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Optimality Criterion for Progressive Type-II Right Censoring Based on Awad Sup-entropy Measures

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Abstract

Censored sampling arises in life-testing experiments whenever the experimenter does not observe the failure times of all units placed on a life-test. There are several censoring schemes that have been discussed in literature. In this paper we consider Type-II progressive right censoring scheme where the unit life-times follow Pareto distribution with shape and scale parameters (θ, λ). A new optimality criterion is suggested; it depends on maximizing the efficiency of censored samples based on Awad sup-entropy measures. We find that the optimal censoring scheme is a one step censoring.

Keywords

Progressive Type-II censoring, Entropy measures, Order statistics, Pareto distribution

1. Introduction

In life-testing and reliability experiments, units may be lost or removed during experimentation before failure. The removal may be unplanned, like in accidental breakage of an experimental unit, or if a unit drops out of the experiment. Most often, the removal is pre-planned in order to save time and cost. Progressive censoring is efficient in exploitation of the available resources.

In other words, when some of the surviving units in the experiment are removed early, they can be used for some other tests. Cohen (1963) discussed the importance of progressive censoring in life-testing reliability experiments. The

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most commonly used censoring schemes are called progressive Type-I and progressive Type-II censoring schemes. In this paper we are interested in progressive Type-II right censoring experiments, where n independent units are placed on a life-test, with the corresponding failure times $X=(X_1, X_2,..., X_n)$. Assume that these failure times are continuous, independent, and identically distributed with cumulative distribution function $F(x)$ and probability density function $f(x)$ $x \epsilon (a, b)$. Suppose further that the experimenter decides to observe only m failures, $(m < n)$, and censor the remaining n-m units progressively in the following manner: At the time of first failure, R_1 of the $n-1$ surviving units are randomly removed from the test; at the time of the next failure, R_2 of the n-2-R_1 surviving units are randomly removed from the test; and so on. Finally, at the time of the m^{th} failure, all the remaining n-m-R_1-...-R_{m-1} surviving units are removed from the test $\left(R_m = n-m- R_1-\ldots-R_{m-1}\right)$. With fixed m and n, we call $(R_1,...,R_m)$ the censoring scheme. Let $X_{i:m:n}$ denote the m observed failure times (life-times), where $i = 1,2,\ldots m$. In particular if we assumed that R_i=0, for $i=1,\ldots,m,$ and $m = n$, then we will obtain a complete sample of order statistics $X_{1:n}, X_{2:n},..., X_{n:n}$, where $X_{1:n} < X_{2:n} < ... < X_{n:n}$. For more information about progressive censoring see Balakrishnan and Aggarwala (2000).

In reliability experiments, one is often interested in maximizing the information obtained from the experiment, subject to some constraints in the experimental time and the number of items used in the experiment. Suppose in a life-testing experiment, we have a fixed number of items available (n). How to allocate the items or how to censor the items become challenging questions in the planning phase of a reliability engineering program. Based on maximum likelihood and asymptotic theories, optimal solutions are obtained numerically by direct search and discrete optimization algorithm under different models. When applying progressive Type-II right censoring in life-testing experiments, the selection of an optimal censoring scheme is an important issue, so there is a need for criteria that lead to optimal designs. Different optimality criteria were discussed in literature, some of these criteria are the variance optimality (for the one-parameter case) and the trace and determinant optimality (for the multi-parameter case); these criteria intend to minimize the variance or the determinant of the variance-covariance matrix of estimator under consideration (Balakrishnan and Aggarwala, 2000; Burkschat et al., 2006 and 2007; Hofmann et al., 2005).

Ng et al. (2004), Zheng and Park (2004) and Abo-Eleneen (2008) used maximum expected Fisher information measure as an optimality criterion. Balakrishnan et al. (2008) found that the optimal censoring scheme for some distributions was the one step censoring, based on maximizing Fisher information measure.

Another criterion was introduced by Burkschat (2008), where the objective functions are the expected test duration, total time on test, and the variance of the test time; hence the optimality was achieved when these objective functions are minimized.

In this paper we are interested in finding an optimal censoring scheme in the sense of maximizing the efficiency of progressive Type-II right censored samples based on Awad sup-entropy measures. Efficiency of a censoring scheme based on a given entropy measure is the ratio of the value of that entropy in the progressive censored scheme to its value in the complete scheme.

The life-times follow Pareto distribution, which is an important life-testing model that is simple to analyze statistically. It was originally developed to describe the distribution of income. For more information about Pareto distribution, see Arnold (1983).

Fisher information in progressive censoring was discussed by some authors, and Kullback-Leibler divergence measure was also studied by Balakrishnan et al. (2007). So until now, few results are available in the area of efficiency based on entropy and information measures under progressive censored samples.

In the next section, Awad sup-entropy measures are defined. In section 3, we give some preliminary results about censored samples. In section 4 Pareto distribution is described and some moments and important Lemma are given. The efficiency based on Awad sup-entropy measures are obtained in section 5, while in section 6 we investigate optimality criterion in the sense of maximizing efficiency based on Awad sup-entropy measures. Finally, numerical illustration is given in section 7.

2. Entropy Measures

Entropy measures play an important role in electrical engineering and in the theory of statistical inference. There are several entropy measures that are defined in the literature. In this section, Awad sup-entropy measures are defined, and will be used in the sequel.

2.1 Awad sup-entropy

Awad (1987) suggested a modification of Shannon (1948) entropy, namely

$$A_1(X;\theta) = -E(\log(\frac{f(x;\theta)}{s})),$$

where $s = \sup_x f(x)$.

2.2 Awad sup-entropy of type α

Awad and Alawneh (1987) suggested a modification of Havarda-Charvat (1967) entropy measure α which is defined as:

$$A_2(X;\theta;\alpha) = \frac{1}{1-\alpha}[E((\frac{f(X;\theta)}{s})^{\alpha-1}) - 1],$$

where α is an entropy parameter, $\alpha \neq 1, \alpha > 0$, and $s = \sup_x f(x)$.

2.3 Awad sup-entropy of order α

Awad (1987) suggested a modification of Renyi (1961) entropy. This entropy measure is given by:

$$A_3(X;\theta;\alpha) = \frac{\log(E(\frac{f(X;\theta)}{s})^{\alpha-1})}{1-\alpha},$$

where α is an entropy parameter, $\alpha \neq 1, \alpha > 0$, and $s = \sup_x f(x)$.

3. Progressive Censored Samples

Let $X_1, X_2, ..., X_n$ be independent and identically distributed life-times, with continuous probability density function (p.d.f.) f, and a cumulative distribution function (c.d.f.) F, let $X_{1:n}, X_{2:n}, ..., X_{n:n}$ be the order life-times, where $X_{1:n} < X_{2:n} < ... < X_{n:n}$.

Consider progressive Type-II right censored sample with censoring scheme $(R_1, ..., R_m)$ and let $X_{1:m:n}, ..., X_{m:m:n}$, be the progressive order life-times, then the likelihood function of progressive Type-II right censored order statistics at $(x_1, x_2, ..., x_m)$ is

$$f_{X_{1:m:n},\ldots,X_{m:m:n}}(x_1,\ldots,x_m) = c\prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{R_{is}}, \qquad a < x_1 < \ldots < x_m < b \qquad (3.1)$$

where c = n(n- R_1-1)...(n- R_1 - R_2 -…- R_{m-1} -m+1), (see Balakrishnan and Aggarwala, 2000).

The cumulative distribution function of $X_{j:m:n}$ at x can be written as

$$F_{X_{j:m:n}}(x) = 1 - C_{j-1}\sum_{i=1}^{j}\frac{a_{i,j}}{r_i}[1-F(x)]^{r_i}, \qquad a<x<b,\ 1 \le j \le m \qquad (3.2)$$

where $r_j = m - j + 1 + \sum_{i=j}^{m} R_i$, $1 \le j \le m$, $C_{j-1} = \prod_{i=1}^{j} r_i$, *for* $j = 1,\ldots,m$ and

$a_{11} = 1,\ \ a_{ij} = \prod_{k=1,k\neq i}^{j}\dfrac{1}{r_k - r_i}$, $1 \le i \le j \le m$, (see Kamps and Cramer, 2001).

Hence the marginal density function of $X_{j:m:n}$ at x will be

$$f_{X_{j:m:n}}(x) = C_{j-1}\sum_{i=1}^{j} a_{ij}[1-F(x)]^{r_i - 1} f(x),\ a < x < b$$

The following remark gives some identities that will be used in the sequel.

Remark 3.1

For progressive Type-II right censored experiment the following identities hold:

1- $C_{j-1}\sum_{i=1}^{j}\dfrac{a_{i,j}}{r_i} = 1$, (Kamps and Cramer, 2001)

2- $C_{j-1}\sum_{i=1}^{j}\dfrac{a_{i,j}}{(r_i)^2} = \sum_{i=1}^{j}\dfrac{1}{r_i}$, (Balakrishnan et al., 2001)

3- $\sum_{j=1}^{m}(1+R_j)C_{j-1}\sum_{i=1}^{j}\dfrac{a_{i,j}}{(r_i)^2} = m$, (Balakrishnan et al., 2008)

4. Pareto Life-times

Pareto distribution with parameters (θ, λ), has been used commonly to model phenomenon in which the distributions of random variables of interest have long tails (see Arnold, 1983). Although it is not one of the distributions that are frequently used in life-test studies, it is an interesting distribution to examine from a mathematical viewpoint. The probability density function of Pareto distribution is given by:

$$f(x,\theta,\lambda)=\theta\lambda^{\theta}x^{-(\theta+1)}, \qquad x\geq\lambda>0,\ \theta>0$$

For a progressive Type-II right censored order statistic $X_{1:m:n},...,X_{m:m:n}$, with Pareto life-times, the likelihood function at $(x_1,x_2,...,x_m)$ is

$$f_{X_{1:m:n},...,X_{m:m:n}}(x_1,...,x_m;\theta,\lambda)=c\,\theta^m\lambda^{n\theta}\prod_{i=1}^{m}x_i^{-\theta(R_i+1)-1}, \qquad \lambda<x_1<...<x_m<\infty,$$

The marginal p.d.f. of $X_{j:m:n}$ at x is

$$f_{X_{j:m:n}}(x)=C_{j-1}\sum_{i=1}^{j}a_{i,j}\theta\lambda^{\theta r_i}x^{-\theta r_i-1}, \qquad \lambda<x<\infty$$

The expected value of $\log X_{j:m:n}$ is given by

$$E(\log X_{j:m:n})=\log\lambda+\frac{1}{\theta}\sum_{i=1}^{j}\frac{1}{r_i} \tag{4.1}$$

In particular, in the complete sample equation (4.1) becomes

$$E(\log X_{j:n})=\log\lambda+\frac{1}{\theta}\sum_{i=1}^{j}\frac{1}{n-i+1}$$

The following Lemma will be useful in finding Awad sup-entropy measures in the case censored samples in section 5.

Lemma 4.1

For a progressive Type-II right censored order statistic $X_{1:m:n},...,X_{m:m:n}$, from Pareto distribution, we have

$$E(f^{\alpha-1}{}_{X_{1:m:n},...,X_{m:m:n}}(x_1,...,x_m;\theta,\lambda))=\frac{c^{\alpha}\theta^{\alpha m}\lambda^{m(1-\alpha)}}{q(\theta,\alpha)}$$

where $q(\theta,\alpha)=m!\prod_{j=1}^{m}(\alpha(\theta r_j+1)-1)$

Proof: By successive integration we get

$$E(f^{\alpha-1}{}_{X_{1:m:n},\dots,X_{m:m:n}}(x_1,\dots,x_m;\theta,\lambda)) = c^{\alpha}\theta^{\alpha m}\lambda^{n\alpha\theta}\int_{\lambda}^{\infty}\dots\int_{x_{m-1}}^{\infty}\prod_{i=1}^{m}x_i^{-(\theta(1+R_i)+1)\alpha}dx_m\dots dx_1$$

$$= \frac{c^{\alpha}\theta^{\alpha m}\lambda^{m(1-\alpha)}}{\prod_{j=1}^{m}[\alpha(\theta r_j + m - j + 1) - m + j - 1]}$$

$$= \frac{c\theta^{m}\lambda^{m(1-\alpha)}}{m!\prod_{j=1}^{m}[\alpha(\theta r_j + 1) - 1]}$$

where r_j was defined in equation (3.2), and here we must assume that $\theta > \frac{1-\alpha}{\alpha}$, in order to obtain an integrable function.

A complete sample of order statistics can be obtained from the above Lemma by assuming that $R_i = 0$, for all i=1,2,…,m, and let m = n, and hence c = n!.
Thus

$$E(f^{\alpha-1}{}_{X_{1:n},\dots,X_{n:n}}(x_1,\dots,x_n;\theta,\lambda)) = (n!)^{\alpha-1}\lambda^{n(1-\alpha)}\frac{\theta^{\alpha n}}{(\alpha(\theta+1)-1)^{n}} \quad (4.2)$$

5. Awad Sup-entropy for Progressive Censored Samples and Efficiency

Our goal is to determine the censoring scheme which maximizes the efficiency. In this section we are going to find Awad sup-entropy measures for complete and progressive Type-II right censored samples together with the efficiency based on each measure.

Lemma 4.1, Remark 3.1, and some facts, are used here to find Awad sup-entropy measures for complete and censored samples.

5.1 Awad Sup-entropy

Consider progressive Type-II right censored sample, with s= $c(\frac{\theta}{\lambda})^{m}$, Awad sup-entropy in this case will be

$$A_{1_{cen}}(X;\theta,\lambda) = -(n\theta+m)\log\lambda + \sum_{i=1}^{m}(\theta(R_i+1)+1)(\log\lambda + \frac{1}{\theta}\sum_{j=1}^{i}\frac{1}{r_j})) = m + \frac{1}{\theta}\sum_{i=1}^{m}\sum_{j=1}^{i}\frac{1}{r_j}$$

For complete sample case $A_{1_{com}}(X;\theta,\lambda) = n + \frac{1}{\theta}n$.

Thus the efficiency based on A_1 entropy measure will be equal to $\dfrac{m+\frac{1}{\theta}\sum_{i=1}^{m}\sum_{j=1}^{i}\frac{1}{r_j}}{n+\frac{n}{\theta}}$.

It is clear that this ratio is always positive, and using identities 2 and 3 in Remark 3.1, we get $\sum_{i=1}^{m}\sum_{j=1}^{i}\frac{1}{r_j}$ ≤ m, so this efficiency will be less than or equal to one, also it can be noticed that this efficiency is free of Pareto parameter λ.

5.2 Awad Sup-entropy of Type α

For progressive Type-II right censored sample, with s = $c(\frac{\theta}{\lambda})^m$, using Lemma 4.1 we have

$$A_{2cen}(X;\theta,\lambda,\alpha) = \frac{1}{1-\alpha}[E((\frac{f_{X_{1:m:n},\dots,X_{m:m:n}}(x_1,\dots,x_m;\theta,\lambda)}{s})^{\alpha-1})-1]$$

$$= \frac{1}{1-\alpha}[\frac{\frac{c^{\alpha}\theta^{\alpha m}\lambda^{m(1-\alpha)}}{q(\theta,\alpha)}}{c^{\alpha-1}\theta^{m(\alpha-1)}\lambda^{m(1-\alpha)}}-1]$$

$$= \frac{1}{1-\alpha}(\frac{c\theta^m}{q(\theta,\alpha)}-1)$$

where $q(\theta,\alpha) = m!\prod_{j=1}^{m}[\alpha(\theta r_j+1)-1]$.

In particular ($m = n$, $R_i = 0$, $i = 1, \dots, m$) the complete sample case will be

$$A_{2com}(X;\theta,\lambda,\alpha) = \frac{1}{1-\alpha}(\frac{\theta^n}{(\alpha\theta+\alpha-1)^n}-1)$$

The efficiency based on Awad sup-entropy of type α will be $\dfrac{\frac{c\theta^m}{q(\theta,\alpha)}-1}{\frac{\theta^n}{(\alpha\theta+\alpha-1)^n}-1}$.

Notice that this efficiency is positive and always less than or equal to one, since by direct computations we have $\frac{c}{q(\theta,\alpha)} < \frac{1}{(\alpha\theta+\alpha-1)^n}$. Also it is free of Pareto parameter λ.

5.3 Awad Sup-entropy of Order α

For a progressive Type-II right censored sample, Awad sup-entropy of order α will be

$$A_{3cen}(X;\theta,\lambda,\alpha) = \frac{\log(E(\frac{f_{X_{1:m:n},\ldots,X_{m:m:n}}(x_1,\ldots,x_m;\theta,\lambda)}{s})^{\alpha-1})}{1-\alpha}$$
$$= \frac{1}{1-\alpha}\log(\frac{c\theta^m}{q(\theta,\alpha)})$$
$$= \frac{1}{1-\alpha}\log\frac{c}{q(\theta,\alpha)} + \frac{m}{1-\alpha}\log\theta$$

where $q(\theta,\alpha) = m!\prod_{j=1}^{m}[\alpha(\theta r_j+1)-1]$.

In particular for complete samples we have

$$A_{3com}(X;\theta,\lambda,\alpha) = \frac{n}{1-\alpha}\log\frac{\theta}{\alpha\theta+\alpha-1}$$

Hence the efficiency will be $\dfrac{\log\frac{c}{q(\theta,\alpha)} + m\log\theta}{n\log(\frac{\theta}{\alpha\theta+\alpha-1})}$.

This efficiency is positive and it is less than or equal to one, also it is free of Pareto parameter λ.

6. Optimal Censoring Scheme

Optimal censoring scheme will be investigated in this section where the idea of obtaining an optimal design is by maximizing the efficiency of progressive Type-II right censored samples based on Awad sup-entropy measures. First we will find the optimal one-stage censoring, and then optimal two-stage censoring, then a comparison is made to determine which censoring scheme is optimal among all censoring schemes.

Lemma 6.1

For Awad sup-entropy measure (A_1), the censoring scheme (0,..., n-m,...,0) (where $n-m$ is in the k^{th} place, and $1 \leq k \leq m$), is called optimal, if censoring occurs after the first failure, i.e. (n-m,0,…,0).

Proof : The efficiency based on A_1 entropy was given by $\dfrac{m+\dfrac{1}{\theta}\sum_{j=1}^{m}\sum_{i=1}^{j}\dfrac{1}{r_i}}{n+\dfrac{n}{\theta}}$.

This efficiency is an increasing function in $r^* = \sum_{j=1}^{m}\sum_{i=1}^{j}\frac{1}{r_i}$.

Notice that $r_i = \begin{cases} m-i+1, & \text{if } k < i \leq m \\ n-i+1, & \text{if } 1 \leq i \leq k \end{cases}$, so r_i is an increasing function in k, hence r^* is a decreasing function in k, thus in order to maximize the efficiency it is enough to minimize k, so taking k =1 will give the optimal censoring scheme which is (n-m,0,…,0).

Lemma 6.2

For Awad sup-entropy measures A_2 and A_3, the censoring scheme (0,...,n-m,...,0) (where n-m is in the k^{th} place, and 1≤k≤m), is called optimal, if censoring occurs after the first failure, i.e. (n-m,0,…,0).

Proof: The efficiency based on A_2 entropy measure depends on the censoring scheme through $\dfrac{c}{q(\theta,\alpha)}$, so we aim to investigate the behavior of $\dfrac{c}{q(\theta,\alpha)}$ with respect to k, which is the position of the censored items (n-m). We will try to show that $\dfrac{c}{q(\theta,\alpha)}$ is a decreasing function in k.

Rewrite c and $q(\theta,\alpha)$ as:

$$c = \prod_{i=1}^{m} r_i = \left(\prod_{i=1}^{k} n-i+1\right)\left(\prod_{i=k+1}^{m} m-i+1\right)$$

$$= \frac{n!}{(n-k)!}\prod_{i=k+1}^{m}(m-i+1)$$

$$q(\theta,\alpha)=m!\prod_{i=1}^{m}[\alpha(\theta r_i+1)-1]$$
$$=\prod_{i=1}^{m}[\alpha(\theta r_i+m-i+1)-m+i-1]$$
$$=\prod_{i=1}^{k}[\alpha(\theta(n-i+1)+m-i+1)-m+i-1]\prod_{i=k+1}^{m}[\alpha(\theta(m-i+1)+m-i+1)-m+i-1]$$

Hence

$$\frac{c}{q(\theta,\alpha)}=\prod_{i=1}^{k}\frac{n!/(n-k)!}{\alpha(\theta(n-i+1)+m-i+1)-m+i-1}\prod_{i=k+1}^{m}\frac{1}{\alpha(\theta+1)-1}$$
$$=(\frac{1}{\theta^*})^m\frac{n!/(n-k)!}{\prod_{i=1}^{k}(\beta-i)}$$

where $\beta=(m+1)+\frac{\alpha\theta}{\theta^*}(n-m),$ and $\theta^*=\alpha(\theta+1)-1>0$.

Let $b(k)=\frac{n!/(n-k)!}{\prod_{i=1}^{k}(\beta-i)}$, then $\frac{b(k)}{b(k+1)}=\frac{\beta-k-1}{n-k}=\frac{m-k+\alpha(n-m)\frac{\theta}{\theta^*}}{n-k}$.

Now if $\alpha>1,$ then $\frac{b(k)}{b(k+1)}<1,$ $\frac{\theta}{\theta^*}<1$, and $(\frac{\theta}{\theta^*})^n-1<0$ hence b(k) is increasing in k, and the efficiency is a decreasing function in k. In this case maximizing the efficiency is equivalent to minimizing k, i.e. $k=1$ gives the optimal design.

Similarly we can show that if $\alpha<1,$ the efficiency will be a decreasing function in k.

The efficiency based on A_3 is a function of $\log\frac{c}{q(\theta,\alpha)}$, hence maximizing the efficiency is equivalent to maximizing $\log\frac{c}{q(\theta,\alpha)}$, but $\log\frac{c}{q(\theta,\alpha)}$ is a decreasing function in k, hence the required result follows using the same argument used for A_2.

Lemma 6.3

Based on sup-entropy measures, a two-stage censoring scheme $R^* = (n-m-1,0,\ldots 0,1,0,\ldots,0)$, (where 1 is in the k^{th} place, and $2 \le k \le m$), is optimal if the second censoring occurs after the second failure, i.e. $(n-m-1,1,0,\ldots,0)$.

Proof: The proof is obvious for A_1 entropy measure, hence we are going to give the proof for A_2, and A_3.

So consider the two-stage censoring scheme R^* = *(n-m-1,0,...0,1,0,...,0),* then $r_i = \begin{cases} n-i+1, & \text{if } i=1 \\ m-i, & \text{if } 2 \le i \le k \\ m-i+1, & \text{if } k < i \le m \end{cases}$, rewriting c and $q(\theta,\alpha)$ as

$c = \frac{m!n}{m-k+1}$, and $q(\theta,\alpha) = (\theta^*)^{m-1}(m-k)!(m(\alpha-1)+\alpha\theta n)\prod_{i=2}^{k}(\gamma-i)$,

where $\gamma = 1+m+\alpha(\frac{\theta}{\theta^*})$.

Hence

$$\frac{c}{q(\theta,\alpha)} = \frac{m!n}{(m-k+1)!}\frac{1}{(\theta^*)^{m-1}(m(\alpha-1)+\alpha\theta n)\prod_{i=2}^{k}(\gamma-i)}$$

Let $a(k) = \frac{c}{q(\theta,\alpha)}$, then $\frac{a(k)}{a(k+1)} = \frac{m-k+\alpha\frac{\theta}{\theta^*}}{m-k+1}$.

Using the same argument in the proof of Lemma 6.2, we conclude that the efficiency based on A_2 and A_3 entropy measures is a decreasing function in k, hence the optimal censoring scheme is obtained when $k=2$, that is when R^* = *(n-m-1,1,0,...,0).*

Lemma 6.4

Based on A_1 sup-entropy measure the efficiency based on the censoring scheme R, is greater than the efficiency based on R^*.

Proof: For the optimal censoring scheme R=(n-m,0,…,0), we have
$r_1^* = \sum_{j=1}^{m}\sum_{i=1}^{j}\frac{1}{r_i} = \frac{m}{n} + m - 1$, and for the optimal two-stage censoring scheme R^* =
(n-m-1,1,0,...,0), $r_2^* = \sum_{j=1}^{m}\sum_{i=1}^{j}\frac{1}{r_i} = \frac{m}{n} + \frac{m-1}{m} + m - 2$.

So comparing these values it is clear that $r_1^* > r_2^*$, and since the efficiency is an increasing function r^* , we will obtain the highest efficiency if we consider the one-step censoring scheme.

Lemma 6.5

Based on A_2 sup-entropy measure the efficiency based on the censoring scheme R, is greater than the efficiency based on R^*.

Proof: Consider the censoring scheme R = (n-m,0,…,0), we obtain

$$\frac{c}{q(\theta,\alpha)} = (\frac{1}{\theta^*})^m \frac{n}{\beta-1} = (\frac{1}{\theta^*})^m \frac{n}{m+\alpha(\frac{\theta}{\theta^*})(n-m)} \qquad (6.1)$$

Now for the optimal two-stage censoring scheme, we have

$$\begin{aligned}\frac{c}{q(\theta,\alpha)} &= \frac{m}{n}\frac{1}{(\theta^*)^{m-1}(m(\alpha-1)+\alpha\theta n)(\gamma-2)} \\ &= \frac{m}{n}\frac{1}{(\theta^*)^{m-1}(m(\alpha-1)+\alpha\theta n)(m+\alpha(\frac{\theta}{\theta^*})-1)}\end{aligned} \qquad (6.2)$$

Comparing between the values of equation (6.1) and (6.2), and using some calculations we conclude that (6.1) is greater than (6.2), hence the efficiency based on A_2 sup-entropy measure is higher when we consider the one step left censoring scheme.

Using similar argument one can prove Lemma 6.5 in the case of A_3 entropy measure.

Searching for optimal censoring scheme, a sequence of arguments like those given in the previous lemmas can be done with more than two-stage censoring, that is one can consider a k-stage censoring scheme and by a similar proof show that the optimal censoring occurs when the items are censored in the first k positions, so that the optimal censoring scheme will be (n-m-(k-1),1,…,1,0,…,0), where the first zero occurs in the $(k+1)^{th}$ position. When a comparison is made between the optimal censoring schemes with different stages, we find that a one step censoring scheme has the highest efficiency with respect to other schemes, so the result will be as follows:

Result 6.1

Based on sup-entropy measures, one step left censoring scheme is more informative than $\mathrm{k-stage}$ censoring for any $k \geq 2$.

7. Numerical Illustration

In this section Mathematica 6.0 is used to obtain numerical values of the efficiency based on sup-entropy measures, with Pareto life-times.

Table 1 gives different censoring schemes with their efficiencies arranged in a descending order. From Table 1, we notice that the highest efficiency is obtained when the censoring scheme is one step left censoring which is at (3,0,0), and this is true for the three sup-entropy measures mentioned in this paper.

Table 1: Censoring schemes and their efficiencies, with n=6, m=3, θ=10, $\lambda = 1$, $\alpha = 1.5$.

	Efficiency		
Censoring Scheme	A_1	A_2	A_3
(3,0,0)	0.492	0.784	0.494
(2,1,0)	0.487	0.780	0.490
(1,2,0)	0.485	0.779	0.488
(0,3,0)	0.483	0.778	0.486
(2,0,1)	0.480	0.775	0.484
(1,1,1)	0.477	0.774	0.481
(0,2,1)	0.476	0.773	0.480
(1,0,2)	0.475	0.772	0.479
(0,1,2)	0.473	0.771	0.478
(0,0,3)	0.472	0.770	0.477

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