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Recurrence Relations for the Product and Single Moments of k^{th} Lower Record Values of Inverse Weibull Distribution

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Abstract

In this paper, we deal with the recurrence relation for the product, ratio and single moments of the k^{th} lower record values from Inverse Weibull Distribution.

Keywords

Moments, Recurrence relations, k^{th} Lower record values, Inverse Weibull distribution

1. Introduction

For a fixed $k \geq 1$, we define the sequence $X^{(k)}_{l(1)}$, $X^{(k)}_{l(2)}$, ---, of the k^{th} lower record times of the sequence X_1, X_2,--- , *as follows:* $X^{(k)}_{l(1)} =1$ and for $n=2, 3,$ --- $X^{(k)}_{1(n)} = \min\left(j > X^{(k)}_{1(n-1)} : X_{k:Lk(n-1)+k-1} > X_{k:J+k-i}\right)$ with $X_{k:L_K(n)+k-1}$, $n \geq 1$. The sequence $\{X^{(k)}_n, n>=1\}$ is called the sequence of the k^{th} record values of the above sequence (see Bieniek and Szynal, 2002).Note that for $k=1$ we have $X^{(1)}_n = X_{1:L1(n)} = R^1_n$ are lower record values of the sequence $\left(X_i, i \geq 1\right)$ and $X^{(k)}_n = X_{k:n} = max(X_1, --- X_n)$.

The aim of this paper is to give a general procedure of obtaining recurrence relations for single and product moments of k^{th} lower record values Z^k_n . We obtain general relation for the moments of the k^{th} lower record values from Inverse Weibull Distribution and use these to give recurrence relations for the moments.

The probability density function of an Inverse Weibull Distribution is given by:

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$$f(x) = \frac{m}{\theta x^{m+1}} \exp\left(-\frac{1}{\theta x^m}\right) \tag{1.1}$$

where x>0 and (θ, m)>0.

And the corresponding distribution function is:

$$F(x) = \exp\left(-\frac{1}{\theta x^m}\right) \tag{1.2}$$

where x> 0 and (θ,m)>0

If we put m=1, it reduces to the Inverse Exponential Distribution. If we put m=2, it reduces to the Inverse Rayleigh Distribution. Some work has been done on Inverse Rayleigh Distribution by Gharraph (1993), Mukarjee and Mait (1996), Mukarjee and Saren (1984) and Voda (1972) and for the product and single moments of lower records values of Inverse Weibull Distribution, see Aleem (2005) and for Recurrence Relations for Quotient Moments of the Weibull Distribution by record values, see Chang (2007).

Let, $p(x) = \frac{m}{\theta x^{m=1}}$, using this relation in (1.1) and (1.2), we obtain:

$$F(x) = \frac{1}{p(x)} . f(x) \tag{1.3}$$

2. Product Moments

The kth lower record values are represented by $X^{(k)}_{l(1)}$, $X^{(k)}_{l(2)}$, ---, $X^{(k)}_{l(n)}$, then the joint pdf of $X^{(k)}_{L(r)}$ and $X^{(k)}_{L(s)}$ $(s>r)$ is given as:

$$f_{(r).(s)}\ (x^{(k)}, y^{(k)}) = k^S C_{r,s}\ [H(x)]^{r-1}\ [H(y) - H(x)]^{s-r-1}\ h(x)\ [F(x)]^{k-1} f(y) \tag{2.1}$$

where $C_{r,s} = -\frac{1}{(r-1)!(s-r-1)!}$ *and* $-\infty < y < x < \infty$

and $H(x) = -\ In\ F(x)$, $0 < F(x) < 1$

$h(x) = -\frac{d}{dx} H(x)$

If g is a Borel measurable function from R^2 to R, then:

$$E\{g(X^{(k)}{}_{L(r)}, X^{(k)}{}_{L(s)})\} = k^s\, C_{r,s} \iint_{0<y<x<\infty} g(x,y)\,[H(x)]^{r-1}[H(y)-H(x)]^{s-r-1}\, h(x)\,[F(x)]^{k-1} f(y)\,dxdy \quad (2.2)$$

Theorem 2.1

For the Distribution function *F(x)* in (1.2), we have:

$$\text{E}\{g(X^{(k)}{}_{L(r)}, X^{(k)}{}_{L(s)})\} = E\{u(X^{(k)}{}_{L(r-1)}, X^{(k)}{}_{L(s-1)})\} - E\{u(X^{(k)}{}_{L(r)}, X^{(k)}{}_{L(s-1)})\} - (k-1)E\{(u(X^{(k)}{}_{L(r)}, X^{(k)}{}_{L(s)})\} \quad (2.3)$$

where

$$u^{\bullet}(\text{x, y}) = g(x,y)\,.\,p(x) \text{ and } u^{\bullet}(\text{x, y}) = \frac{\partial}{\partial x}u(x,y)$$

Proof: Using (1.3) in (2.2), we have:

$$E\{g(X^{(k)}{}_{L(r)}, X^{(k)}{}_{L(s)})\} = k^s C_{r,s} \iint_{0<y<x<\infty} u^{\bullet}(x,y)[H(x)]^{r-1}[H(y)-H(x)]^{s-r-1}\,[F(x)]^{k-1} f(y)\,dx\,.dy \quad (2.4)$$

Upon integrating the R.H.S. of (2.4) we get:

$$= k^s C_{r-1,s-1} \iint_{0<y<x<\infty} u(x,y)[H(x)]^{r-2}[H(y)-H(x)]^{s-r-1} h(x)(F(x))^{k-1} f(y)\;dx\;dy$$

$$-\;k^s C_{r,s-1} \iint_{0<y<x<\infty} u(x,y)[H(x)]^{r-1}[H(y)-H(x)]^{s-r-2} h(x)(F(x))^{k-1} f(y)\;dx\;dy$$

$$-\;(\text{k-1})\,k^s C_{r,s} \iint_{0<y<x<\infty} u(x,y)[H(x)]^{r-1}[H(y)-H(x)]^{s-r-1} h(x)\;(F(x))^{k-1} f(y)\;dx\;dy$$

and

$$\text{E}\{g(X^{(k)}{}_{L(r)}, X^{(k)}{}_{L(s)})\} = E\{u(X^{(k)}{}_{L(r-1)}, X^{(k)}{}_{L(s-1)})\} - E\{u(X^{(k)}{}_{L(r)}, X^{(k)}{}_{L(s-1)})\} - (k-1)E\{(u(X^{(k)}{}_{L(r)}, X^{(k)}{}_{L(s)})\} \quad (2.5)$$

Hence, the theorem.

Theorem 2.2

For the Distribution function *F(x)* in (1.2), the Recurrence Relation for the product moments of Inverse Weibull Distribution is given by:

$$u^{(k)}{}_{(r),(s)}^{J,L} = \frac{m}{\theta(J-m)}[u^{(k)}{}_{(r-1),(s-1)}^{(J-m),L} - u^{(k)}{}_{(r),(s-1)}^{(J-m),L} - (k-1)u^{(k)}{}_{(r),(s)}^{(J-m),L}] \tag{2.6}$$

where

u (x,y) $= g(x,y)\,.\,p\;(x)$ and u (x, y)= $\dfrac{\partial}{\partial x}u(x,y)$

Proof: Now $p(x) = \dfrac{m}{\theta\, x^{m+1}}$ and *g(x,y)* = $x^J y^L$. This gives:

$u(x,y) = \dfrac{m}{\theta(J-m)} x^{J-m} y^L$, Now putting in Theorem (2.1), we get the required Recurrence Relation (2.6).

Corollary 2.2

By respected application of the Recurrence Relation (2.5), we obtain for *r, s≥1. j=0,1,2….* and *v=1,2,…, (r-1)*

$$u_{(r),(s-1)}^{J,L} = u_{(v),(s-1)}^{J-m,L} - \frac{\theta(j-m)}{m}\sum_{p=1}^{v}[u_{(p),(s)}^{(J-m),L} - \tfrac{1}{(k-1)}u_{(p),(s)}^{(J),L}] - \frac{\theta(j-m)}{m}[u_{(r),(s)}^{(J-m),L} - \tfrac{1}{(k-1)}u_{(r),(s)}^{(J),k}] \tag{2.7}$$

Remarks:

For *k=1*, the Recurrence Relations (2.5) & (2.6) become identical to Aleem (2005)

Note, for the ratio, let *k*= *-j***,** then $u_{(r),(s)}^{J,-J} = E\left(X_{(r)} / X_{(s)} \right)^J \;\forall J$

3. Single Moments

The lower record values are represented by $X^{(k)}{}_{L(1)}$, $X^{(k)}{}_{L(2)}$, ---, $X^{(k)}{}_{L(N)}$. The pdf of $X^{(k)}{}_{L(n)}\,(n \ge 2)$ is:

$$f_{(n)}(x) = \frac{k^{(n)}[H(x)]^{n-1}}{(n-1)!}(F(x))^{k-1} f(x) \tag{3.1}$$

where *H(x) = -LnF(x)* *0<F(x)<1*

$$h(x) = -\frac{d}{dx}H(x)$$

If g is a Borel measurable function from R^2 to R, then:

$$E\{g(X^{(k)}{}_{L(n)})\} = k^n C_n \int_{0<x<\infty} g(x)[H(x)]^{n-1}(F(x))^{k-1} f(x)dx \tag{3.2}$$

where $C_n = \frac{1}{(n-1)!}$

Theorem 3.1

For the Distribution function *F(x)* in (1.2), we have:

$$E\{g(X^{(k)}{}_{L(n)})\} = E\{u(x^{(k)}{}_{L(n-1)})\} - KE\{u(x^{(k)}{}_{L(n)})\} \tag{3.3}$$

Proof: Using (1.3) in (3.2), we have:

$$E\{g(X^{(k)}{}_{L(n)})\} = K^n C_n \int_{0<x<\infty} u^{\bullet}(x)[H(x)]^{n-1}\{F(x)\}^k dx \tag{3.4}$$

Upon integrating the R.H.S. of (3.4) we get:

$$= C_{n-1} \int_{0<x<\infty} u(x)[H(x)]^{n-2} f(x)dx - k C_n \int_{0<x<\infty} u(x)[H(x)]^{n-1} f(x)dx$$

and

$$E\{g(X^{(k)}{}_{L(n)})\} = E\{u(x^{(k)}{}_{L(n-1)})\} - kE\{u(x^{(k)}{}_{L(n)})\} \tag{3.5}$$

Hence, the theorem.

Theorem 3.2

For the Distribution function F(x) in (1.2) the Recurrence Relation for the single moments of Inverse Weibull Distribution is given by:

$$u_{(n)}^{J} = \frac{m}{\theta(J-m)}\left[u_{(n-1)}^{J-m} - k\,u_{(n)}^{J-m}\right] \tag{3.6}$$

Proof: Now $p(x) = \dfrac{m}{\theta\, x^{m+1}}$ and $g(X_{L(n)}) = x^{J}$ this gives:

$u(X_{L(n)}) = \dfrac{m}{\theta(J-m)}\, x^{J-m}$, putting in (3.1) we get the Recurrence Relation.

Corollary 3.2
By repeated application of the Recurrence Relation (3.2), we obtain for $n \geq 1$, $J=0,1,2,\ldots$ and $v=0,1,2,\ldots,n-2$

$$u_{(n)}^{j-m} = \frac{1}{k^{V+1}}\, u_{(n-v-1)}^{(J-m)} - \theta\,(j-m)\sum_{p=1}^{v} \frac{1}{k^{P+1}}\, u^{j}{}_{(n-p)} \tag{3.7}$$

Remarks:

(i) This Recurrence Relation in Theorem (2.2) between the moments of ratio of two k^{th} lower record values, Quasi-ranges, joint moment generating function, (and) characteristic functions can be obtained by setting respectively g(x,y) equal to:

$\left(x^{J} y^{-K}\right), (y-x), e^{T(X+Y)}, e^{iT(X+Y)}$.

(ii) For k=1 the Recurrence Relations (3.6) and (3.7) become identical to Aleem (2005).

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