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X- Chart Using ANOM Approach

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Abstract

Control limits for individual measurements (X) chart are available in the literature using moving range as an estimate of process standard deviation. This paper presents new control limits for X - chart using analysis of means (ANOM) approach. In deriving the control limits, sample standard deviation is used as an estimate of process standard deviation. The expected length of the interval between existing (old) control limits is compared with the expected length of the interval between new control limits for the same confidence coefficient and recommendations are made to the practitioner when to use the new control limits.

Keywords

Analysis of means, X-chart, Control limits

1. Introduction

For testing the equality of several population means, a graphical procedure, namely, Analysis of Means (ANOM) was introduced by Ott (1967) as an alternative to Analysis of Variance (ANOVA). In Ott's procedure, the results are summarized in an ANOM chart. This chart is similar in appearance to a control chart. Instead of control limits, decision lines are used in ANOM procedure. The main difference between ANOM chart and control chart is that the value of k (number of samples) is usually as large as 20 or more in control charts, whereas $k \geq 2$ in an ANOM chart. When there are exactly 2 means $(k = 2)$ the ANOM is simply a graphical form of Student's t - test. In ANOM chart, the sample mean values are compared to the overall grand mean, about which the upper and lower

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decision lines have been constructed. If a sample mean falls outside these decision lines, it is declared significantly different from the grand mean.

There are several advantages of ANOM plotting, e.g., (i) it provides a comparison of the relative importance and magnitude of the factors as well as their statistical significance, (ii) it provides a pin-pointing of sources of non-randomness, and (iii) it encourages the translation of conclusions into scientific action and for taking managerial decisions. Hence, ANOM plots reveal the statistical significance as well as practical significance of samples being compared.

Several authors extended the ANOM technique for comparing several (i) proportions (ii) counts (iii) treatment effects (iv) interaction effects (v) Linear contrasts (vi) variances (vii) correlation coefficients (viii) Regression coefficients (ix) intercepts (x) autocorrelation coefficients (xi) coefficients of variation. Recently, Nelson et al. (2005) wrote a book exclusively on ANOM graphical method for comparing means, rates and proportions. Rao (2005) reviewed papers on analysis of means starting from the first paper in 1967 upto 2004.

Kiani et al. (2008) developed new control limits for $\overline{X}$ - chart with unknown σ based on the t - distribution. With k subgroups, each with n observations, the control limits obtained by them are:

$$\overline{\overline{X}} \pm t_{\alpha/2,k(n-1)}\, S_b \sqrt{(k-1)/kn} \qquad (1.1)$$

where

$$S_b = \left(\sum_{i=1}^{k} \sum_{j=1}^{n} \left(X_{ij} - \overline{X}_i \right)^2 \Big/ k(n-1) \right)^{1/2}$$

$$\overline{\overline{X}} = \sum_{i=1}^{k} \overline{X}_i \Big/ k, \qquad \overline{X}_i = \sum_{j=1}^{n} X_{ij} \Big/ n$$

X_{ij} are random variables for $i = 1, 2, ..., k$ subgroups and $j = 1, 2, ..., n$ observations in each subgroup and $t_{\alpha/2,\ k(n-1)}$ is the critical value obtained from t - distribution with level of significance α and degrees of freedom $k(n-1)$.

It may be remarked here that the derivation of Shewhart control limits for $\overline{X}$ - chart depends on:

$$E\left(\overline{X}_i\right) \pm Z_{\alpha/2}\ S.E.\left(\overline{X}_i\right) \tag{1.2}$$

whereas the derivation of Kiani et al.(2008) control limits depends on:

$$E\left(\overline{X}_i\right) \pm Z_{\alpha/2}\ S.E.\left(\overline{X}_i - \overline{\overline{X}}\right) \tag{1.3}$$

as in ANOM approach.

Control limits for individual measurements(X) chart are available in the literature using moving range as an estimate of process standard deviation. This paper presents new control limits for X - chart using ANOM approach. The expected length of the interval between existing (old) control limits is compared with the expected length of the interval between new control limits for the same confidence coefficient.

2. Review of X-Chart and Existing Control Limits

The purpose of the X- chart or individual measurements chart is same as that of the $\overline{X}$ - chart to monitor the process mean. The assumptions for an X - chart are the same as the assumptions for an $\overline{X}$ - chart - normality and independence. The normality assumption is far more important when individual observations are plotted than it is when averages are plotted, since there is no central limit theorem- type effect with individual observations. Juran and Godfrey (2004, p. 45.11) suggest to collect 20 or more observations for a trial study. There are many situations in which the subgroup size used for process monitoring is 1, that is, the subgroup consists of an individual unit. Some examples of these situations are given in Montgomery (2001)

Existing (old) Control Limits

The Shewhart control limits for X - chart with known parameters are given by

$$\mu \pm Z_{\alpha/2}\,\sigma\ . \tag{2.1}$$

When the parameters μ and σ are unknown, the control limits for X- chart with k individual units $X_i\left(i = 1, 2, ..., k\right)$ are given by:

$$\left.\begin{aligned} UCL &= \overline{X} + Z_{\alpha/2}\frac{\overline{MR}}{d_2} \\ CL &= \overline{X} \\ LCL &= \overline{X} - Z_{\alpha/2}\frac{\overline{MR}}{d_2} \end{aligned}\right\} \tag{2.2}$$

where

$$\hat{\mu} = \overline{\mathrm{X}}, \quad \overline{X} = \frac{\sum_{i=1}^{k} X_i}{k}, \quad \hat{\sigma} = \frac{\overline{\mathrm{MR}}}{\mathrm{d}_2},$$

$$\mathrm{MR}_{\mathrm{i}} = \left\{ max\left(X_i\right) - min\left(X_i\right) \right\}, \quad for \quad i = 1, 2, \ldots, k \tag{2.3}$$

In the chosen $\mathrm{n}(\geq 2)$ successive observations and $\mathrm{n} < k$, d_2 is a function of n successive observations to obtain moving ranges. $\overline{\mathrm{MR}}$ is the average of all the MR_{i} computed from the sample of size k. $Z_{\alpha/2}$ is the critical value from Standard Normal Distribution with level of significance α.

3. Derivation of New Control Limits for X- Chart

In deriving the new control limits we will use the method of Kiani et al.(2008). Let $X_i\left(i = 1, 2, \ldots, k\right)$ denote the k individual measurements drawn from a normal population with mean μ and variance σ^2. Let $\overline{X} = \sum_{i=1}^{k} X_i \big/ k$ be the mean of $X_1, X_2, \ldots, X_k$ and $\overline{X} \sim N\left(\mu, \sigma^2/k\right)$. Considering $\left(X_i - \overline{X}\right)$ as a variate, we define:

$$Z_i = \frac{\left(X_i - \overline{X}\right) - E\left(X_i - \overline{X}\right)}{S.E.\left(X_i - \overline{X}\right)}. \tag{3.1}$$

On deriving $E\left(X_i - \overline{X}\right)$ and $S.E.\left(X_i - \overline{X}\right)$, we obtain:

$$E\left(X_i - \overline{X}\right) = 0$$

$$S.E.\left(X_i - \overline{X}\right) = \sigma\sqrt{(k-1)/k} \quad . \tag{3.2}$$

We know that:

$$Z_i = \frac{\left(X_i - \overline{X}\right)}{S.E.\left(X_i - \overline{X}\right)} \sim N(0,1). \tag{3.3}$$

Substituting (3.2) in (3.3), we get:

$$Z_i = \frac{\left(X_i - \overline{X}\right)}{\sigma\sqrt{\frac{k-1}{k}}} \sim N(0,1). \tag{3.4}$$

We know that:

$$\frac{(k-1)s^2}{\sigma^2} \sim \chi^2_{k-1}, \tag{3.5}$$

where

$$s^2 = \frac{1}{k-1}\sum_{i=1}^{k}\left(X_i - \overline{X}\right)^2 \text{ with degrees of freedom } \nu = k-1.$$

We have:

$$\sqrt{\frac{\chi^2}{\nu}} = \sqrt{\frac{s^2}{\sigma^2}} = \frac{s}{\sigma} \tag{3.6}$$

Since sample mean and sample variance are independently distributed, Z_i and χ^2 are independent random variables. Hence, Fisher's t is given by:

$$t_i = \frac{Z_i}{\sqrt{\chi^2/\nu}} \sim t_\nu. \tag{3.7}$$

Substituting (3.4) and (3.6) in (3.7), we get:

$$t_i = \frac{\left(X_i - \overline{X}\right)}{s\sqrt{\left(\frac{k-1}{k}\right)}} \sim t_{k-1}. \tag{3.8}$$

As a result, the new control limits for X-chart are obtained as:

$$\left.\begin{aligned} LCL &= \overline{X} - t_{\alpha/2,\, k-1} s\sqrt{(k-1)/k} \\ CL &= \overline{X} \\ UCL &= \overline{X} + t_{\alpha/2,\, k-1} s\sqrt{(k-1)/k} \end{aligned}\right\} \tag{3.9}$$

where

$$P\left(\left|t_i\right| \le t_{\alpha/2,\, k-1}\right) = 1-\alpha . \tag{3.10}$$

Plot the individual measurements $\left(X_i\right)$ against the control limits in (3.9) and if any point falls outside the control limits, the process is said to be out of control with respect to the process mean.

4. Comparison of New Control Limits with Existing Control Limits

The new control limits of X - chart using ANOM approach are given in (3.9) and the length of the interval between the control limits is given by:

$$L_1 = 2t_{\alpha/2,k-1} s\sqrt{(k-1)/k} . \tag{4.1}$$

and the expected length is:

$$E\left(L_1\right) = 2t_{\alpha/2,k-1}\, c_4 \sigma\sqrt{k-1/k} \quad , \tag{4.2}$$

since $E\left(s\right) = c_4\sigma$, where c_4 is a function of k (sample size) and is tabulated in several quality control books for $k \le 25$. When k >25, c_4~ (see Montgomery, 2001). The existing (old) control limits of X- chart using moving ranges are given in (2.2) and the length of the interval between the control limits is given by:

$$L_2 = 2Z_{\alpha/2}\frac{\overline{MR}}{d_2} \tag{4.3}$$

and the expected length is:

$$E(L_2) = 2Z_{\alpha/2}\sigma. \tag{4.4}$$

We wish to compare the expected lengths $E(L_1)$ and $E(L_2)$ for the same confidence coefficient $(1-\alpha)$. Since 2σ is common in the two equations (4.2) and (4.4), we will compare $E(L_1)/2\sigma$ and $E(L_2)/2\sigma$ effectively, where:

$$\frac{E(L_1)}{2\sigma} = t_{\alpha/2,\text{k-1}}c_4\sqrt{(k-1)/k} \tag{4.5}$$

and

$$\frac{E(L_2)}{2\sigma} = Z_{\alpha/2}. \tag{4.6}$$

We have compared the expressions in (4.5) and (4.6) by computing the difference $\left(\frac{E(L_1)}{2\sigma} - \frac{E(L_2)}{2\sigma}\right)$ for various values of k and $\alpha = 0.01, 0.05, 0.10$ and is compiled in Tables 1-3 until the difference becomes zero. Even though the computation of expected lengths is made for many values of k not appearing in the Tables, the results presented only for few values of k are to show the trend of difference between expected lengths. When the difference is zero, the X- chart with new control limits and the X- chart with old control limits have equal expected lengths for the same confidence coefficient.

Table 1: Comparison of Expected Lengths at $\alpha = 0.01$

k	d.f.	t-value	C4	E(L1)/2σ	E(L2)/2σ	E(L1)/2σ - E(L2)/2σ
20	19	2.861	0.987	2.752	2.576	0.176
100	99	2.626	0.997	2.607	2.576	0.031
200	199	2.601	0.999	2.591	2.576	0.015
500	499	2.586	0.999	2.582	2.576	0.006
1500	1499	2.579	1.000	2.578	2.576	0.002
2000	1999	2.578	1.000	2.577	2.576	0.001
3000	2999	2.577	1.000	2.577	2.576	0.001
4000	3999	2.577	1.000	2.577	2.576	0.001
5000	4999	2.577	1.000	2.576	2.576	0.000
6000	5999	2.577	1.000	2.576	2.576	0.000

Table 2: Comparison of Expected Lengths at $\alpha = 0.05$

k	d.f.	t-value	C4	E(L1)/2σ	E(L2)/2σ	E(L1)/2σ - E(L2)/2σ
20	19	2.093	0.987	2.014	1.960	0.054
50	49	2.010	0.995	1.979	1.960	0.019
100	99	1.984	0.997	1.969	1.960	0.009
200	199	1.972	0.999	1.965	1.960	0.005
300	299	1.968	0.999	1.963	1.960	0.003
500	499	1.965	0.999	1.962	1.960	0.002
1000	999	1.962	1.000	1.961	1.960	0.001
1500	1499	1.962	1.000	1.961	1.960	0.001
2000	1999	1.961	1.000	1.960	1.960	0.000
2500	2499	1.961	1.000	1.960	1.960	0.000

Table 3: Comparison of Expected Lengths at α = 0.10

k	d.f.	t-value	C4	E(L1)/2σ	E(L2)/2σ	E(L1)/2σ - E(L2)/2σ
20	19	1.729	0.987	1.663	1.645	0.018
50	49	1.677	0.995	1.651	1.645	0.006
100	99	1.660	0.997	1.648	1.645	0.003
200	199	1.653	0.999	1.646	1.645	0.001
300	299	1.650	0.999	1.646	1.645	0.001
400	399	1.649	0.999	1.646	1.645	0.001
500	499	1.648	0.999	1.645	1.645	0.000
1000	999	1.646	1.000	1.645	1.645	0.000

From the Tables we observe that $\frac{E(L_1)}{2\sigma} > \frac{E(L_2)}{2\sigma}$ in general, and as k increases the difference $\left(\frac{E(L_1)}{2\sigma} - \frac{E(L_2)}{2\sigma}\right)$ reduces and becomes zero.

5. Recommendations and Concluding Remarks

I. In the X- chart with new control limits, standard deviation (s) is computed from the sample only once, whereas in the X- chart with old control limits several MR_i are to be computed finally to get $\overline{MR}$. Hence, one can save some time in computing the new control limits which is the advantage in using these limits. Moreover, the concept of moving range enters along with the concept of dependence of subgroups but independence of subgroups is essential in the development of any control chart. Calculation of s instead of $\overline{MR}$ overcomes this objection in the present X- chart.

Based on the results in the Tables 1 – 3, the practitioner is advised to use the new control limits in the following situations:

(i) $\alpha = 0.01$ and $k \geq 5000$

(ii) $\alpha = 0.05$ and $k \geq 2000$

(iii) $\alpha = 0.10$ and $k \geq 500$

II. For degrees of freedom $k-1=\infty$, $z_{\alpha/2}=t_{\alpha/2,\text{ k-1}}$ whatever α may be. Hence the expected length $E\left(L_1\right)$ between new control limits is always less than the expected length $E\left(L_2\right)$ between old control limits since $c_4 \le 1$ and $\sqrt{(k-1)/k} < 1$. In this situation, the X – chart with new control limits is always preferred over the X – chart with old control limits.

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References

1. Juran, J. M. and Godfrey, A. B. (2004). *Juran's Quality Handbook*, Fifth Edition. McGraw -Hill, New York.
2. Kiani,M., Panaretos, J. and Psarakis, S. (2008). A new procedure to monitor the mean of a quality characteristic. *Communications in Statistics - Simulation and Computation,* **37**, 1870 - 1880.
3. Montgomery, D. C. (2001). *Introduction to Statistical Quality Control* , Third Edition, John Wiley & Sons, Inc., New York.
4. Nelson, P.R., Wludyka, P.S. and Copeland, K.A.F.(2005). *The Analysis of Means: A Graphical Method for Comparing Means, Rates and Proportions*. ASA- SIAM Series on Statistics and Applied Probability, Philadelphia.
5. Ott, E. R. (1967). Analysis of means — a graphical procedure. *Industrial Quality Control*, **24**, 101–109.
6. Rao, C.V. (2005). Analysis of means - a review. *Journal of Quality Technology*, **37**, 308-315.