these distributions.

4. Summary

In this paper, we have investigated the finite sample properties of 10 tests for Asymmetry through Monte Carlo Simulations. The tests are a selection of the most common ones within the families of tests based on Moments, Characteristic Functions, Gaps and Ordered Statistics, respectively. From these tests, many have only been sparsely investigated in the literature previously. The power properties of the tests are examined through a diversity of distributions such as Finite Range, Multimodal and Asymmetric Distributions with zero skewness coefficient and distributions without finite kurtosis. Moreover, modifications of the tests that have not been considered before are proposed in the paper. These include Trimmed Estimates of the first positive zero within the Empirical Characteristic Function and also Bootstrapped versions of Order Statistics. The main findings are that no single test stands out as uniformly superior to the others. Several tests diverge from the nominal size as the sample size increases if the Symmetric Distribution has finite range or multiple modes. Moreover, other tests are inconsistent against Asymmetric Distributions with skewness coefficient equal to zero. It is also concluded that the size properties of several tests may be improved considerably by using Bootstrapped critical values. All tests should be used with caution when the target variable has finite range or multiple modes. Finally, the only tests that possess acceptable over-all size and power properties are those based on the Empirical Characteristic Function.

References

1. Antille, A., G. Kersting and W. Zucchini (1982). Testing symmetry. *Journal of the American Statistical Association*, **77(379)**, 639-646.

2. Baklizi, A. and Kibria, B. M. G. (2009). One and two sample confidence intervals for estimating the mean of skewed populations: An empirical comparative study. *Journal of Applied Statistics,* **36**, 601-609.
3. Bowman, K. O. and L. A. Shenton (1975). Omnibus test contours for departures from normality based on $\sqrt{b_1}$ and b_2. *Biometrika*, **62(2)**, 243-250.
4. Braker, H. U. and J. Husler (1991). On the first zero of an empirical characteristic function. *Journal of Applied Probability*, **28**, 593-601.
5. Cheng, Wei-Hou and N. Balakrishnan (2004). A modified sign test for symmetry. *Communications in Statistics-Simulation and Computation*, **33(3)**, 703-709.
6. Churchill, E. (1946). Information given by odd moments. *Annals of Mathematical Statistics*, **17**, 244-246.
7. Csorgo, S. and Heathcote, C. R. (1987). Testing for symmetry. *Biometrika*, **74(1)**, 177-184.
8. David, F. N. and N. L. Johnson (1956). Some tests of significance with ordered variables. *Journal of the Royal Statistical Society*, **18(1)**, 1-31.
9. Feuerverger, A. and R. A. Mureika (1977). The empirical characteristic function and its applications. *The Annals of Statistics,* **5(1)**, 89-97.
10. Finch, S. J. (1977). Robust univariate test of symmetry. *Journal of the American Statistical Association*, **72(358)**, 387-392.
11. Fisher, R. A. (1929). Moments and product moments of sampling distributions. Proceedings of the London mathematical society (Series 2), 3, 199-238.
12. Gupta, M. K. (1967). An asymptotically nonparametric test of symmetry. Annals of Mathematical Statistics, 38(3), 849-866.
13. Hammarstedt, M. and Shukur, G. (2006). Immigrants relative earnings in Sweden-A cohort analysis. *Labour*, 20, 285-323.
14. Heathcote, C. R. and J. Husler (1990). The first zero of an empirical characteristic function. Stochastic processes and their Applications, 35, 347-360.
15. Holgersson, H. E. T. (2006). Robust testing for skewness: Communications in Statistics. *Theory and Methods*, **36(3)**, 485-498.
16. Horsewell, R. L. and Looney, S. W. (1993). Diagnostic limitations of skewness coefficients in assessing departures from univariate and multivariate normality. *Communications in statistics-simulation and computation*, **22(2)**, 437-459.
17. Johnson, M. E. (1987). Multivariate Statistical Simulation. Wiley.

18. Johnson, N. L., Kotz, S. and N. Balakrishnan (1994, 1995). *Continuous Univariate Distributions*, **1 and 2**, Wiley.
19. Karian, Z. A. and Dudewicz, E. J. (2000). Fitting Statistical distributions: The Generalized Lambda Distribution and Generalized Bootstrap Methods, CRC Press.
20. Kotz, N. L. et al. (Eds.). (2005). Encyclopedia of Statistical Sciences, 2nd edition, Wiley.
21. Ord, J. K. (1972). Families of frequency distributions, Griffin´s statistical monographs and courses.
22. Pearson, K. (1905). Das fehlergesetz und seine verallgemeinerungen durch Fechner und Pearson. A rejoinder. *Biometrika*, **4(1/2)**, 169-212.
23. Resek, R. W. (1974). Alternative tests for skewness: Efficiencey comparison under realistic alternative hypothesis. *Proc. Bus. and Econ. Stat. Section. Am. Stat. Assoc*, 546-551.
24. Stuart and Ord. (1993). Kendalls Advanced Theory of Statistics, **1**, Distribution theory.
25. Ushakov, N. G. (1999). Selected Topics in Characteristic Functions. VSP.
26. Welsh, A. H. (1986). Implementing Empirical Characteristic Function Procedures. *Statistics and Probability letters*, **4**, 65-67.
27. Yule, G. U. (1917). An introduction to the theory of statistics. C. Griffen and company.

Table 1: Symmetric Distributions

Distribution	Skewness	Kurtosis	Shape	Range
Normal $(0,1)$	0	3	Unimodal	$\mathbb{R}$
Uniform $(0,1)$	0	1.8	Flat	$(0,1]$
Kintchine (3.33, 2)	0	21.7	Unimodal	$\mathbb{R}$
Beta $(0.5, 0.5)$	0	1.5	Bimodal	$(0,1]$
Laplace (1)	0	6	Unimodal	$\mathbb{R}$
Cauchy (0)	Undefined	Undefined	Unimodal	$\mathbb{R}$
Students $t_{(2)}$	0	Undefined	Unimodal	$\mathbb{R}$
Trimodal Normal	0	1.53	Trimodal	$\mathbb{R}$

Table 2: Asymmetric Distributions

Distribution	Skewness	Kurtosis	Shape	Range
Gamma $(1/8, 1)$	5.65	48	Unimodal	$\mathbb{R}^+$
Beta(0.7, 0.2)	1.27	0.28	Bimodal	$(0,1]$
Burr	-	Undefined	Unimodal	$\mathbb{R}^+$
Type-I-extreme	1.14	5.40	Unimodal	$(0,1]$
Double centered gamma	0	6.20	Unimodal	$\mathbb{R}$
Stieltje	0	458	Multimodal	$\mathbb{R}$

Table 3: Rejection Frequencies for the Standard Normal Distribution (Symmetric)

N	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
10	0.016	0.014	0.034	0.042	0.037	0.035	0.034	0.050	0.014	0.013
20	0.031	0.045	0.030	0.035	0.032	0.046	0.043	0.045	0.013	0.044
30	0.035	0.048	0.031	0.036	0.033	0.046	0.047	0.052	0.023	0.056
50	0.042	0.052	0.034	0.037	0.035	0.051	0.049	0.052	0.025	0.072
75	0.045	0.051	0.037	0.039	0.038	0.060	0.059	0.051	0.025	0.084
100	0.047	0.053	0.041	0.043	0.042	0.049	0.049	0.049	0.030	0.043
200	0.051	0.049	0.044	0.045	0.045	0.050	0.050	0.054	0.036	0.031
500	0.048	0.048	0.046	0.046	0.046	0.046	0.047	0.047	0.040	0.031

Table 4. Rejection Frequencies for the Standard Uniform Distribution (Symmetric)

N	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
10	0.003	0.015	0.033	0.052	0.037	0.098	0.098	0.069	0.015	0.015
20	0.003	0.050	0.042	0.059	0.049	0.137	0.094	0.063	0.021	0.035
30	0.002	0.056	0.050	0.062	0.062	0.155	0.104	0.068	0.038	0.047
50	0.002	0.056	0.055	0.060	0.060	0.166	0.100	0.065	0.036	0.047
75	0.001	0.056	0.056	0.058	0.058	0.189	0.112	0.070	0.044	0.051
100	0.001	0.056	0.056	0.058	0.057	0.183	0.102	0.067	0.044	0.049
200	0.001	0.055	0.056	0.055	0.056	0.197	0.010	0.067	0.045	0.050
500	0.001	0.050	0.050	0.050	0.050	0.209	0.107	0.067	0.043	0.047

Table 5. Rejection Frequencies for the Beta Distribution (Symmetric)

N	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
10	0.007	0.044	0.064	0.088	0.061	0.201	0.201	0.178	0.041	0.041
20	0.006	0.082	0.071	0.100	0.073	0.300	0.218	0.162	0.055	0.067
30	0.004	0.074	0.066	0.087	0.070	0.344	0.233	0.200	0.067	0.070
50	0.004	0.077	0.058	0.080	0.069	0.405	0.241	0.196	0.063	0.067
75	0.004	0.067	0.048	0.071	0.069	0.451	0.254	0.210	0.066	0.067
100	0.003	0.065	0.046	0.067	0.069	0.481	0.254	0.203	0.057	0.058
200	0.003	0.057	0.041	0.059	0.061	0.536	0.251	0.200	0.055	0.057
500	0.002	0.053	0.052	0.059	0.054	0.591	0.248	0.206	0.053	0.053

Table 6. Rejection Frequencies for the Students $t_{(2)}$ Distribution (Symmetric)

N	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
10	0.250	0.060	0.005	0.004	0.006	0.020	0.020	0.093	0.057	0.057
20	0.469	0.080	0.010	0.003	0.016	0.021	0.026	0.072	0.036	0.131
30	0.579	0.075	0.014	0.010	0.023	0.023	0.031	0.096	0.048	0.148
50	0.697	0.057	0.013	0.021	0.026	0.024	0.031	0.099	0.037	0.169
75	0.766	0.046	0.015	0.033	0.024	0.025	0.030	0.091	0.039	0.178
100	0.808	0.034	0.014	0.031	0.0250	0.023	0.031	0.087	0.035	0.065
200	0.883	0.024	0.018	0.040	0.030	0.023	0.029	0.096	0.039	0.042
500	0.940	0.018	0.026	0.043	0.040	0.022	0.029	0.094	0.042	0.041

Table 7. Rejection Frequencies for the Trimodal Normal Distribution (Symmetric)

N	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
10	0.008	0.038	0.010	0.041	0.022	0.210	0.210	0.149	0.041	0.041
20	0.004	0.073	0.071	0.113	0.091	0.278	0.215	0.149	0.049	0.055
30	0.004	0.072	0.137	0.121	0.134	0.283	0.209	0.156	0.052	0.054
50	0.004	0.073	0.194	0.114	0.151	0.311	0.200	0.157	0.047	0.046
75	0.004	0.068	0.218	0.101	0.141	0.327	0.190	0.138	0.038	0.039
100	0.004	0.066	0.228	0.092	0.133	0.342	0.177	0.136	0.033	0.031
200	0.004	0.058	0.245	0.075	0.108	0.375	0.192	0.133	0.034	0.028
500	0.003	0.057	0.236	0.066	0.088	0.398	0.213	0.133	0.039	0.037

Table 8. Rejection Frequencies for the Cauchy Distribution (Symmetric)

N	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
10	0.513	0.129	0.007	0.003	0.009	0.016	0.0166	0.167	0.115	0.115
20	0.756	0.086	0.012	0.002	0.021	0.018	0.0215	0.150	0.067	0.183
30	0.840	0.066	0.016	0.004	0.025	0.016	0.0216	0.183	0.076	0.190
50	0.905	0.040	0.016	0.014	0.029	0.016	0.0201	0.189	0.053	0.190
75	0.937	0.028	0.018	0.019	0.025	0.020	0.0258	0.190	0.059	0.190
100	0.957	0.024	0.020	0.021	0.028	0.018	0.0219	0.189	0.056	0.091
200	0.975	0.014	0.020	0.022	0.024	0.016	0.0188	0.194	0.053	0.059
500	0.990	0.011	0.017	0.019	0.020	0.015	0.0202	0.187	0.051	0.054

Table 9. Rejection Frequencies for the Khintchine Distribution (Symmetric)

N	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
10	0.277	0.085	0.008	0.006	0.011	0.019	0.019	0.116	0.073	0.073
20	0.490	0.104	0.014	0.005	0.029	0.019	0.022	0.119	0.057	0.145
30	0.573	0.091	0.021	0.011	0.041	0.019	0.023	0.139	0.062	0.160
50	0.654	0.071	0.019	0.034	0.044	0.023	0.027	0.148	0.043	0.158
75	0.714	0.055	0.020	0.045	0.051	0.026	0.031	0.153	0.046	0.158
100	0.742	0.050	0.018	0.044	0.051	0.024	0.028	0.155	0.047	0.065
200	0.802	0.038	0.021	0.048	0.057	0.029	0.034	0.166	0.044	0.040
500	0.830	0.034	0.020	0.046	0.054	0.029	0.037	0.174	0.043	0.042

Table 10. Rejection Frequencies for the Laplace Distribution (Symmetric)

N	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
10	0.101	0.040	0.004	0.008	0.005	0.020	0.020	0.067	0.033	0.033
20	0.210	0.076	0.009	0.005	0.014	0.022	0.026	0.061	0.027	0.089
30	0.275	0.079	0.009	0.013	0.016	0.019	0.025	0.067	0.040	0.111
50	0.333	0.068	0.012	0.025	0.020	0.019	0.022	0.068	0.032	0.125
75	0.378	0.064	0.011	0.036	0.025	0.018	0.024	0.067	0.036	0.135
100	0.406	0.058	0.013	0.038	0.029	0.019	0.022	0.068	0.038	0.053
200	0.454	0.049	0.016	0.046	0.035	0.015	0.021	0.065	0.041	0.035
500	0.491	0.047	0.016	0.051	0.044	0.018	0.022	0.072	0.041	0.041

Table 11. Rejection Frequencies for the skewed Beta Distribution (Asymmetric)

N	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
10	0.306	0.532	0.354	0.141	0.291	0.431	0.431	0.821	0.493	0.493
20	0.649	0.870	0.842	0.559	0.854	0.712	0.578	0.956	0.825	0.867
30	0.861	0.967	0.951	0.896	0.957	0.853	0.755	0.998	0.955	0.965
50	0.986	0.998	0.994	0.999	0.997	0.976	0.913	1	0.997	0.998
75	1	1	1	1	1	0.999	0.985	1	1	1
100	1	1	1	1	1	1	0.992	1	1	1
200	1	1	1	1	1	1	1	1	1	1
500	1	1	1	1	1	1	1	1	1	1

Table 12. Rejection Frequencies for the Burr Distribution (Asymmetric)

N	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
10	0.542	0.522	0.155	0.082	0.108	0.164	0.164	0.502	0.477	0.476
20	0.918	0.666	0.614	0.113	0.559	0.343	0.232	0.673	0.793	0.941
30	0.990	0.576	0.810	0.161	0.784	0.477	0.317	0.899	0.969	0.996
50	1	0.483	0.928	0.421	0.923	0.713	0.443	0.984	0.997	1
75	1	0.435	0.968	0.653	0.961	0.917	0.682	0.999	1	1
100	1	0.426	0.982	0.731	0.978	0.948	0.703	1	1	1
200	1	0.418	0.994	0.871	0.991	0.999	0.929	1	1	1
500	1	0.434	0.997	0.944	0.996	1	1	1	1	1

Table 13. Rejection Frequencies for the Chi-square (9 d.f.) Distribution (Asymmetric)

N	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
10	0.065	0.055	0.062	0.084	0.072	0.049	0.049	0.094	0.042	0.042
20	0.216	0.196	0.092	0.121	0.104	0.087	0.065	0.113	0.089	0.226
30	0.358	0.314	0.175	0.204	0.187	0.104	0.076	0.168	0.226	0.418
50	0.609	0.515	0.377	0.394	0.384	0.139	0.082	0.256	0.376	0.706
75	0.803	0.697	0.614	0.608	0.612	0.269	0.147	0.360	0.588	0.902
100	0.911	0.807	0.790	0.774	0.785	0.246	0.111	0.418	0.631	0.853
200	0.998	0.954	0.981	0.978	0.980	0.432	0.176	0.711	0.915	0.971
500	1	0.996	1	1	1	0.799	0.351	0.977	1	1

Table 14. Rejection Frequencies for the Gamma Distribution (Asymmetric)

N	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
10	0.824	0.915	0.222	0.034	0.180	0.593	0.593	0.962	0.882	0.882
20	0.997	0.708	0.598	0.048	0.587	0.859	0.759	0.998	1	1
30	1	0.562	0.804	0.134	0.799	0.962	0.924	1	1	1
50	1	0.505	0.963	0.542	0.962	0.999	0.989	1	1	1
75	1	0.515	0.996	0.804	0.996	1	1	1	1	1
100	1	0.539	1	0.913	1	1	1	1	1	1
200	1	0.625	1	0.996	1	1	1	1	1	1
500	1	0.808	1	1	1	1	1	1	1	1

Table 15. Rejection Frequencies for the Stieltje Distribution (Asymmetric)

N	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
10	0.660	0.110	0.040	0.000	0.030	0.100	0.100	0.300	0.147	0.147
20	0.870	0.300	0.020	0.050	0.060	0.120	0.160	0.350	0.113	0.270
30	0.920	0.240	0.050	0.090	0.050	0.110	0.230	0.430	0.159	0.330
50	0.950	0.250	0.080	0.220	0.100	0.310	0.530	0.530	0.163	0.350
75	0.910	0.130	0.050	0.220	0.060	0.520	0.760	0.720	0.272	0.304
100	0.900	0.180	0.020	0.290	0.060	0.520	0.820	0.730	0.354	0.180
200	0.950	0.050	0.030	0.220	0.030	0.910	0.970	0.950	0.703	0.870
500	1	0.080	0.010	0.380	0.030	1	1	1	0.984	0.950

Table 16. Rejection Frequencies for the Double Centred Gamma Distribution (Asymmetric)

N	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
10	0.199	0.108	0.013	0.010	0.013	0.051	0.051	0.162	0.104	0.096
20	0.420	0.201	0.087	0.021	0.095	0.106	0.098	0.250	0.146	0.227
30	0.522	0.209	0.147	0.084	0.169	0.145	0.138	0.363	0.193	0.244
50	0.603	0.191	0.266	0.174	0.295	0.252	0.214	0.529	0.198	0.223
75	0.658	0.175	0.408	0.222	0.443	0.289	0.215	0.691	0.247	0.202
100	0.682	0.168	0.520	0.255	0.552	0.493	0.394	0.810	0.371	0.140
200	0.731	0.143	0.782	0.388	0.770	0.810	0.654	0.972	0.559	0.097
500	0.756	0.121	0.938	0.714	0.922	0.997	0.961	1	0.883	0.163

Table 17. Rejection Frequencies for the Type-I Extreme Value Distribution (Asymmetric)

N	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
10	0.083	0.063	0.008	0.014	0.012	0.051	0.051	0.103	0.055	0.055
20	0.272	0.214	0.025	0.028	0.050	0.080	0.063	0.118	0.105	0.252
30	0.434	0.328	0.033	0.111	0.078	0.099	0.070	0.188	0.243	0.445
50	0.680	0.506	0.061	0.346	0.182	0.137	0.081	0.266	0.398	0.704
75	0.863	0.673	0.113	0.650	0.327	0.267	0.150	0.387	0.616	0.887
100	0.946	0.771	0.144	0.823	0.456	0.252	0.119	0.456	0.680	0.853
200	0.999	0.917	0.256	0.992	0.808	0.452	0.182	0.754	0.937	0.966
500	1.000	0.986	0.407	1.000	0.998	0.832	0.394	0.987	1	1.000