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A Weighted Resampling for the Linear Estimator of ARCH Model

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Abstract

A Weighted Resampling Method is used to approximate the true distribution of the Linear Estimator (LE) of ARCH Models in finite sample. The Weighted Bootstrapped Linear Estimator is obtained by solving linear equations and hence the approach is easy to implement. Using a class of Weighted Resampling Schemes, it is found that there are schemes that can match and even perform better than the commonly used Paired Bootstrap Scheme. Using the Linear Estimator, instead of the Quasi-Maximum Likelihood Estimator for fitting ARCH Model enables us to obtain these results in very little time.

Keywords

ARCH model, Linear estimator (LE), Quasi-maximum likelihood estimator (QMLE), Bootstrapping.

1. Introduction

The Autoregressive Conditional Heteroscedastic (ARCH) Model was introduced by Engle (1982) to describe the volatility of the current return of an asset as a linear function of the squares of its past returns. The estimation of ARCH Model is often carried out using the Quasi-Maximum Likelihood Estimator (QMLE). The Asymptotic properties of the QMLE for ARCH Model under the existence of fourth-order moment on the ARCH process were established by Weiss (1986). The QMLE does not admit a closed form expression. Numerical optimization methods must be used to obtain the estimates.

The Linear Estimator (LE) for the parameters of ARCH model was proposed by Bose and Mukherjee (2003).

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The Linear Estimator has a closed form and is obtained by solving linear equations. The quick computation of the LE enables one to perform computer intensive tasks such as Bootstrapping ARCH Models in little time.

Efron (1979) introduced the idea of bootstrapping which is a general approach to statistical inference based on building a sampling distribution for a statistic by Resampling from the data at hand. For a comprehensive introduction of Bootstrap Methods, see Efron and Tibshirani (1993). Chatterjee and Bose (2005) introduced a Bootstrap Technique for estimators obtained by solving estimating equations. They call it Generalized Bootstrap (GBS) because Classical Bootstrap, the Delete-d Jackknife and variations of the Bayesian Bootstrap are shown to be some special cases of GBS. Examples of GBS weights and their implementation in Heteroscedastic Time Series, Generalized Linear Models and Nonlinear Regression Models are also discussed.

In this paper, we use the idea of Weighted Resampling and develop suitable Bootstrap versions for the Linear Estimator of ARCH models. We Bootstrap the Linear Estimators and our goal is to approximate the sampling distribution of the parameters with this new approach to Resampling.

The rest of the paper is organized as follows. In Section 2, we define the Linear Estimator for ARCH Model. In Section 4, Monte Carlo Simulations are performed to check the accuracy of LE in estimating the parameters of ARCH Models. The forecasts of volatility are also obtained and application to real data sets is presented. Finally, Section 5 concludes the results.

2. The Linear Estimator for ARCH Model

Consider the following ARCH Model where one observes;

$\{X_t; 1-p \leq t \leq T\}$

Satisfying,

$$X_t = h_t^{1/2}(\beta)\epsilon_t; \quad 1 \leq t \leq T \tag{1.1}$$

where,

$\beta = [\beta_0, \beta_1, \dots, \beta_p]'$ is the unknown parameters to be estimated with

$\beta_0 > 0, \beta_j \geq 0, 1 \leq j \leq p.$

$h_t = \beta_0 + \beta_1 X_{t-1}^2 + \cdots + \beta_p X_{t-p}^2$ with $\{\epsilon_t; 1 \le t \le T\}$ are independently and identically distributed with mean zero and unit variance. It is assumed that $\{\epsilon_t; 1 \le t \le T\}$ are independent of $\{X_t; 1-p \le t \le 0\}$. It is also assumed that (1.1) holds $\{X_t; t \ge 1-p\}$ is a stationary and ergodic and $E(\epsilon^4) < \infty$.

Let,
$Y_t = X_t^2, 1-p \le t \le T,$
$\mathrm{Z}_{\mathrm{t}-1} = \left[1, Y_{t-1}, \dots, Y_{t-p}\right]' = \left[1, X_{t-1}^2, \dots, X_{t-p}^2\right]'$
And
$\eta_t = \epsilon_t^2 - 1, 1 \le t \le T.$
Then squaring both sides of (1.1) and using the form $h_{t-1}(\beta) = \mathrm{Z}_{t-1}'\,\beta$, we get;

$$Y_t = \mathrm{Z}_{t-1}'\beta + h_{t-1}(\beta)\,\eta_t \qquad 1 \le t \le T \tag{1.2}$$

where,

$$E\{h_{t-1}(\beta)\eta_t\} = E\{h_{t-1}(\beta)\}E\{\eta_t\} = 0$$

Bose and Mukherjee (2003) define a preliminary Least Squares Estimator $\hat{\beta}_{pr}$ of β as the solution of

$$\sum_{t=1}^{T}[\mathrm{Z}_{t-1}'\,\{Y_t - \mathrm{Z}_{t-1}'\,\beta\,\}] = 0 \tag{1.3}$$

which yields the estimator

$$\hat{\beta}_{pr} = (\mathrm{Z}'\mathrm{Z})^{-1}\mathrm{Z}'\mathrm{Y}$$

where,
Z is the $T\times(1+p)$ matrix whose t^th^ row equals Z_{t-1}' and Y is the vector with entry t^th^ $Y_t,\ 1 \le t \le T$.

An improved estimator $\hat{\beta}$ of β can be obtained as follows. Dividing (1.2) by $h_{t-1}(\beta)$, we get;

$$\frac{Y_t}{\mathrm{Z}_{t-1}'\,\beta} = \left(\frac{\mathrm{Z}_{t-1}}{\mathrm{Z}_{t-1}'\beta}\right)'\beta + \eta_t$$

Now replacing $\mathrm{Z}_{t-1}'\beta$ by $\mathrm{Z}_{t-1}'\hat{\beta}_{pr}$ yields

$$\frac{Y_t}{\mathrm{Z}_{t-1}'\,\hat{\beta}_{pr}} \approx \left(\frac{\mathrm{Z}_{t-1}}{\mathrm{Z}_{t-1}'\hat{\beta}_{pr}}\right)'\beta + \eta_t$$

Therefore, a Linear Estimator of β is defined as the solution of

$$\sum_{t=1}^{T}\left[\left\{\frac{\mathrm{Z}_{t-1}}{\mathrm{Z}_{t-1}\hat{\beta}_{pr}}\right\}\left\{\frac{Y_t}{\mathrm{Z}_{t-1}'\,\hat{\beta}_{pr}} - \left(\frac{\mathrm{Z}_{t-1}}{\mathrm{Z}_{t-1}'\hat{\beta}_{pr}}\right)'\beta\right\}\right] = 0 \tag{1.4}$$

yielding the Linear Estimator

$$\hat{\beta}_T = \left[\sum_{t=1}^{T}\left\{\frac{\mathrm{Z}_{t-1}\mathrm{Z}_{t-1}'}{\left(\mathrm{Z}_{t-1}'\hat{\beta}_{pr}\right)^2}\right\}\right]^{-1}\left[\sum_{t=1}^{T}\left\{\frac{\mathrm{Z}_{t-1}Y_t}{\left(\mathrm{Z}_{t-1}'\hat{\beta}_{pr}\right)^2}\right\}\right] \tag{1.5}$$

It is shown in Bose and Mukherjee (2003) that under the model assumptions
$T^{1/2}(\hat{\beta}_T - \beta) \rightarrow N[0, \mathrm{Var}(\epsilon_1^2)\{\mathrm{E}(\mathrm{Z}_0\mathrm{Z}_0'(\beta'\mathrm{Z}_0)^{-2})\}^{-1}]$

3. A Weighted Resampling for Linear Estimator

Chatterjee and Bose (2005) developed the idea of Weighted Bootstrap of estimators that have been obtained as solution of minimizing problems or as solution of equations in general dependent models.

Let,
$\{w_{Tt}; 1 \le t \le T, T \ge 1\}$
be a triangular array of random variables such that for each
$T \ge 1, \{w_{Tt}; 1 \le t \le T\}$
are exchangeable, independent of $\{X_t; t \ge 1-p\}$ and $\{\epsilon_t; t \ge 1\}$ and $E(w_{Tt}) = 1$.
These are called Bootstrap Weights.

The Bootstrap Preliminary Least Squares Estimator $\hat{\beta}_{pr}^*$ of β is defined by mimicking (2.3), as the solution of

$$\sum_{t=1}^{T} w_{Tt}[\mathrm{Z}_{t-1}'\{Y_t - \mathrm{Z}_{t-1}'\beta\}] = 0 \quad (1.6)$$

Similarly, as in (1.4), the Bootstrap Linear Estimator $\hat{\beta}_T^*$ may be defined as a solution of

$$\sum_{t=1}^{T} w_{Tt}\left[\left\{\frac{\mathrm{Z}_{t-1}}{\mathrm{Z}_{t-1}\hat{\beta}^*_{pr}}\right\}\left\{\frac{Y_t}{\mathrm{Z}_{t-1}'\hat{\beta}^*_{pr}} - \left(\frac{\mathrm{Z}_{t-1}}{\mathrm{Z}_{t-1}'\hat{\beta}^*_{pr}}\right)'\beta\right\}\right] = 0 \quad (1.7)$$

which gives

$$\hat{\beta}_T^* = \left[\sum_{t=1}^{T} w_{Tt}\left\{\frac{\mathrm{Z}_{t-1}\mathrm{Z}_{t-1}'}{(\mathrm{Z}_{t-1}'\hat{\beta}^*_{pr})^2}\right\}\right]^{-1}\left[\sum_{t=1}^{T} w_{Tt}\left\{\frac{\mathrm{Z}_{t-1}Y_t}{(\mathrm{Z}_{t-1}'\hat{\beta}^*_{pr})^2}\right\}\right]$$

We assume the following basic conditions (Conditions Bootstraps Weight (BW) of Chatterjee and Bose (2005)) where $\sigma_T^2 = V_B(w_{Tt})$ and $k_1 > 0$ is a constant. $V_B(w_{Tt})$ is a variance of (w_{Tt}) based on Bootstraps samples. The conditions on Weights are as under:
$E_B(w_{T1}) = 1, \quad 0 < k_1 < \sigma_T^2 = O(T)$ and $\mathrm{corr}_B(w_{T1}w_{T2}) = O(T^{-1})$.

Similar to Chatterjee and Bose (2005, Theorem 3.2), it can be shown that under some technical assumptions on the correlation structure of Bootstrap Weights, the

distribution of $\sqrt{\mathrm{T}}\,(\hat{\beta}_{\mathrm{T}} - \beta)$ can be approximated by the distribution of $\sigma_T^{-1}\sqrt{\mathrm{T}}\,\left(\hat{\beta}_T^{*} - \hat{\beta}_{\mathrm{T}}\right)$ outside a set of probability zero.
where,
σ_T^2 denotes the variance of w_{T1}.
We approximate such distribution via Weighted Bootstrap.

Three different schemes for weights are considered. These are;

- Scheme M, when weights have a Multinomial (T,1/T,...,1/T) Distribution.
- Scheme U, when $w_{Tt} = U_t/\bar{U}$ where $U_t{}'$s are independently and identically distributed Uniform (0.5, 1.5) and $\bar{U} = \sum_{t=1}^{T} U_t/T$.
- Scheme E, when $w_{Tt} = E_t/\bar{E}$ where $E_t{}'$s are independently and identically distributed Exponential (1) and $\bar{E} = \sum_{t=1}^{T} E_t/T$.

Note that, Scheme M corresponds to the commonly used Paired Bootstrap in Heteroscedastic Models. We empirically study Schemes U and E as possible alternatives to the Paired Bootstrap. We also consider residual Bootstrap when Standardized residuals are Bootstrapped to form a new Bootstrapped return series. Using this Bootstrapped series, the Bootstrapped parameters are estimated and the Bootstrapped Distributions of the parameters are obtained. It is also possible to obtain quantiles of the Bootstrap Distribution of $\sigma_T^{-1}\sqrt{\mathrm{T}}\,\left(\hat{\beta}_T^{*} - \hat{\beta}_{\mathrm{T}}\right)$ using Simulation and then using the Bootstrap approximation, we can construct the Bootstrap Confidence Intervals of β.

4. Simulation Results

This section reports the results of a Monte Carlo Simulation. We investigate the quality of Bootstrap approximation to the finite sample distribution of
$\sqrt{\mathrm{T}}\,(\hat{\beta}_{\mathrm{T}} - \beta)$. We use a sample of size T, and assume that the underlying Error Distributions of $\{\epsilon_t\}$ be standard normal. An ARCH (p) model is fitted to the data set using the Linear Estimator.

In our first study, we generate *K*=10,000 samples each of size *T*=50, 250, and 500 from the ARCH (3) model with $\beta = [0.01, 0.1, 0.2, 0.2]'$.

Let,
$\hat{\beta}_{T(k)} = [\hat{\beta}_{T0}, \hat{\beta}_{T1}, \hat{\beta}_{T2}, \hat{\beta}_{T3}]'$
denote the vector of estimated parameters computed from the kth sample;
$1 \le k \le K$.

For each replication we compute;
$\sqrt{T}\left(\hat{\beta}_{Tj}-\beta_j\right),\ 0 \le j \le p = 3$

The mean and average of the squares of the three sets over K replications represent the mean and the Mean Squared Error (MSE) of $\sqrt{T}\left(\hat{\beta}_{Tj}-\beta_j\right)$, $0 \le j \le p = 3$.

The estimates of means under Normal approximation are zero. The estimate of MSE using the Normal approximation is obtained by averaging over K estimated MSEs where the k$^{\text{th}}$ $(1 \le k \le K)$ estimate is obtained from the diagonals of the matrix $\hat{V}\left[T^{-1}\sum_{t=1}^{T}\left\{Z_{t-1}Z'_{t-1}\left(\hat{\beta}'_T Z_{t-1}\right)^{-2}\right\}\right]^{-1}$
where,
$\hat{V}$ is the variance of $\{\epsilon_1^2, \ldots, \epsilon_T^2\}$ and $\hat{\epsilon}_t = X_t/\left(\hat{\beta}'_T Z_{t-1}\right)^{1/2},\ 1 \le t \le T,\ \hat{\beta}_T$ being the estimate based on the k$^{\text{th}}$ replication.

Table 1 shows results of means, MSEs and MSE under Normal approximations. It can be seen that the true means of the distributions of all parameters except β_0, are significantly different from the Normal approximation values. The MSEs for small sample sizes are also found different than the Normal approximations. For T=500, the values of the MSE match that of the MSE under Normal approximations.

Next, we turn our attention to Bootstrap approximations. To approximate the distribution of $\sqrt{\mathrm{T}}\,(\hat{\beta}_{\mathrm{T}}-\beta)$, we proceed as follows. We choose and fix $\hat{\beta}_{T(r)}\ (1 \le r \le R),\ R \le K$.

In this study, we generate B=999 Bootstrap samples. Bootstrap results are based on R=100 replications. For Weighted Resampling, these Bootstrap samples are generated based on weights under three schemes, Scheme M, Scheme U and Scheme E, after fixing $\hat{\beta}_{T(r)}\ (1 \le r \le R)$. For the b$^{\text{th}}$ sample, $1 \le b \le B$, we compute $\sigma_T^{-1}\sqrt{T}(\hat{\beta}^*_{T(b)}-\hat{\beta}_{T(r)})$. For residual Bootstrap, we generate B=999 Bootstrap samples and for the b$^{\text{th}}$ sample compute $\sigma_T^{-1}\sqrt{T}(\hat{\beta}^*_{T(b)}-\hat{\beta}_{T(r)})$, after fixing $\hat{\beta}_{T(r)}\ (1 \le r \le R)$.

Table 2 reports the results of means and the MSEs of the distribution of the Standardized Bootstrap Estimators under residual Bootstrap and three different schemes. Entries in bold represent MSEs those provide the closest approximations to those in Table 1. The Bootstrapped approximations of means do not match the corresponding estimated means in most of the cases except under the residual Bootstrap at T=500. The residual Bootstrap method seems to capture the sign of the means of all parameters correctly for almost all sample sizes.

The summary of results based on the MSE of the distributions of all parameters under all cases is as follows;

For T=50, all schemes provide close estimate for $\hat{\beta}_0$, scheme E for $\hat{\beta}_1$ and $\hat{\beta}_3$, and residual Bootstrap for $\hat{\beta}_2$. For T=250, again the MSEs of the distribution of $\hat{\beta}_0$ are very well approximated by all schemes, scheme E for $\hat{\beta}_1$, and scheme U for both $\hat{\beta}_2$ and $\hat{\beta}_3$ can be considered better than other schemes. And finally, for T=500, the Bootstrap approximation for the MSE of $\hat{\beta}_0$, under all schemes, provide accurate results. For the same sample size, scheme E, residual Bootstrap and scheme U provide close approximations for $\hat{\beta}_1$, $\hat{\beta}_2$ and $\hat{\beta}_3$, respectively. These results conclude that although there is no clear cut selection for schemes, the widely used scheme M is out performed by other schemes we used in our analysis.

We have performed experiments with different sample sizes and ARCH orders and the results of those Simulations have been found similar to our first study. These results can be obtained from the author upon request.

Using our own MATLAB and FORTRAN code, we checked the CPU time (in sec) taken by both LE and the QMLE for estimating an ARCH (3) Model. Experiment was performed on a Pentium CPU with Intel Core 2 Duo process running at 2 Ghz and having 2 GB of random access memory (RAM). The sample size used was T = 1,000 and the experiment was repeated K = 10,000 times. Linear Estimator took 215.55 second, whereas, the QMLE took 902.43 second for estimating the same data sets. We also computed the MSE for the parameters and the difference between the two estimators for this large sample size was negligible. This clearly reveals the advantage of using the LE for estimating the parameters of ARCH Models. The LE takes around one-fourth of the time than the QMLE and also is not only efficient but also estimates the parameters as accurately as the QMLE. This difference becomes very significant when a Computer Intensive Method such as Resampling is used for ARCH Models.

5. Conclusion

A Weighted Resampling Method for the Linear Estimator for the parameters of ARCH Models is presented. It is found in this study that Weighted Bootstrap Schemes work well for ARCH Models when LE is used for estimation. We also found that schemes such as scheme U and scheme E are good alternative to scheme M. Finally, using LE instead of the QMLE for fitting ARCH Models enables us to obtain these results in very quick time.

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Table 1: Means and the MSEs of the Distributions of $\sqrt{T}\,(\hat{\beta}_T - \beta)$ for ARCH (3) Model and the MSE due to Normal approximation of the Distribution.

	T=50			**T=250**			**T=500**		
	Mean	**MSE**	**MSE_N**	**Mean**	**MSE**	**MSE_N**	**Mean**	**MSE**	**MSE_N**
$\boldsymbol{\beta_0 = 0.01}$	0.0043	0.0010	0.0009	0.0096	0.0011	0.0010	0.0096	0.0011	0.0010
$\boldsymbol{\beta_1 = 0.10}$	0.3106	0.7253	2.3588	0.1155	1.1062	1.6271	0.0126	1.3101	1.5818
$\boldsymbol{\beta_2 = 0.20}$	-0.1028	0.8923	2.6574	-0.4087	2.0168	2.0555	-0.3579	2.2632	2.0880
$\boldsymbol{\beta_3 = 0.20}$	-0.1119	0.9461	2.6099	-0.3669	2.0833	2.0862	-0.3583	2.2986	2.1263

Table 2: Means and the MSEs of the Distribution of the Standardized Bootstrap Estimators for ARCH (3) Model under different Schemes.

B=999	**Scheme M**		**Scheme U**		**Scheme E**		**Residual Boot**	
T=50	**Mean**	**MSE**	**Mean**	**MSE**	**Mean**	**MSE**	**Mean**	**MSE**
$\boldsymbol{\beta_0 = 0.01}$	-0.0079	**0.0009**	-0.0044	0.0013	-0.0086	0.0008	0.0076	0.0035
$\boldsymbol{\beta_1 = 0.10}$	0.2084	0.8097	0.1008	**1.4651**	0.1756	0.7690	0.0759	0.8191
$\boldsymbol{\beta_2 = 0.20}$	0.1654	0.9205	0.0769	**1.7891**	0.1307	0.8601	0.0736	0.8906
$\boldsymbol{\beta_3 = 0.20}$	0.0227	1.0108	0.0317	**2.1055**	0.0044	1.0082	-0.0898	1.1000
T=250	**Mean**	**MSE**	**Mean**	**MSE**	**Mean**	**MSE**	**Mean**	**MSE**
$\boldsymbol{\beta_0 = 0.01}$	-0.0052	0.0010	-0.0008	**0.0011**	-0.0045	0.0009	0.0047	0.0010
$\boldsymbol{\beta_1 = 0.10}$	0.2303	1.2915	0.0588	1.4598	0.1945	**1.2032**	0.1082	1.2416
$\boldsymbol{\beta_2 = 0.20}$	0.0982	1.6402	0.0106	**1.8967**	0.0613	1.5235	-0.1197	1.5939
$\boldsymbol{\beta_3 = 0.20}$	-0.0288	1.8742	-0.0430	**2.1741**	-0.0551	1.7376	-0.2123	1.8346
T=500	**Mean**	**MSE**	**Mean**	**MSE**	**Mean**	**MSE**	**Mean**	**MSE**
$\boldsymbol{\beta_0 = 0.01}$	-0.0003	0.0010	0.0007	0.0010	0.0001	0.0010	0.0079	**0.0010**
$\boldsymbol{\beta_1 = 0.10}$	0.1959	1.3506	0.0411	1.5165	0.1721	**1.3066**	0.0711	1.2910
$\boldsymbol{\beta_2 = 0.20}$	-0.0966	2.0314	-0.0494	2.1212	-0.1060	1.9078	-0.3006	**2.1318**
$\boldsymbol{\beta_3 = 0.20}$	-0.1216	1.9385	-0.0555	**2.0342**	-0.1213	1.8245	-0.2604	1.9985