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On Some Ridge Regression Estimators: A Monte Carlo Simulation Study Under Different Error Variances

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Abstract

Following Khalaf and Shukur (2005), Alkhamisi et al. (2006) and Muniz et al. (2010), this paper considers several estimators for estimating the Ridge parameter k. This paper differs from aforementioned papers in three ways: 1. the number of regressors considered is between 4 to 12 instead of 2 to 4 which are the usual practice. 2. Both mean square error (MSE) and prediction sum of square (PRESS) are considered as the performance criterion. 3. Different error variances are used (σ is between 0.5 and 5). To compare the performance of the estimators, a simulation study has been conducted. It is evident that increasing the correlation between the independent variables has negative effect on the MSE and PRESS, while increasing the number of regressors has positive effect on MSE and PRESS. When the sample size increases the MSE decreases even when the correlation between the independent variables is large. It is interesting to note that the dominance pictures of the estimators remain the same under both the MSE and PRESS criterion. However, the performance of the estimators depends on the amount of error variances specifically, when the sample sizes are small.

Keywords

Estimation, Linear model, LSE, MSE, Multicollinearity, PRESS, Ridge regression, Simulation

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1. Introduction

To describe the problem of this paper, we consider the following Multiple Linear Regression Model:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e} \tag{1.1}$$

where $\mathbf{y}$ is an $n \times 1$ vector of responses, $\mathbf{X}$ is an $n \times p$ observed matrix of the regressors, $\boldsymbol{\beta}$ is an $p \times 1$ vector of unknown parameters, and $\mathbf{e}$ is an $n \times 1$ vector of random errors which are distributed as normal with mean vector 0 and covariance matrix $\sigma^2\mathbf{I_n}$, and $\mathbf{I_n}$ is an identity matrix of order *n*.

The Ordinary Least Square estimator (OLS) of $\boldsymbol{\beta}$ is obtained as $\hat{\boldsymbol{\beta}} = \mathbf{(X'\,X)}^{-1}\mathbf{X'\,y}$, and the covariance matrix of $\hat{\boldsymbol{\beta}}$ is obtained as $Cov(\hat{\boldsymbol{\beta}}) = \sigma^2\mathbf{(X'\,X)}^{-1}$. We observed that both $\hat{\boldsymbol{\beta}}$ and $Cov(\hat{\boldsymbol{\beta}})$ highly depend on the characteristics of the matrix $\mathbf{X'X}$. If the independent variables are correlated, i.e., the $\mathbf{X'X}$ is ill-conditioned, the OLS estimators are sensitive to a number of errors. For example, some of the Regression coefficients may be statistically insignificant or they may have the wrong signs. With these errors, it is difficult to make a valid statistical inference. The problem of multicollinearity can be solved by various methods. However, the Ridge Regression (RR) method, which has been proposed by Hoerl and Kennard (1970) is one of the most popular methods. The idea of Ridge Regression method is that a small positive number ($k \geq 0$) to be added to the diagonal elements of the $X'X$ matrix, and the resulting estimators are obtained as:

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X'X} + k\mathbf{I_p}\right)^{-1}\mathbf{X'y} \qquad k \geq 0 \tag{1.2}$$

which is known as a Ridge Regression estimator. The constant k ($k \geq 0$) is known as a “ridge” or “biased” parameter and will be estimated from observed data. Hoerl and Kennard (1970) showed that for a positive value of *k*, this estimator provides a smaller Mean Squared Error (MSE) compared to Least Squared Estimator (LSE). For more on RR method, we refer to Hoerl and Kennard (1970), Saleh and Kibria (1993), Zhang and Ibrahim (2005), Saleh (2006), Alkhamisi and Shukur (2008) and, very recently, Mansson et al. (2010) among others.

Estimating the value of the Ridge parameter k is an important problem in Ridge Regression method. Many different techniques for estimating k have been proposed or suggested by various researchers. To mention a few, Hoerl and Kennard (1970), McDonald and Galarneau (1975), Lawless and Wang (1976), Dempster et al. (1977), Gibbons (1981), Kibria (2003), Khalaf and Shukur (2005), Alkhamisi et al. (2006), Alkhamisi and Shukur (2008), Muniz and Kibria (2009) and, recently, Mansson et al. (2010). The performance of the Ridge estimators was compared based on the simulation study. Most of the researchers have generated data from a normal population, number of regressors was between 2 to 4 and only the MSE was used as a performance criterion. The objective of this paper is to compare the performance of several RR estimators under different error variances, with large numbers of regressors and both the prediction sum of square (PRESS) and MSE are used as a performance criterion.

This paper is organized as follows: The model, proposed estimators and performance criterion are presented in Section 2. A Monte Carlo simulation study has been conducted in Section 3. Some concluding remarks are given in Section 4.

2. Methodology

In this section we describe the RR method introduced by Hoerl & Kennard (1970) and then we also describe how to evaluate the performance of different estimators proposed in some recent papers.

2.1 Ridge Regression

The Multiple Linear Regression Model can be expressed as:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u} \tag{2.1}$$

where $\mathbf{y}$ is an $n \times 1$ vector of observations, $\mathbf{X}$ is an $n \times (p+1)$ data matrix where p is the number of explanatory variables, $\boldsymbol{\beta}$ is a $(p+1) \times 1$ vector of coefficients, and $\mathbf{u}$ is an $n \times 1$ vector of residuals. The most common method of estimating $\boldsymbol{\beta}$ is to use OLS where the residual sum of squares is minimized using the following equation:

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\left(\mathbf{X}'\mathbf{y}\right) \tag{2.2}$$

As a remedy to the problem caused by multicollinearity, Hoerl and Kennard (1970) proposed that one should use the following RR estimator:

$$\tilde{\boldsymbol{\beta}} = \left(\mathbf{X'X} + k\mathbf{I}\right)^{-1}\left(\mathbf{X'y}\right) \tag{2.3}$$

instead of applying OLS. The Mean Squared Error (MSE) of this biased estimator equals:

$$\begin{aligned} MSE(\tilde{\boldsymbol{\beta}}) &= Var(\tilde{\boldsymbol{\beta}}) + [Bias(\tilde{\boldsymbol{\beta}})]^2 \\ &= \sigma^2(\mathbf{X'X} + k\mathbf{I_p})^{-1}(\mathbf{X'X})(\mathbf{X'X} + k\mathbf{I_p})^{-1} + k^2\boldsymbol{\beta}'(\mathbf{X'X} + k\mathbf{I_p})^{-2}\boldsymbol{\beta} \end{aligned} \tag{2.4}$$

Since $\mathbf{X'X}$ is a positive definite matrix, there exists an orthogonal matrix $\mathbf{Q}$ such that $\mathbf{X'QX} = \mathbf{T}$, where, $\mathbf{T} = diag(t_1, t_2, \ldots\ldots, t_p)$ and $t_1, \ldots, t_p$, are the eigen values of $\mathbf{X'X}$. Now letting that $\boldsymbol{\alpha} = \mathbf{Q'}\boldsymbol{\beta}$, the MSE of $\tilde{\beta}$ becomes:

$$MSE(\tilde{\boldsymbol{\beta}}) = \sigma^2 \sum_{i=1}^{p} \frac{t_i}{(t_i + k)^2} + k^2 \sum_{i=1}^{p} \frac{\alpha_i^2}{(t_i + k)^2}$$

where α_i is the ith element of the vector $\boldsymbol{\alpha} = \mathbf{Q'}\boldsymbol{\beta}$,. The objective of Ridge Regression is to find a value of *k* such that the reduction in the variance term is greater than the increase in the squared bias. Hence, one wants to find a *k* that is large enough to solve the problem of near singularity of the matrix of cross products but not so big that it causes a lot of bias. It follows from Hoerl and Kennard (1970) that the value of k_i that minimizes the MSE equals $k_i = \frac{\sigma^2}{\beta_i^2}$.

Since both the parameters in k_i are unknown they have to be estimated and Hoerl and Kennard (1970) suggested the following estimator:

$$K1 = \hat{k}_{HK} = \frac{s^2}{\hat{\beta}_{\max}^2}$$

where, $s^2 = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{n - p - 1}$ and $\hat{\beta}_{\max}^2$ is the maximum element of $\hat{\boldsymbol{\beta}}$. Hence, this estimator of *k* is based on the unbiased estimators of the variance of the error term and the coefficient vector $\boldsymbol{\beta}$ obtained from the Regression estimated by OLS.

However, in general, $\hat{k}_{HK}$ underestimates the optimal k and, therefore, Kibria (2003) proposed the following estimator:

$$K2 = \hat{k}_{GM} = \frac{\hat{\sigma}^2}{\left(\prod_{i=1}^{p}\hat{\beta}_i^2\right)^{\frac{1}{p}}}, \qquad K3 = \hat{k}_{MED} = Median\left\{m_i^2\right\},$$

where $m_i = \sqrt{\frac{\hat{\sigma}^2}{\hat{\beta}_i^2}}$. Furthermore, in Alkhamisi et al. (2006), the following estimators were suggested:

$$K5 = \hat{k}_{\max}^{KS} = \max\left(s_i\right), \qquad K6 = \hat{k}_{md}^{KS} = median\left(s_i\right).$$

where $s_i = \frac{t_i\hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + t_i\hat{\beta}_i^2}$. Based on these previous estimators and the idea of square root transformations taken from Alkhamisi and Shukur (2008), the following estimators were suggested by Muniz and Kibria (2009):

$$K7 = \hat{k}_{gm}^{KS} = \left(\prod_{i=1}^{p} s_i\right)^{\frac{1}{p}}, \qquad K8 = \hat{k}_{KM2} = \max\left(\frac{1}{m_i}\right), \qquad K9 = \hat{k}_{KM4} = \left(\prod_{i=1}^{p}\frac{1}{m_i}\right)^{\frac{1}{p}}$$

$$K10 = \hat{k}_{KM5} = \left(\prod_{i=1}^{p} m_i\right)^{\frac{1}{p}}, \qquad K11 = \hat{k}_{KM6} = median\left(\frac{1}{m_i}\right)$$

Finally, in Muniz et al. (2010), the following estimators were proposed:

$$K12 = \hat{k}_{KM8} = \max\left(\frac{1}{q_i}\right), \qquad K13 = k_{KM9} = \max\left(q_i\right), \qquad K14 = \hat{k}_{KM10} = \left(\prod_{i=1}^{p}\frac{1}{q_i}\right)^{\frac{1}{p}}$$

$$K15 = \hat{k}_{KM11} = \left(\prod_{i=1}^{p} q_i\right)^{\frac{1}{p}}, \qquad K16 = \hat{k}_{KM12} = median\left(\frac{1}{q_i}\right)$$

where $q_i = \frac{t_{\max}\hat{\sigma}^2}{\left(n-p\right)\hat{\sigma}^2 + t_{\max}\hat{\beta}_i^2}$ and $t_{\max}$ is the maximum eigen value of $\mathbf{X'X}$ matrix. All 15 proposed different Ridge parameters were investigated and evaluated in this paper.

2.2 Judging the Performance of the Estimators

The most commonly applied method of evaluating estimators is to look at the MSE and determine how close the estimated parameters are to the true parameters using the following equation:

$$\text{MSE} = \frac{\sum_{i=1}^{R} \left(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}\right)'\left(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}\right)}{R}$$

where $\tilde{\boldsymbol{\beta}}$ is the vector of estimated parameters estimated by OLS or RR and R is the number of replicates in the Monte Carlo simulation. In addition to this commonly applied method, we also used the following maximum MSE (MMSE):

$$MMSE = \frac{\sum_{i=1}^{R} \max_{j}\left(MSE\left(\tilde{\beta}_j\right)\right)}{R}$$

when we have many explanatory variables in order to see if the same dominance structure exists among the different estimation methods. Another method for evaluating different estimation methodologies is by testing the predictive ability through cross-validation, i.e., by estimating the model using all the data except one observation and then predict the left-out observation. Then do this n times, leaving out a different observation each time and finally calculate the prediction sum of square (PRESS) using the following equation:

$$\text{PRESS} = \frac{\sum_{i=1}^{n} \left(\hat{y}_i - y_i\right)}{n}$$

This is done in every replicate and then the average PRESS for all the replicates is calculated. Finally, we will also estimate the average value of k. If several Ridge parameters minimize the MSE then the one with the lowest average value of k should be chosen since it induces the smallest amount of bias.

3. The Monte Carlo Simulation

The aim of this paper is to evaluate the MSE and the predictive ability of different RR estimators and the OLS estimator. Furthermore, we want to investigate whether there is a connection between MSE and PRESS. The performance of the

OLS and the different RR estimators is evaluated using Monte Carlo simulations where 10000 replicates (*R*) are used. In this section we give a brief description of the factors that vary in our simulation study.

3.1 The number of explanatory variables (p)

In most simulation studies (see Alkhamisi and Shukur (2008), Muniz and Kibria (2009) and Muniz et al. (2010)), the MSE of the proposed Ridge estimators is calculated using a fairly low number of explanatory variables (two and four is the most common value of p). Hence, there is a need to conduct an investigation where more variables are used to see which Ridge estimator is the best option both with respect to MSE and PRESS. Therefore, besides four explanatory variables we also simulated models where p is equal to eight and twelve.

3.2 The sample size

Increasing the sample size is supposed to have a positive effect both on the MSE and PRESS (i.e. increasing the sample size leads to lower value of both loss functions). Since the number of explanatory variables will be quite high, we choose to fix the number of degrees of freedom instead of the number of observations. Otherwise many combinations of different n and p will not be possible to estimate since the degrees of freedom will exceed the number of explanatory variables.

3.3 The strength of correlation among the explanatory variables (ρ^2)

The most obvious factor that may affect the properties of the different estimation methods is the degree of correlation between the explanatory variables and, therefore, we use $\rho = 0.75,\ 0.85$ and 0.95. To be able to change the strength of the correlation we use the subsequent data generating process for the explanatory variables:

$$x_{ij} = (1-\rho^2)^{(1/2)} z_{ij} + \rho z_{ij}$$

where ρ^2 represents the correlation between the explanatory variables, and z_{ij} are generated using the Standard Normal Distribution. Furthermore, the n observations for the dependent variable are generated by using the following equation:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_p x_{ip} + e_i, \qquad i = 1,2,\ldots n$$

where e_i are i.i.d. $N\left(0,\sigma^2\right)$ and β_0 is equal to zero. We choose the parameter values so that $\sum_j^p \beta_j^2 = 1$ since Newhouse and Oman (1971) stated that if the mean squared error (MSE) is a function of β, σ^2, and k, and if the explanatory variables are fixed, then the MSE is minimized when we choose this coefficient vector.

3.4 Variability of the Error Term

Most studies (e.g. Kibria (2003) and Muniz et al. (2010)) have shown that increasing the variance of e_i has a negative impact on the MSE, especially for the OLS estimator. Increasing the variance is also expected to have a negative impact on PRESS since the dependent variable is harder to predict when more outliers are generated. Therefore, we use $\sigma = 0.5$, 1 and 5.

3.5. Results and Discussion

The simulated results are presented in Tables 1 to 10. In Table 1 we have provided the estimated PRESS when p=4 (4 independent variables), in Table 4 when p=8 and in Table 7 when p=12. Furthermore, in Table 2 we have presented the MSE when p=4, in Table 5 we have MSE when p=8 and in Table 8 we have MSE when p=12. The dominant effect on the MSE and PRESS is the variance of the error term of the dependent variable which leads to a higher MSE and PRESS. The effect of increasing the variance on PRESS is very natural since it leads to more outliers which are harder to predict. Increasing the strength of the correlation does not have an impact on the PRESS of the OLS estimator. However, it has a (small) positive effect when PRESS is calculated using RR as estimation methodology. This cannot be connected with the lower MSE since the MSE increases for the RR when the strength of the correlation increases.

When comparing the MSE with the PRESS between the different estimation methodologies and fixing the factors (variance, number of degrees of freedom, etc.) we can see the expected connection between PRESS and MSE, since a lower MSE tends to lead to lower PRESS (see Figure 1 for an illustration). Thus using the Ridge Regression (RR) leads to both lower MSE and a lower PRESS. The best estimators using RR have about 50 % lower value of PRESS than the OLS estimator. The gain in PRESS seems to increase when the number of explanatory variables increases. It is worthwhile to note that based on both MSE and PRESS, we found that the estimators K2, K3, K4, K9 and K12 performed well compared to the rest. This pictures also holds when looking at Table 10 where MSE is used as measure of performance and p=12. Furthermore, we also observed from the estimated values of k in Tables 3, 6 and 9, that the estimator K12 has most often the lowest values of *k*. The other estimators of the Ridge parameter have sometimes very high value of *k*. For instance, the K2, K3 and K4 estimators have high average values when ρ is less than 0.95 while K9 has high average values when the number of explanatory variables equals 4. Therefore, the number of explanatory variables, estimated sample variance and amount of correlation have moderate to significant effect on the estimated values of k.

4. Some Concluding Remarks

Based on the work of Khalaf and Shukur (2005), Alkhamisi et al. (2006) and Muniz et al. (2010), this paper considered several estimators for estimating the Ridge parameter k. However, this paper differs from aforementioned papers in three ways: 1. The number of regressors considered are between 4 to 12 instead of only 2 to 4 which are the usual practice. 2. Mean Square Error (MSE), maximum MSE and prediction sum of square (PRESS) are considered as the performance criterion. 3. Different error variances are used (σ is between 0.5 and 5). Since a theoretical comparison is not possible, a simulation study has been conducted to compare the performance of the estimators. Based on the simulation study, we found that increasing the correlation between the independent variables has negative effect on the MSE and PRESS. However, increasing the number of regressors has positive effect on MSE and PRESS. When the sample size increases the MSE decreases even when the correlation between the independent variables is large. It is important to note that the dominance pictures of the estimators remain the same under both the MSE and PRESS criterion. However, the performance of the estimators also depends on the error variance of the distribution. It appears that when looking at the estimated MSE and PRESS, the proposed estimators K2, K3, K4, K9 and K12 are performing better than the rest. Among these estimators of *k*, it is only K12 that always have a low average value of *k* for all simulation conditions. Hence, this estimator gives stable low values of k, therefore, induces a small amount of bias and may be recommended to practitioners.

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Table 1: Estimated PRESS when p=4

$\sigma = 0.5$																
	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75																
10	0.376	0.356	0.335	0.340	0.334	0.354	0.353	0.317	0.328	0.327	0.324	0.317	0.319	0.323	0.331	0.323
20	0.306	0.300	0.294	0.295	0.296	0.301	0.301	0.290	0.294	0.293	0.293	0.290	0.291	0.293	0.295	0.292
30	0.287	0.284	0.282	0.282	0.283	0.285	0.285	0.280	0.282	0.282	0.281	0.280	0.281	0.281	0.282	0.281
50	0.271	0.270	0.269	0.269	0.269	0.270	0.270	0.268	0.269	0.269	0.269	0.268	0.269	0.269	0.269	0.269
ρ=0.85																
10	0.374	0.349	0.324	0.335	0.323	0.348	0.347	0.303	0.314	0.313	0.309	0.303	0.305	0.308	0.316	0.308
20	0.188	0.183	0.177	0.178	0.179	0.185	0.184	0.174	0.178	0.177	0.177	0.174	0.175	0.177	0.178	0.176
30	0.287	0.283	0.278	0.279	0.280	0.285	0.285	0.277	0.280	0.279	0.279	0.277	0.277	0.279	0.280	0.279
50	0.271	0.270	0.268	0.268	0.269	0.270	0.270	0.267	0.268	0.268	0.268	0.267	0.267	0.268	0.268	0.268
ρ=0.95																
10	0.373	0.338	0.313	0.335	0.309	0.342	0.340	0.292	0.289	0.293	0.287	0.292	0.287	0.286	0.294	0.287
20	0.307	0.294	0.281	0.293	0.282	0.300	0.298	0.271	0.278	0.276	0.276	0.271	0.273	0.275	0.278	0.275
30	0.285	0.279	0.270	0.274	0.272	0.283	0.282	0.266	0.271	0.269	0.270	0.266	0.267	0.269	0.271	0.269
50	0.055	0.055	0.054	0.054	0.054	0.055	0.055	0.054	0.054	0.054	0.054	0.054	0.054	0.054	0.054	0.054
$\sigma = 1$																
	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75																
10	1.504	1.368	1.310	1.351	1.265	1.387	1.384	1.242	1.332	1.243	1.314	1.237	1.227	1.290	1.274	1.292
20	1.223	1.180	1.157	1.173	1.152	1.199	1.196	1.165	1.194	1.141	1.188	1.162	1.146	1.179	1.162	1.179
30	1.141	1.119	1.101	1.108	1.108	1.131	1.130	1.118	1.130	1.101	1.127	1.116	1.106	1.123	1.114	1.123
50	1.084	1.074	1.063	1.065	1.071	1.081	1.080	1.077	1.080	1.069	1.079	1.076	1.071	1.078	1.074	1.078
ρ=0.85																
10	1.489	1.338	1.270	1.318	1.231	1.369	1.362	1.182	1.264	1.202	1.248	1.178	1.180	1.227	1.224	1.229
20	1.224	1.172	1.147	1.178	1.135	1.198	1.194	1.139	1.178	1.122	1.171	1.136	1.124	1.159	1.143	1.160
30	1.140	1.112	1.094	1.111	1.093	1.129	1.127	1.104	1.122	1.086	1.119	1.102	1.091	1.113	1.101	1.113
50	1.082	1.070	1.055	1.061	1.063	1.079	1.078	1.070	1.077	1.059	1.075	1.069	1.063	1.073	1.067	1.073
ρ=0.95																
10	1.492	1.315	1.215	1.264	1.201	1.363	1.351	1.112	1.150	1.150	1.140	1.111	1.124	1.132	1.160	1.132
20	1.217	1.149	1.115	1.158	1.101	1.188	1.181	1.075	1.112	1.085	1.106	1.074	1.077	1.096	1.096	1.097
30	1.143	1.103	1.085	1.120	1.074	1.131	1.127	1.068	1.095	1.066	1.090	1.066	1.064	1.082	1.076	1.083
50	1.082	1.062	1.048	1.071	1.047	1.078	1.076	1.050	1.065	1.043	1.063	1.049	1.044	1.058	1.051	1.058
$\sigma = 5$																
	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75																
10	37.28	32.32	28.56	28.82	30.55	34.20	34.09	31.47	33.42	29.76	32.96	31.20	30.15	32.27	31.19	32.27
20	30.77	28.61	27.01	27.15	28.43	30.09	29.99	29.67	30.20	27.96	30.10	29.50	28.69	29.77	28.98	29.78
30	28.58	27.26	26.36	26.46	27.38	28.30	28.24	28.20	28.40	27.01	28.37	28.11	27.62	28.20	27.74	28.20
50	27.11	26.39	25.98	26.05	26.60	27.01	26.99	27.01	27.07	26.34	27.06	26.97	26.75	26.99	26.78	26.99
ρ=0.85																
10	37.47	32.42	28.68	28.95	30.18	34.29	34.09	29.99	31.93	29.37	31.40	29.81	29.29	30.90	30.41	30.86
20	30.69	28.51	26.95	27.10	28.01	30.00	29.85	28.94	29.68	27.65	29.52	28.79	28.06	29.19	28.45	29.19
30	28.58	27.24	26.32	26.43	27.10	28.30	28.21	27.90	28.23	26.85	28.16	27.80	27.27	27.97	27.46	27.97
50	27.10	26.37	25.92	26.01	26.42	27.01	26.97	26.91	27.01	26.24	26.99	26.86	26.58	26.90	26.64	26.91
ρ=0.95																
10	37.54	32.57	28.99	29.20	29.82	34.26	33.94	27.72	28.80	28.59	28.43	27.67	28.05	28.31	29.02	28.24
20	30.60	28.46	26.89	27.01	27.48	29.84	29.64	27.17	28.00	27.09	27.75	27.11	26.99	27.57	27.47	27.55
30	28.63	27.25	26.28	26.36	26.72	28.31	28.19	26.91	27.47	26.53	27.32	26.84	26.58	27.15	26.87	27.14
50	27.24	26.45	25.91	25.98	26.24	27.13	27.07	26.57	26.85	26.13	26.78	26.52	26.26	26.66	26.40	26.66

Table 2: Estimated MSE when p=4

$\sigma = 0.5$																
	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75																
10	0.257	0.202	0.120	0.131	0.143	0.198	0.198	0.102	0.146	0.126	0.132	0.101	0.106	0.129	0.140	0.126
20	0.123	0.107	0.083	0.088	0.094	0.110	0.109	0.077	0.091	0.086	0.088	0.076	0.079	0.087	0.091	0.085
30	0.079	0.072	0.063	0.064	0.067	0.074	0.074	0.059	0.065	0.064	0.064	0.059	0.060	0.063	0.066	0.063
50	0.047	0.044	0.042	0.042	0.043	0.046	0.045	0.040	0.041	0.041	0.041	0.040	0.041	0.041	0.042	0.041
ρ=0.85																
10	0.435	0.314	0.148	0.162	0.187	0.315	0.311	0.106	0.182	0.148	0.161	0.104	0.117	0.153	0.174	0.151
20	0.126	0.102	0.063	0.070	0.079	0.109	0.107	0.057	0.131	0.112	0.073	0.056	0.059	0.072	0.076	0.070
30	0.134	0.114	0.084	0.090	0.098	0.122	0.121	0.080	0.060	0.055	0.093	0.078	0.080	0.092	0.096	0.090
50	0.079	0.071	0.061	0.063	0.066	0.075	0.075	0.058	0.063	0.062	0.063	0.057	0.059	0.062	0.065	0.062
ρ=0.95																
10	1.346	0.812	0.277	0.290	0.364	0.879	0.845	0.074	0.194	0.205	0.168	0.072	0.131	0.146	0.245	0.149
20	0.633	0.428	0.156	0.166	0.235	0.514	0.492	0.090	0.204	0.160	0.177	0.088	0.117	0.158	0.194	0.159
30	0.428	0.311	0.128	0.143	0.189	0.374	0.361	0.097	0.186	0.141	0.165	0.094	0.111	0.152	0.172	0.151
50	0.050	0.040	0.020	0.023	0.028	0.047	0.045	0.019	0.144	0.115	0.027	0.019	0.020	0.026	0.027	0.025
$\sigma = 1$																
	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75																
10	1.036	0.673	0.295	0.312	0.424	0.741	0.732	0.434	0.648	0.355	0.617	0.419	0.360	0.552	0.470	0.558
20	0.483	0.355	0.174	0.191	0.281	0.414	0.407	0.330	0.414	0.248	0.389	0.320	0.269	0.365	0.313	0.366
30	0.323	0.254	0.138	0.155	0.220	0.293	0.289	0.257	0.286	0.193	0.283	0.251	0.217	0.271	0.240	0.271
50	0.191	0.161	0.108	0.118	0.151	0.181	0.179	0.170	0.180	0.141	0.178	0.167	0.151	0.173	0.160	0.173
ρ=0.85																
10	1.733	1.045	0.396	0.421	0.565	1.195	1.163	0.455	0.820	0.452	0.752	0.438	0.403	0.654	0.588	0.666
20	0.807	0.541	0.213	0.226	0.366	0.675	0.654	0.419	0.585	0.306	0.564	0.404	0.331	0.507	0.416	0.514
30	0.532	0.383	0.161	0.177	0.289	0.475	0.463	0.351	0.438	0.248	0.425	0.341	0.279	0.395	0.331	0.397
50	0.318	0.247	0.124	0.142	0.212	0.298	0.293	0.254	0.282	0.187	0.280	0.248	0.211	0.268	0.235	0.268
ρ=0.95																
10	5.382	2.792	0.940	1.075	1.251	3.451	3.281	0.318	0.827	0.711	0.720	0.306	0.452	0.608	0.878	0.615
20	2.536	1.415	0.451	0.497	0.703	2.031	1.904	0.424	0.930	0.503	0.830	0.406	0.420	0.686	0.671	0.707
30	1.678	0.996	0.313	0.331	0.535	1.453	1.374	0.463	0.857	0.423	0.803	0.444	0.405	0.672	0.580	0.694
50	0.986	0.635	0.206	0.213	0.394	0.908	0.871	0.443	0.675	0.326	0.646	0.425	0.354	0.562	0.456	0.575
$\sigma = 5$																
	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75																
10	26.28	13.11	4.496	5.092	9.049	18.19	17.87	12.68	17.04	7.504	16.32	11.97	8.480	14.58	10.94	14.56
20	12.33	6.305	2.190	2.488	5.825	10.35	10.06	9.375	10.66	4.768	10.49	8.915	6.602	9.609	7.366	9.637
30	7.827	4.077	1.432	1.627	4.431	7.018	6.840	6.799	7.492	3.616	7.236	6.526	5.140	6.770	5.456	6.789
50	4.698	2.567	0.936	1.042	3.165	4.411	4.326	4.393	4.506	2.441	4.534	4.263	3.625	4.331	3.726	4.339
ρ=0.85																
10	43.21	20.76	7.155	8.209	12.08	29.05	28.08	13.41	20.92	9.753	18.99	12.63	9.309	16.97	13.58	16.75
20	20.28	9.888	3.222	3.744	7.425	16.73	16.01	12.09	15.39	6.419	14.67	11.40	7.852	13.18	9.585	13.18
30	13.30	6.627	2.186	2.500	5.888	11.76	11.33	9.877	11.22	4.775	11.16	9.377	6.742	10.19	7.622	10.22
50	7.967	4.098	1.344	1.546	4.337	7.418	7.212	6.900	7.387	3.525	7.355	6.625	5.146	6.880	5.478	6.900
ρ=0.95																
10	135.6	63.29	21.24	23.92	28.95	85.93	81.32	9.010	21.51	17.03	16.69	8.515	10.60	15.30	20.93	14.50
20	64.27	29.89	9.453	10.64	15.86	51.09	47.62	12.99	24.02	11.79	20.46	12.15	9.961	18.08	16.06	17.67
30	42.14	19.89	6.285	7.162	11.79	36.34	34.11	13.89	22.17	9.477	19.78	12.97	9.485	17.37	13.63	17.20
50	24.74	11.67	3.499	4.063	8.12	22.70	21.61	13.08	17.73	6.878	16.56	12.25	8.373	14.63	10.57	14.64

Table 3: Estimated average *k* when p=4

$\sigma = 0.5$															
	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75															
10	0.730	4.406	12.77	1.893	0.417	0.434	1.383	0.477	4.406	0.778	1.463	1.900	0.890	1.173	0.917
20	0.923	3.316	3.253	1.653	0.397	0.437	1.142	0.439	3.316	0.736	1.243	1.713	0.866	1.177	0.887
30	1.021	2.855	2.861	1.486	0.387	0.431	1.055	0.457	2.855	0.730	1.165	1.595	0.868	1.167	0.884
50	1.159	2.315	2.202	1.444	0.376	0.433	0.967	0.474	2.315	0.721	1.088	1.486	0.866	1.163	0.877
ρ=0.85															
10	0.600	4.494	8.468	2.002	0.284	0.316	1.563	0.539	4.494	0.810	1.631	1.974	0.916	1.155	0.940
20	0.766	3.974	4.782	1.797	0.269	0.317	1.276	0.418	3.974	0.743	1.362	1.850	0.865	1.182	0.888
30	0.880	3.462	3.202	1.661	0.263	0.316	1.150	0.411	3.462	0.727	1.245	1.761	0.853	1.189	0.874
50	1.011	2.670	2.406	1.557	0.256	0.316	1.048	0.438	2.670	0.718	1.154	1.622	0.853	1.181	0.868
ρ=0.95															
10	0.348	3.587	9.468	1.803	0.109	0.138	2.210	1.075	3.587	1.040	2.260	1.838	1.131	0.995	1.155
20	0.479	4.332	11.66	1.830	0.103	0.143	1.697	0.587	4.332	0.824	1.763	1.867	0.959	1.098	0.964
30	0.591	4.228	8.032	1.845	0.100	0.145	1.474	0.450	4.228	0.760	1.550	1.871	0.898	1.145	0.908
50	0.705	4.115	4.568	1.767	0.097	0.145	1.303	0.395	4.115	0.734	1.390	1.844	0.867	1.171	0.884
$\sigma = 1$															
	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75															
10	1.123	9.180	18.64	2.463	0.468	0.503	1.155	0.290	9.180	0.589	1.251	2.044	0.767	1.365	0.765
20	1.459	8.920	16.72	2.296	0.440	0.507	0.932	0.209	8.920	0.528	1.054	1.947	0.726	1.404	0.726
30	1.652	7.267	8.533	2.135	0.427	0.503	0.849	0.205	7.267	0.517	0.984	1.873	0.720	1.406	0.721
50	1.957	6.199	5.560	2.152	0.416	0.510	0.759	0.210	6.199	0.507	0.910	1.791	0.712	1.414	0.715
ρ=0.85															
10	0.888	9.124	23.31	2.448	0.307	0.352	1.336	0.377	9.124	0.644	1.416	2.072	0.808	1.319	0.803
20	1.184	9.008	17.73	2.395	0.288	0.359	1.059	0.233	9.008	0.543	1.163	2.009	0.739	1.389	0.730
30	1.400	9.094	14.05	2.392	0.279	0.362	0.938	0.200	9.094	0.520	1.055	1.970	0.718	1.414	0.714
50	1.653	7.456	8.815	2.296	0.271	0.361	0.834	0.192	7.456	0.508	0.965	1.910	0.705	1.430	0.706
ρ=0.95															
10	0.482	5.283	13.11	1.964	0.113	0.145	1.982	0.921	5.283	0.956	2.040	1.892	1.057	1.076	1.083
20	0.688	7.487	16.27	2.157	0.105	0.153	1.479	0.462	7.487	0.695	1.557	1.919	0.873	1.221	0.858
30	0.851	8.802	18.48	2.280	0.103	0.157	1.281	0.313	8.802	0.595	1.370	1.959	0.798	1.302	0.777
50	1.075	9.378	18.59	2.289	0.099	0.159	1.090	0.240	9.378	0.545	1.194	1.949	0.753	1.356	0.737
$\sigma = 5$															
	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75															
10	2.546	26.50	64.88	3.076	0.513	0.572	33.00	8.159	26.50	77.28	33.10	2.125	90.33	1.595	77.28
20	3.734	39.28	556.93	3.179	0.482	0.590	0.644	0.085	39.28	0.309	0.815	2.059	0.605	1.695	0.584
30	4.477	41.73	88.80	3.225	0.464	0.592	0.561	0.063	41.73	0.270	0.751	2.034	0.586	1.734	0.567
50	5.621	48.91	75.96	3.200	0.451	0.597	0.480	0.046	48.91	0.244	0.696	2.000	0.574	1.757	0.560
ρ=0.85															
10	1.838	24.05	44.939	2.891	0.329	0.389	1.061	0.283	24.05	0.537	1.166	2.119	0.729	1.488	0.726
20	2.781	29.83	180.88	3.087	0.306	0.402	0.770	0.131	29.83	0.374	0.911	2.091	0.634	1.635	0.614
30	3.454	35.07	52.057	3.192	0.297	0.410	0.659	0.090	35.07	0.312	0.820	2.069	0.602	1.697	0.580
50	4.373	44.35	83.350	3.307	0.288	0.415	0.555	0.059	44.35	0.265	0.739	2.060	0.576	1.758	0.556
ρ=0.95															
10	0.817	8.814	20.36	2.205	0.115	0.152	1.732	0.824	8.814	0.929	1.801	1.940	1.004	1.152	1.061
20	1.324	17.16	39.41	2.461	0.107	0.162	1.210	0.360	17.16	0.623	1.308	1.986	0.800	1.350	0.804
30	1.815	19.80	41.45	2.662	0.105	0.168	1.012	0.242	19.80	0.511	1.128	2.001	0.727	1.453	0.718
50	2.363	26.99	84.89	2.936	0.102	0.174	0.816	0.144	26.99	0.391	0.954	2.027	0.655	1.577	0.634

Table 4: Estimated PRESS when p=8

$\sigma = 0.5$																
	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75																
10	0.501	0.450	0.376	0.388	0.378	0.437	0.437	0.352	0.383	0.374	0.374	0.352	0.350	0.375	0.381	0.372
20	0.261	0.252	0.238	0.241	0.240	0.253	0.253	0.232	0.241	0.239	0.239	0.232	0.232	0.239	0.241	0.239
30	0.322	0.316	0.308	0.310	0.309	0.317	0.317	0.304	0.309	0.309	0.309	0.304	0.304	0.308	0.310	0.308
50	0.292	0.290	0.288	0.288	0.288	0.291	0.291	0.286	0.288	0.288	0.288	0.286	0.286	0.287	0.288	0.287
ρ=0.85																
10	0.498	0.437	0.348	0.361	0.357	0.424	0.425	0.322	0.353	0.344	0.344	0.322	0.321	0.344	0.351	0.342
20	0.360	0.343	0.313	0.319	0.319	0.345	0.345	0.305	0.322	0.315	0.318	0.304	0.302	0.318	0.319	0.316
30	0.322	0.313	0.300	0.303	0.303	0.316	0.316	0.296	0.304	0.301	0.303	0.295	0.294	0.303	0.303	0.302
50	0.292	0.288	0.284	0.285	0.285	0.290	0.290	0.282	0.285	0.284	0.285	0.282	0.282	0.285	0.285	0.284
ρ=0.95																
10	0.500	0.422	0.319	0.330	0.336	0.414	0.415	0.295	0.304	0.307	0.299	0.295	0.291	0.299	0.311	0.298
20	0.282	0.262	0.227	0.232	0.235	0.282	0.262	0.227	0.232	0.235	0.228	0.218	0.218	0.228	0.230	0.227
30	0.321	0.307	0.281	0.285	0.287	0.313	0.313	0.274	0.288	0.282	0.285	0.274	0.273	0.285	0.286	0.284
50	0.083	0.082	0.078	0.079	0.079	0.083	0.083	0.077	0.079	0.079	0.079	0.077	0.077	0.079	0.079	0.079
$\sigma = 1$																
	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75																
10	1.998	1.695	1.362	1.382	1.365	1.680	1.682	1.385	1.604	1.365	1.564	1.381	1.299	1.536	1.423	1.537
20	1.450	1.353	1.213	1.226	1.249	1.384	1.383	1.304	1.379	1.248	1.367	1.300	1.236	1.353	1.287	1.354
30	1.287	1.237	1.154	1.166	1.187	1.261	1.260	1.230	1.261	1.189	1.256	1.228	1.186	1.250	1.213	1.250
50	1.151	1.130	1.093	1.101	1.113	1.143	1.142	1.135	1.143	1.115	1.142	1.134	1.114	1.140	1.125	1.140
ρ=0.85																
10	1.994	1.664	1.291	1.320	1.314	1.657	1.658	1.271	1.476	1.291	1.438	1.269	1.216	1.417	1.336	1.417
20	1.443	1.331	1.171	1.184	1.203	1.372	1.370	1.237	1.335	1.197	1.320	1.234	1.176	1.304	1.235	1.306
30	1.282	1.222	1.123	1.135	1.153	1.254	1.253	1.193	1.240	1.154	1.233	1.191	1.146	1.224	1.180	1.225
50	1.169	1.142	1.089	1.099	1.112	1.160	1.159	1.139	1.155	1.116	1.153	1.138	1.114	1.150	1.129	1.150
ρ=0.95																
10	1.981	1.627	1.221	1.258	1.263	1.627	1.629	1.120	1.222	1.192	1.198	1.120	1.120	1.193	1.212	1.190
20	1.443	1.309	1.124	1.141	1.155	1.364	1.360	1.115	1.214	1.130	1.196	1.114	1.096	1.186	1.152	1.186
30	1.284	1.206	1.086	1.098	1.111	1.252	1.249	1.107	1.179	1.101	1.168	1.106	1.081	1.158	1.121	1.159
50	1.167	1.128	1.059	1.067	1.078	1.157	1.155	1.093	1.130	1.076	1.125	1.092	1.069	1.119	1.092	1.120
$\sigma = 5$																
	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75																
10	49.94	39.74	29.62	30.55	31.90	41.25	41.36	35.51	41.09	32.21	40.12	35.30	31.55	39.02	34.38	39.14
20	36.11	32.01	27.51	27.91	29.65	34.30	34.24	33.18	34.88	29.59	34.66	33.01	30.45	34.07	31.40	34.15
30	32.15	29.61	26.80	27.02	28.58	31.42	31.37	31.10	31.77	28.40	31.69	30.99	29.42	31.38	29.84	31.41
50	29.28	27.90	26.38	26.47	27.69	29.05	29.02	29.00	29.19	27.44	29.17	28.95	28.25	29.04	28.38	29.06
ρ=0.85																
10	49.69	39.68	29.52	30.46	31.24	40.91	40.97	32.31	37.55	31.19	36.56	32.19	29.78	35.83	32.66	35.88
20	36.18	32.02	27.28	27.70	28.97	34.26	34.18	31.64	33.97	28.99	33.64	31.51	29.17	33.04	30.39	33.14
30	32.11	29.56	26.58	26.83	27.95	31.35	31.28	30.29	31.37	27.97	31.24	30.19	28.51	30.88	29.13	30.93
50	29.22	27.82	26.16	26.29	27.21	28.98	28.94	28.68	29.03	27.14	29.00	28.63	27.74	28.83	27.97	28.85
ρ=0.95																
10	50.22	40.24	29.63	30.57	30.86	41.00	41.02	27.99	31.04	29.58	30.37	27.97	27.76	30.20	30.20	30.13
20	36.03	32.00	27.20	27.61	28.27	34.03	33.89	28.06	30.58	27.89	30.12	28.02	27.20	29.78	28.54	29.81
30	32.13	29.59	26.44	26.71	27.28	31.31	31.20	27.97	29.75	27.19	29.47	27.92	26.93	29.14	27.82	29.19
50	29.16	27.77	25.93	26.09	26.56	28.89	28.84	27.53	28.40	26.58	28.29	27.48	26.65	28.06	27.11	28.11

Table 5: Estimated MSE when p=8

	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
$\sigma = 0.5$																
ρ=0.75																
10	0.586	0.446	0.245	0.280	0.252	0.414	0.414	0.200	0.291	0.255	0.270	0.199	0.188	0.271	0.275	0.264
20	0.201	0.171	0.128	0.137	0.135	0.173	0.173	0.116	0.162	0.152	0.135	0.115	0.111	0.136	0.138	0.133
30	0.182	0.161	0.136	0.142	0.143	0.165	0.166	0.128	0.143	0.139	0.140	0.127	0.124	0.140	0.142	0.139
50	0.107	0.099	0.092	0.093	0.094	0.102	0.102	0.088	0.093	0.092	0.092	0.088	0.087	0.092	0.093	0.091
ρ=0.85																
10	0.974	0.682	0.283	0.339	0.311	0.627	0.630	0.199	0.358	0.296	0.320	0.198	0.185	0.320	0.325	0.310
20	0.463	0.368	0.218	0.247	0.247	0.379	0.380	0.187	0.270	0.235	0.250	0.186	0.173	0.250	0.250	0.245
30	0.306	0.256	0.182	0.198	0.199	0.270	0.270	0.164	0.207	0.191	0.198	0.162	0.154	0.198	0.199	0.195
50	0.180	0.159	0.133	0.138	0.140	0.168	0.168	0.124	0.136	0.132	0.138	0.123	0.119	0.137	0.139	0.136
ρ=0.95																
10	3.108	1.868	0.451	0.573	0.569	1.692	1.705	0.136	0.414	0.390	0.352	0.135	0.161	0.347	0.443	0.337
20	1.176	0.792	0.241	0.304	0.336	0.869	0.862	0.137	0.241	0.186	0.292	0.136	0.135	0.285	0.295	0.278
30	0.986	0.703	0.267	0.325	0.350	0.807	0.800	0.190	0.381	0.295	0.341	0.188	0.178	0.334	0.333	0.328
50	0.155	0.118	0.060	0.069	0.072	0.138	0.137	0.049	0.102	0.086	0.073	0.049	0.046	0.072	0.072	0.071
$\sigma = 1$																
	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75																
10	2.344	1.507	0.480	0.581	0.647	1.485	1.491	0.821	1.375	0.687	1.289	0.810	0.548	1.207	0.852	1.214
20	1.107	0.811	0.331	0.398	0.504	0.901	0.898	0.685	0.906	0.515	0.864	0.674	0.489	0.824	0.627	0.827
30	0.724	0.562	0.282	0.333	0.411	0.638	0.635	0.541	0.637	0.421	0.620	0.533	0.412	0.600	0.489	0.600
50	0.422	0.350	0.228	0.257	0.298	0.394	0.392	0.362	0.399	0.308	0.386	0.358	0.300	0.379	0.333	0.379
ρ=0.85																
10	3.996	2.405	0.597	0.758	0.835	2.338	2.349	0.833	1.711	0.809	1.551	0.822	0.545	1.444	1.023	1.454
20	1.866	1.260	0.375	0.472	0.630	1.455	1.445	0.821	1.279	0.621	1.217	0.808	0.539	1.136	0.792	1.147
30	1.220	0.876	0.317	0.393	0.519	1.043	1.034	0.714	0.966	0.525	0.929	0.703	0.494	0.882	0.651	0.887
50	0.718	0.553	0.267	0.318	0.396	0.658	0.653	0.532	0.628	0.409	0.613	0.525	0.399	0.593	0.477	0.594
ρ=0.95																
10	12.53	6.841	1.278	1.682	1.820	6.579	6.594	0.583	1.897	1.256	1.608	0.577	0.513	1.532	1.496	1.509
20	6.097	3.608	0.686	0.908	1.218	4.391	4.310	0.819	2.055	0.932	1.854	0.806	0.569	1.686	1.231	1.711
30	3.892	2.398	0.494	0.660	0.926	3.151	3.084	0.891	1.887	0.816	1.719	0.876	0.577	1.556	1.058	1.585
50	2.312	1.511	0.374	0.485	0.692	2.048	2.011	0.876	1.459	0.662	1.397	0.861	0.570	1.278	0.873	1.300
$\sigma = 5$																
	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75																
10	57.86	29.98	6.070	8.245	11.86	34.48	34.78	22.96	37.56	13.12	34.60	22.40	11.77	31.56	18.11	31.93
20	27.86	15.47	2.998	4.041	8.957	22.05	21.86	19.10	24.15	8.926	23.53	18.60	11.20	21.74	13.73	21.98
30	18.13	10.38	2.037	2.715	7.271	15.66	15.49	14.65	16.87	6.866	16.60	14.30	9.607	15.59	10.84	15.72
50	10.67	6.320	1.287	1.727	5.588	9.841	9.752	9.688	10.36	4.931	10.27	9.499	7.246	9.844	7.667	9.881
ρ=0.85																
10	97.60	50.69	9.591	13.00	16.80	55.71	55.99	23.61	45.92	16.85	41.89	23.12	12.15	38.26	22.52	38.65
20	46.29	25.04	4.392	5.947	11.18	35.33	34.86	23.01	34.83	11.67	33.01	22.42	12.17	29.86	17.08	30.41
30	30.19	16.72	2.928	4.026	8.986	25.55	25.13	19.85	26.38	9.34	25.15	19.34	11.40	23.09	14.19	23.41
50	18.20	10.47	1.831	2.500	7.006	16.54	16.32	14.61	16.79	6.74	16.70	14.28	9.50	15.67	10.73	15.81
ρ=0.95																
10	311.6	159.5	27.01	35.64	40.25	59.60	159.86	16.15	49.52	28.99	42.31	15.90	11.78	39.77	34.59	39.43
20	149.1	80.05	12.49	17.03	24.69	106.94	104.33	23.31	54.95	20.49	49.02	22.82	13.09	44.02	27.74	44.82
30	96.49	52.31	7.895	10.87	17.28	77.73	75.52	25.40	51.30	16.80	46.30	24.79	13.48	40.94	23.58	42.04
50	58.95	32.62	4.869	6.715	12.90	51.95	50.69	24.74	40.22	12.59	38.82	24.12	13.46	34.33	19.53	35.28

Table 6: Estimated average k when p=8

$\sigma = 0.5$															
	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75															
10	0.525	3.713	2.704	2.996	0.425	0.416	1.611	0.632	1.797	0.770	1.644	2.804	0.796	1.321	0.837
20	0.702	3.134	2.374	2.328	0.404	0.404	1.301	0.625	1.704	0.728	1.349	2.409	0.780	1.310	0.810
30	0.809	2.754	2.238	2.028	0.392	0.397	1.179	0.640	1.624	0.717	1.236	2.199	0.783	1.296	0.807
50	0.952	2.395	2.137	1.811	0.380	0.391	1.066	0.664	1.532	0.712	1.132	1.948	0.790	1.277	0.807
ρ=0.85															
10	0.404	3.703	2.808	3.241	0.294	0.292	1.889	0.656	1.774	0.805	1.915	2.990	0.821	1.297	0.864
20	0.563	3.594	2.540	2.661	0.277	0.283	1.478	0.606	1.802	0.734	1.516	2.670	0.774	1.327	0.808
30	0.668	3.201	2.423	2.377	0.269	0.279	1.319	0.612	1.727	0.721	1.364	2.485	0.769	1.322	0.800
50	0.809	2.709	2.211	2.058	0.260	0.274	1.169	0.635	1.618	0.711	1.223	2.242	0.771	1.309	0.795
ρ=0.95															
10	0.206	2.660	2.106	2.961	0.116	0.118	2.825	0.851	1.457	1.040	2.842	2.984	1.015	1.096	1.086
20	0.313	3.508	2.844	2.950	0.108	0.117	2.089	0.669	1.730	0.817	2.115	2.840	0.852	1.235	0.881
30	0.391	3.780	2.852	2.897	0.104	0.116	1.805	0.618	1.822	0.756	1.836	2.737	0.807	1.284	0.829
50	0.506	3.659	2.663	2.635	0.100	0.114	1.535	0.594	1.825	0.722	1.573	2.625	0.775	1.316	0.801
$\sigma = 1$															
	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75															
10	0.737	7.152	5.462	3.988	0.485	0.476	1.394	0.478	2.460	0.587	1.432	3.010	0.668	1.578	0.674
20	1.023	7.455	5.305	3.441	0.452	0.462	1.096	0.427	2.581	0.523	1.153	2.713	0.633	1.616	0.634
30	1.227	6.942	5.001	3.189	0.437	0.456	0.977	0.424	2.526	0.510	1.045	2.565	0.627	1.617	0.631
50	1.519	5.776	4.505	2.931	0.422	0.448	0.854	0.441	2.352	0.503	0.936	2.390	0.626	1.609	0.631
ρ=0.85															
10	0.340	4.333	3.477	3.617	0.123	0.127	2.454	0.722	1.805	0.880	2.474	3.131	0.893	1.268	0.934
20	0.598	7.932	6.521	4.027	0.114	0.129	1.686	0.494	2.497	0.599	1.720	3.014	0.694	1.538	0.686
30	0.838	11.34	9.198	4.311	0.109	0.129	1.370	0.402	3.017	0.492	1.413	2.959	0.619	1.690	0.600
50	1.230	16.70	13.98	4.727	0.105	0.130	1.090	0.320	3.707	0.390	1.146	2.905	0.552	1.864	0.525
ρ=0.95															
10	0.255	3.519	2.921	3.365	0.120	0.123	2.651	0.779	1.644	0.955	2.668	3.069	0.949	1.187	1.005
20	0.407	5.336	4.445	3.550	0.111	0.123	1.907	0.571	2.098	0.696	1.936	2.942	0.765	1.387	0.773
30	0.520	6.344	5.063	3.605	0.107	0.123	1.618	0.502	2.319	0.609	1.653	2.870	0.705	1.476	0.699
50	0.704	7.366	5.608	3.612	0.103	0.122	1.343	0.442	2.544	0.540	1.387	2.794	0.653	1.568	0.643
$\sigma = 5$															
	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75															
10	1.316	16.92	13.77	5.241	0.542	0.538	1.121	0.353	3.599	0.432	1.170	3.141	0.560	1.895	0.546
20	2.112	26.25	21.68	5.084	0.502	0.529	0.808	0.256	4.661	0.314	0.886	2.864	0.502	2.042	0.478
30	2.702	31.88	25.74	5.061	0.482	0.523	0.692	0.223	5.199	0.273	0.787	2.749	0.486	2.092	0.461
50	3.634	38.16	29.00	4.949	0.463	0.517	0.578	0.196	5.769	0.242	0.695	2.641	0.473	2.131	0.453
ρ=0.85															
10	0.896	11.62	9.196	4.965	0.349	0.351	1.393	0.429	2.985	0.530	1.430	3.263	0.620	1.745	0.619
20	1.480	19.71	15.90	5.127	0.321	0.348	0.997	0.307	3.957	0.377	1.055	3.014	0.531	1.950	0.509
30	1.972	25.19	20.82	5.200	0.308	0.345	0.831	0.257	4.587	0.316	0.903	2.907	0.499	2.050	0.474
50	2.673	32.57	26.30	5.292	0.295	0.342	0.692	0.219	5.268	0.268	0.781	2.812	0.474	2.135	0.450
ρ=0.95															
10	0.340	4.333	3.477	3.617	0.123	0.127	2.454	0.722	1.805	0.880	2.474	3.131	0.893	1.268	0.934
20	0.598	7.932	6.521	4.027	0.114	0.129	1.686	0.494	2.497	0.599	1.720	3.014	0.694	1.538	0.686
30	0.838	11.34	9.198	4.311	0.109	0.129	1.370	0.402	3.017	0.492	1.413	2.959	0.619	1.690	0.600
50	1.230	16.70	13.98	4.727	0.105	0.130	1.090	0.320	3.707	0.390	1.146	2.905	0.552	1.864	0.525

Table 7: Estimated PRESS when p=12

	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
$\sigma = 0.5$																
ρ=0.75																
10	0.628	0.478	0.313	0.326	0.331	0.471	0.476	0.388	0.481	0.347	0.466	0.386	0.324	0.455	0.371	0.456
20	0.416	0.360	0.286	0.290	0.307	0.383	0.384	0.363	0.395	0.313	0.392	0.362	0.317	0.384	0.335	0.386
30	0.358	0.326	0.279	0.281	0.297	0.346	0.346	0.339	0.352	0.301	0.351	0.338	0.310	0.347	0.319	0.347
50	0.312	0.296	0.270	0.271	0.285	0.308	0.308	0.307	0.311	0.285	0.310	0.307	0.294	0.309	0.297	0.309
ρ=0.85																
10	0.627	0.479	0.308	0.323	0.324	0.467	0.472	0.343	0.430	0.333	0.415	0.343	0.302	0.409	0.349	0.408
20	0.418	0.361	0.282	0.288	0.301	0.383	0.384	0.341	0.382	0.306	0.376	0.340	0.300	0.370	0.322	0.371
30	0.357	0.323	0.273	0.276	0.288	0.343	0.343	0.325	0.345	0.292	0.343	0.324	0.295	0.338	0.307	0.339
50	0.312	0.294	0.265	0.267	0.278	0.308	0.308	0.302	0.309	0.280	0.308	0.302	0.286	0.306	0.291	0.307
ρ=0.95																
10	0.625	0.481	0.304	0.319	0.318	0.463	0.468	0.283	0.331	0.307	0.321	0.282	0.276	0.321	0.314	0.319
20	0.416	0.359	0.275	0.282	0.291	0.380	0.380	0.287	0.331	0.287	0.324	0.287	0.272	0.320	0.295	0.321
30	0.356	0.322	0.267	0.271	0.279	0.342	0.341	0.288	0.320	0.279	0.315	0.288	0.271	0.312	0.287	0.312
50	0.313	0.295	0.262	0.265	0.271	0.309	0.308	0.286	0.302	0.273	0.300	0.286	0.271	0.297	0.280	0.298
$\sigma = 1$																
	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75																
10	2.494	1.869	1.203	1.264	1.297	1.862	1.884	1.548	1.917	1.364	1.858	1.541	1.281	1.811	1.464	1.817
20	1.666	1.421	1.109	1.135	1.219	1.533	1.535	1.462	1.586	1.244	1.572	1.455	1.269	1.541	1.338	1.547
30	1.424	1.276	1.074	1.090	1.173	1.373	1.372	1.350	1.400	1.183	1.395	1.346	1.229	1.379	1.262	1.381
50	1.251	1.171	1.054	1.061	1.132	1.235	1.234	1.231	1.245	1.132	1.244	1.229	1.176	1.237	1.187	1.238
ρ=0.85																
10	2.507	1.903	1.217	1.277	1.292	1.868	1.889	1.384	1.730	1.328	1.671	1.381	1.210	1.645	1.395	1.644
20	1.664	1.425	1.101	1.129	1.189	1.527	1.529	1.364	1.526	1.211	1.505	1.360	1.196	1.477	1.280	1.483
30	1.423	1.278	1.067	1.085	1.141	1.369	1.368	1.299	1.377	1.160	1.369	1.295	1.175	1.350	1.221	1.354
50	1.247	1.167	1.044	1.053	1.105	1.230	1.229	1.210	1.236	1.114	1.234	1.208	1.142	1.225	1.160	1.227
ρ=0.95																
10	2.501	0.925	1.215	1.275	1.270	1.852	1.873	1.129	1.325	1.226	1.286	1.129	1.103	1.287	1.254	1.278
20	1.670	1.437	1.099	1.127	1.162	1.524	1.525	1.155	1.329	1.149	1.301	1.153	1.092	1.287	1.181	1.288
30	1.428	1.288	1.067	1.085	1.115	1.370	1.368	1.158	1.283	1.117	1.266	1.157	1.088	1.251	1.149	1.254
50	1.248	1.170	1.039	1.049	1.075	1.229	1.228	1.141	1.203	1.083	1.196	1.139	1.078	1.185	1.112	1.188
$\sigma = 5$																
	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75																
10	62.34	46.72	29.92	31.52	32.39	46.55	47.08	38.79	47.93	41.56	46.46	38.63	32.10	45.31	36.66	45.44
20	41.57	35.30	27.26	28.02	30.33	38.24	38.30	36.54	39.61	30.96	39.25	36.37	31.65	38.48	33.36	38.62
30	35.74	31.82	26.47	26.98	29.32	34.44	34.42	33.93	35.15	29.59	35.03	33.81	30.83	34.60	31.63	34.67
50	31.27	29.09	25.89	26.19	28.26	30.87	30.85	30.80	31.13	28.20	31.10	30.73	29.38	30.93	29.64	30.95
ρ=0.85																
10	63.21	47.67	30.26	31.86	32.21	46.94	47.48	34.70	43.46	33.19	41.95	34.62	30.22	41.30	34.91	41.27
20	41.50	35.46	27.35	28.09	29.61	38.07	38.11	34.03	38.05	30.18	37.54	33.92	29.82	36.83	31.91	36.97
30	35.69	31.94	26.52	27.01	28.56	34.34	34.31	32.63	34.56	29.02	34.36	32.53	29.45	33.89	30.59	33.98
50	31.24	29.16	25.95	26.24	27.65	30.82	30.79	30.33	30.97	27.86	30.91	30.27	28.62	30.69	29.06	30.73
ρ=0.95																
10	62.88	48.14	30.28	31.80	31.66	46.40	46.93	28.25	33.18	30.61	32.19	28.24	27.56	32.20	31.30	31.99
20	41.49	35.73	27.41	28.11	28.96	37.90	37.91	28.77	33.07	28.63	32.37	28.74	27.24	32.04	29.43	32.06
30	35.83	32.24	26.68	27.15	27.93	34.38	34.33	29.10	32.22	27.98	31.79	29.06	27.27	31.41	28.80	31.49
50	31.37	29.40	26.08	26.35	27.02	30.91	30.87	28.69	30.25	27.22	30.08	28.65	27.12	29.80	27.95	29.87

Table 8: Estimated MSE when p=12

$\sigma = 0.5$																
	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75																
10	0.908	0.492	0.084	0.118	0.143	0.478	0.491	0.308	0.549	0.182	0.511	0.304	0.138	0.478	0.242	0.483
20	0.429	0.255	0.044	0.061	0.113	0.319	0.321	0.264	0.362	0.128	0.351	0.260	0.140	0.328	0.186	0.332
30	0.283	0.177	0.032	0.045	0.096	0.237	0.236	0.215	0.260	0.105	0.255	0.212	0.129	0.242	0.154	0.244
50	0.167	0.111	0.023	0.032	0.077	0.152	0.151	0.146	0.161	0.079	0.159	0.144	0.103	0.154	0.113	0.154
ρ=0.85																
10	1.542	0.835	0.126	0.178	0.201	0.761	0.783	0.315	0.676	0.235	0.614	0.312	0.137	0.581	0.299	0.583
20	0.729	0.423	0.063	0.090	0.148	0.521	0.523	0.320	0.527	0.168	0.499	0.316	0.151	0.462	0.236	0.470
30	0.477	0.288	0.043	0.062	0.119	0.386	0.384	0.280	0.397	0.136	0.384	0.276	0.146	0.358	0.198	0.363
50	0.281	0.177	0.029	0.041	0.093	0.250	0.248	0.211	0.258	0.103	0.253	0.208	0.127	0.240	0.151	0.242
ρ=0.95																
10	4.896	2.624	0.346	0.484	0.511	2.154	2.225	0.222	0.756	0.405	0.650	0.221	0.125	0.644	0.476	0.626
20	2.313	1.310	0.161	0.232	0.333	1.513	1.514	0.315	0.821	0.291	0.737	0.312	0.148	0.689	0.376	0.695
30	1.517	0.877	0.105	0.152	0.240	1.145	1.134	0.349	0.757	0.239	0.698	0.345	0.161	0.639	0.325	0.653
50	0.903	0.536	0.065	0.096	0.170	0.769	0.758	0.342	0.607	0.188	0.574	0.337	0.168	0.521	0.268	0.534
$\sigma = 1$																
	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75																
10	3.627	1.926	0.307	0.434	0.544	1.889	1.945	1.243	2.199	0.701	2.050	1.227	0.543	1.910	0.945	1.933
20	1.733	0.988	0.147	0.214	0.430	1.287	1.294	1.089	1.479	0.495	1.434	1.071	0.563	1.336	0.739	1.355
30	1.124	0.659	0.096	0.141	0.361	0.937	0.936	0.865	1.039	0.388	1.022	0.850	0.510	0.964	0.600	0.974
50	0.670	0.409	0.061	0.089	0.285	0.607	0.604	0.591	0.648	0.284	0.643	0.583	0.407	0.617	0.442	0.621
ρ=0.85																
10	6.176	3.257	0.470	0.663	0.775	3.017	3.110	1.278	2.732	0.912	2.492	1.265	0.540	2.341	1.170	2.360
20	2.912	1.643	0.220	0.320	0.560	2.063	2.071	1.286	2.113	0.640	2.009	1.267	0.593	1.847	0.916	1.884
30	1.902	1.102	0.148	0.217	0.449	1.536	1.529	1.136	1.602	0.519	1.553	1.118	0.582	1.438	0.776	1.463
50	1.121	0.670	0.089	0.133	0.350	0.996	0.989	0.851	1.036	0.384	1.021	0.839	0.506	0.962	0.594	0.972
ρ=0.95																
10	19.95	10.66	1.388	1.928	2.041	8.640	8.941	0.886	3.031	1.606	2.608	0.880	0.500	2.575	1.891	2.510
20	9.218	5.150	0.621	0.898	1.320	6.017	6.025	1.290	3.301	1.148	2.966	1.276	0.596	2.772	1.487	2.799
30	6.128	3.507	0.410	0.599	0.955	4.630	4.581	1.434	3.085	0.952	2.845	1.416	0.651	2.598	1.307	2.659
50	3.613	2.114	0.239	0.352	0.648	3.072	3.029	1.381	2.453	0.723	2.327	1.362	0.669	2.098	1.053	2.159
$\sigma = 5$																
	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75																
10	92.59	48.54	7.57	10.88	13.46	47.82	49.17	31.34	55.73	17.50	51.84	30.92	13.51	48.30	23.67	48.85
20	43.17	24.11	3.36	5.00	10.48	32.08	32.22	27.40	37.03	12.06	35.97	26.93	14.00	33.43	18.31	33.93
30	28.12	16.03	2.20	3.27	8.85	23.46	23.41	21.78	26.06	9.53	25.67	21.41	12.75	24.19	14.95	24.44
50	16.64	9.69	1.27	1.94	7.02	15.10	15.02	14.81	16.13	6.87	16.02	14.59	10.18	15.38	10.98	15.46
ρ=0.85																
10	151.9	79.38	11.66	16.53	19.00	74.70	76.86	32.09	67.73	22.51	61.63	31.75	13.42	58.03	28.93	58.37
20	73.71	41.28	5.44	7.96	13.92	52.16	52.33	32.57	53.46	16.10	50.86	32.10	14.95	46.68	23.12	47.65
30	47.43	26.97	3.43	5.10	11.03	38.29	38.10	28.50	40.06	12.68	38.89	28.05	14.44	35.93	19.18	36.58
50	28.50	16.66	2.05	3.08	8.54	25.29	25.10	21.65	26.43	9.49	26.04	21.31	12.75	24.47	14.91	24.75
ρ=0.95																
10	483.3	256.7	33.10	46.26	49.24	213.0	220.0	22.51	76.30	39.49	65.71	22.37	12.36	64.83	46.61	63.22
20	234.3	131.6	15.47	22.30	32.72	152.3	152.4	32.31	83.41	28.70	74.87	31.96	14.82	69.84	37.34	70.62
30	151.5	86.14	9.93	14.50	23.21	114.7	113.5	35.78	76.66	23.42	70.81	35.32	16.09	64.63	32.17	66.14
50	91.12	52.73	5.71	8.55	16.19	77.40	76.29	34.91	62.04	17.91	58.78	34.41	16.77	52.91	26.33	54.47

Table 9: Estimated average *k* when p=12

$\sigma = 0.5$															
	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75															
10	1.235	18.29	13.00	7.899	0.621	0.577	1.197	0.320	3.854	0.395	1.225	4.146	0.482	2.217	0.470
20	2.174	32.74	23.34	7.658	0.556	0.549	0.823	0.221	5.282	0.273	0.873	3.635	0.420	2.446	0.395
30	3.040	45.89	32.66	7.633	0.523	0.534	0.677	0.181	6.307	0.224	0.744	3.420	0.401	2.538	0.375
50	4.493	66.69	47.79	7.564	0.490	0.518	0.537	0.146	7.677	0.181	0.627	3.222	0.386	2.614	0.364
ρ=0.85															
10	0.764	11.62	8.387	7.265	0.396	0.369	1.565	0.400	3.068	0.492	1.584	4.359	0.547	1.984	0.548
20	1.396	21.59	15.52	7.577	0.354	0.354	1.066	0.274	4.268	0.338	1.101	3.884	0.453	2.290	0.431
30	1.937	30.13	21.59	7.710	0.332	0.344	0.872	0.225	5.090	0.278	0.918	3.676	0.421	2.430	0.395
50	2.964	46.69	33.69	8.013	0.311	0.336	0.684	0.177	6.379	0.219	0.747	3.491	0.392	2.588	0.365
ρ=0.95															
10	0.261	4.104	2.947	5.227	0.139	0.131	2.871	0.683	1.814	0.832	2.880	4.288	0.812	1.388	0.863
20	0.470	7.669	5.590	5.760	0.124	0.127	1.944	0.460	2.544	0.563	1.961	3.985	0.619	1.716	0.618
30	0.693	11.21	8.168	6.084	0.117	0.124	1.563	0.372	3.097	0.455	1.588	3.822	0.545	1.912	0.528
50	1.105	18.07	13.11	6.699	0.110	0.122	1.206	0.288	3.953	0.352	1.240	3.674	0.477	2.154	0.449
$\sigma = 1$															
	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75															
10	1.284	18.87	13.31	7.854	0.625	0.580	1.178	0.315	3.908	0.390	1.207	4.151	0.478	2.233	0.467
20	2.347	34.97	24.71	7.690	0.559	0.551	0.803	0.215	5.433	0.266	0.855	3.631	0.415	2.472	0.390
30	3.388	50.80	36.39	7.788	0.525	0.537	0.651	0.174	6.607	0.215	0.721	3.419	0.395	2.575	0.370
50	5.121	77.38	55.02	7.759	0.492	0.521	0.512	0.138	8.212	0.170	0.605	3.227	0.380	2.657	0.358
ρ=0.85															
10	0.783	11.93	8.49	7.445	0.397	0.371	1.555	0.401	3.089	0.493	1.574	4.370	0.546	1.992	0.548
20	1.452	22.54	16.36	7.659	0.354	0.355	1.050	0.269	4.351	0.331	1.086	3.887	0.449	2.308	0.426
30	2.102	32.14	23.07	7.840	0.332	0.345	0.852	0.220	5.246	0.270	0.900	3.682	0.416	2.461	0.389
50	3.304	51.06	36.63	8.158	0.312	0.338	0.658	0.169	6.674	0.209	0.724	3.492	0.385	2.629	0.360
ρ=0.95															
10	0.257	4.136	2.972	5.318	0.139	0.131	2.873	0.678	1.823	0.828	2.883	4.296	0.808	1.391	0.859
20	0.488	7.775	5.645	5.796	0.124	0.127	1.931	0.458	2.558	0.561	1.949	3.980	0.617	1.721	0.616
30	0.722	11.69	8.364	6.248	0.117	0.125	1.559	0.366	3.156	0.449	1.584	3.827	0.541	1.929	0.522
50	1.136	18.46	13.52	6.810	0.110	0.123	1.201	0.284	3.995	0.347	1.235	3.681	0.474	2.167	0.445
$\sigma = 5$															
	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75															
10	1.341	19.72	13.97	7.999	0.627	0.582	1.167	0.312	3.971	0.387	1.196	4.148	0.475	2.248	0.464
20	2.534	37.14	26.67	7.925	0.562	0.556	0.786	0.210	5.577	0.259	0.840	3.629	0.412	2.495	0.386
30	3.754	55.28	39.20	7.982	0.527	0.540	0.629	0.168	6.870	0.208	0.701	3.428	0.390	2.610	0.365
50	6.019	88.03	63.01	8.014	0.493	0.524	0.486	0.130	8.745	0.160	0.584	3.235	0.374	2.701	0.353
ρ=0.85															
10	0.827	12.33	8.891	7.534	0.398	0.372	1.534	0.394	3.143	0.485	1.554	4.373	0.541	2.011	0.541
20	1.540	23.37	16.43	7.667	0.355	0.355	1.037	0.267	4.418	0.329	1.073	3.896	0.447	2.325	0.424
30	2.256	33.86	24.56	7.948	0.333	0.347	0.831	0.215	5.370	0.264	0.881	3.678	0.412	2.483	0.385
50	3.682	56.14	39.66	8.303	0.313	0.339	0.638	0.163	6.972	0.202	0.707	3.497	0.381	2.661	0.355
ρ=0.95															
10	0.261	4.132	2.977	5.239	0.140	0.131	2.859	0.678	1.825	0.828	2.869	4.299	0.808	1.393	0.859
20	0.492	7.839	5.616	5.762	0.125	0.127	1.939	0.457	2.569	0.559	1.957	3.988	0.616	1.727	0.614
30	0.739	12.02	8.653	6.364	0.117	0.125	1.533	0.362	3.193	0.444	1.558	3.837	0.537	1.942	0.518
50	1.211	19.23	14.00	6.907	0.110	0.123	1.180	0.279	4.080	0.339	1.215	3.684	0.469	2.191	0.438

Table 10: Estimated Maximum MSE when p=12

	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
$\sigma = 0.5$																
ρ=0.75																
10	0.680	0.366	0.052	0.073	0.092	0.329	0.339	0.205	0.389	0.118	0.362	0.203	0.093	0.333	0.161	0.339
20	0.320	0.183	0.025	0.036	0.072	0.225	0.227	0.181	0.264	0.082	0.255	0.178	0.094	0.235	0.124	0.239
30	0.210	0.123	0.018	0.026	0.064	0.168	0.168	0.149	0.190	0.066	0.186	0.147	0.086	0.173	0.103	0.175
50	0.124	0.076	0.012	0.017	0.049	0.110	0.109	0.105	0.118	0.049	0.117	0.103	0.070	0.112	0.077	0.112
ρ=0.85																
10	1.236	0.646	0.079	0.111	0.129	0.528	0.547	0.209	0.474	0.153	0.432	0.208	0.092	0.400	0.196	0.406
20	0.583	0.322	0.037	0.053	0.095	0.375	0.378	0.210	0.385	0.107	0.364	0.208	0.100	0.327	0.155	0.336
30	0.380	0.214	0.025	0.037	0.077	0.286	0.284	0.189	0.300	0.087	0.288	0.187	0.098	0.261	0.131	0.267
50	0.227	0.133	0.016	0.023	0.060	0.192	0.191	0.150	0.202	0.066	0.198	0.148	0.086	0.183	0.103	0.185
ρ=0.95																
10	4.436	2.276	0.226	0.308	0.327	1.524	1.585	0.151	0.524	0.269	0.456	0.150	0.086	0.444	0.317	0.438
20	2.131	1.139	0.103	0.146	0.219	1.165	1.167	0.215	0.572	0.192	0.516	0.213	0.101	0.473	0.251	0.484
30	1.397	0.759	0.068	0.098	0.162	0.926	0.916	0.235	0.539	0.158	0.495	0.233	0.110	0.442	0.218	0.457
50	0.819	0.450	0.041	0.059	0.111	0.644	0.631	0.227	0.463	0.122	0.434	0.224	0.114	0.375	0.178	0.392
$\sigma = 1$																
	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75																
10	1.364	0.726	0.101	0.140	0.183	0.657	0.678	0.413	0.786	0.233	0.732	0.407	0.184	0.670	0.318	0.684
20	0.643	0.362	0.047	0.068	0.144	0.452	0.455	0.368	0.533	0.164	0.516	0.361	0.190	0.473	0.250	0.482
30	0.419	0.242	0.032	0.045	0.120	0.336	0.335	0.300	0.381	0.129	0.374	0.295	0.172	0.348	0.204	0.352
50	0.254	0.153	0.020	0.029	0.098	0.224	0.223	0.214	0.243	0.096	0.241	0.210	0.141	0.229	0.155	0.230
ρ=0.85																
10	2.485	1.304	0.156	0.217	0.253	1.056	1.090	0.420	0.956	0.303	0.870	0.416	0.183	0.804	0.392	0.818
20	1.163	0.637	0.072	0.103	0.190	0.750	0.754	0.426	0.776	0.213	0.734	0.421	0.201	0.658	0.309	0.679
30	0.759	0.425	0.048	0.069	0.153	0.573	0.570	0.384	0.605	0.173	0.582	0.378	0.198	0.526	0.264	0.539
50	0.448	0.256	0.029	0.042	0.115	0.381	0.377	0.299	0.401	0.127	0.393	0.294	0.169	0.362	0.202	0.368
ρ=0.95																
10	8.754	4.442	0.440	0.610	0.651	3.018	3.134	0.300	1.044	0.528	0.903	0.299	0.169	0.882	0.625	0.866
20	4.136	2.196	0.202	0.287	0.435	2.298	2.300	0.437	1.149	0.382	1.040	0.432	0.204	0.950	0.500	0.974
30	2.805	1.521	0.134	0.191	0.325	1.864	1.840	0.473	1.090	0.314	1.000	0.468	0.221	0.888	0.435	0.921
50	1.639	0.895	0.077	0.113	0.218	1.287	1.261	0.451	0.930	0.239	0.869	0.446	0.226	0.751	0.352	0.785
$\sigma = 5$																
	OLS	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15
ρ=0.75																
10	6.554	3.453	0.484	0.676	0.892	3.203	3.301	2.073	3.850	1.149	3.598	2.047	0.922	3.305	1.571	3.368
20	3.221	1.790	0.223	0.322	0.708	2.258	2.273	1.836	2.671	0.802	2.589	1.805	0.942	2.367	1.234	2.416
30	2.092	1.186	0.141	0.207	0.589	1.670	1.667	1.492	1.900	0.626	1.865	1.465	0.854	1.732	1.009	1.756
50	1.255	0.727	0.085	0.125	0.473	1.108	1.101	1.063	1.204	0.461	1.194	1.045	0.697	1.132	0.760	1.141
ρ=0.85																
10	12.23	6.385	0.772	1.075	1.284	5.250	5.425	2.132	4.785	1.519	4.363	2.112	0.925	4.034	1.966	4.105
20	5.848	3.209	0.351	0.506	0.933	3.772	3.795	2.142	3.898	1.062	3.687	2.113	1.009	3.312	1.549	3.411
30	3.845	2.142	0.222	0.327	0.731	2.880	2.861	1.901	3.048	0.840	2.935	1.871	0.974	2.640	1.298	2.711
50	2.275	1.287	0.133	0.194	0.578	1.933	1.914	1.525	2.047	0.628	2.011	1.498	0.857	1.845	1.018	1.878
ρ=0.95																
10	44.44	22.82	2.267	3.100	3.287	15.21	15.82	1.532	5.291	2.705	4.591	1.523	0.864	4.484	3.195	4.412
20	21.25	11.33	1.019	1.444	2.183	11.59	11.62	2.177	5.746	1.911	5.184	2.155	1.016	4.750	2.501	4.858
30	14.00	7.536	0.649	0.934	1.574	9.301	9.175	2.405	5.492	1.569	5.052	2.377	1.118	4.480	2.185	4.656
50	8.164	4.417	0.369	0.536	1.070	6.410	6.279	2.269	4.666	1.178	4.373	2.240	1.131	3.763	1.746	3.940

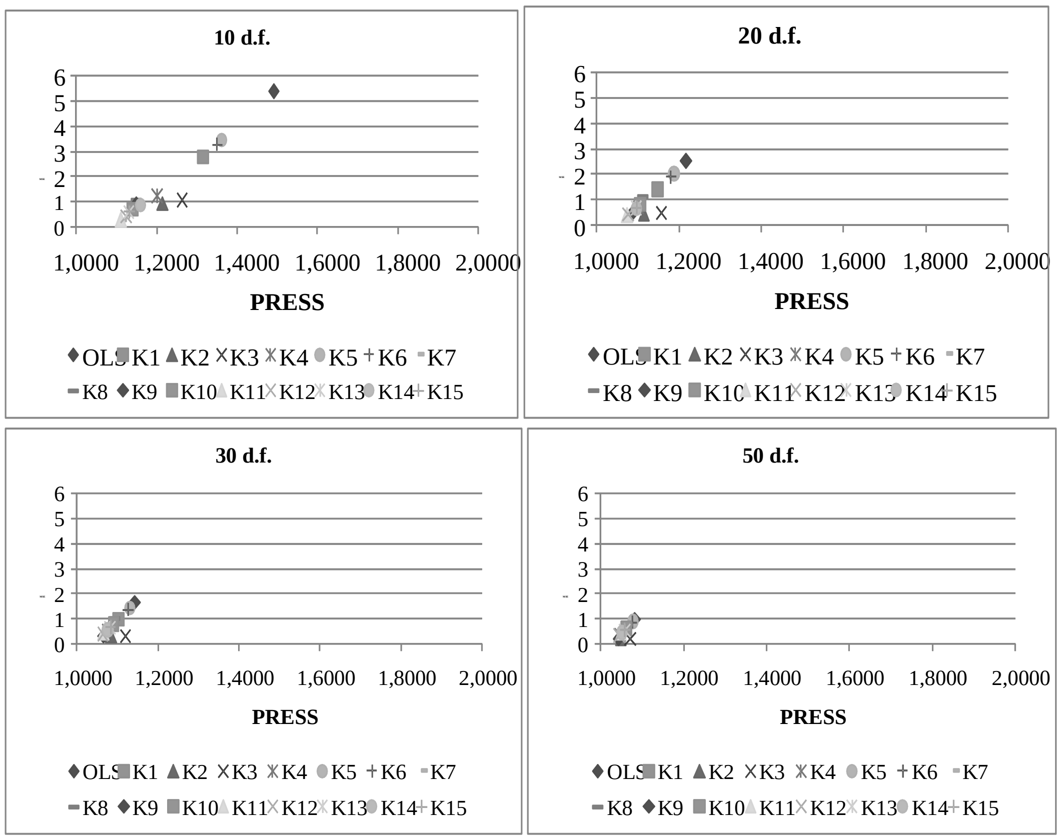

Figure 1: The relation between PRESS and MSE for the different Ridge parameters with different degrees of freedom.