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Moments of Generalized Order Statistics from a General Class of Distributions

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Abstract

Order statistics, record values and several other models of ordered random variables can be viewed as special case of generalized order statistics (*gos*) [Kamps, 1995]. In this paper explicit expressions for single and product moments of generalized order statistics from a family of distributions have been obtained. Further, some deductions and particular cases are discussed.

Keywords

Generalized order statistics, Record values, Single moments, Product moments, Burr and Weibull distributions, *AMS-Subject Classification:* 62G30, 62E15

1. Introduction

Let $X_1, X_2, \cdots, X_n$ be a sequence of independent and identically distributed random variables with absolutely continuous distribution function (df) $F(x)$ and probability density function (pdf) $f(x)$, $x \in (\alpha, \beta)$. Let $n \in N$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, 2, \cdots, n-1\}$. Then $X(r, n, \tilde{m}, k)$, $r = 1, 2, \cdots n$ are called generalized order statistics (*gos*) if their joint pdf is given by

$$k\left(\prod_{j=1}^{n-1} \gamma_j\right)\left(\prod_{i=1}^{n-1} [1 - F(x)]^{m_i} f(x_i)\right)[1 - F(x_n)]^{k-1} f(x_n) \tag{1.1}$$

on the cone $F^{-1}(0) \leq x_1 < \cdots < x_n \leq F^{-1}(1)$ (Kamps, 1995).

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Let B_j, $1 \le j \le n$, be independent random variables having $Beta(\gamma_j, 1)$ distribution, then it can be seen that (Burkschat et al., 2003)

$$X(r,n,\tilde{m},k) \sim F^{-1}(1-W_r),\ r = 1,2,\cdots n. \tag{1.2}$$

where $W_r = \prod_{j=1}^{r} B_j$

Khan et al. (2008) have obtained explicit expressions for exact moments of generalized order statistics from a general form of distribution. In this paper, we have extended the results of Khan et al. (2008) and have obtained exact moments of *gos* for a family of distributions $\bar{F}(x) = [ax^p + b]^c$. For some additional results, one may refer to Keseling (1999), Kamps and Cramer (2001), Cramer and Kamps (2003), Cramer (2003), Raqab (2004), Athar and Islam (2004) and references therein.

2. Moments for Generalized Order Statistics

Let the general class of the distribution be:

$$\bar{F}(x) = [ax^p + b]^c, \quad p > 0, \eta < x < \omega, \tag{2.1}$$

where a, b and c are so chosen that $F(x)$ is a *df* over (η, ω). Then from (1.2), we have for $m_1 = m_2 = \cdots = m_{n-1} = m$,

$$X(r,n,m,k) \sim \left[\frac{1}{a} \left(\prod_{j=1}^{r} B_j^{1/c} - b \right) \right]^{1/p}. \tag{2.2}$$

2.1 Relations for Single Moments:

Theorem 2.1: For the distribution given in (2.1) and $\alpha = 1, 2, \ldots$

$$E[X^{\alpha}(r,n,m,k)] = \left(\frac{-b}{a} \right)^{[\alpha/p]} \sum_{i=0}^{[\alpha/p]} (-1)^i \frac{1}{b^i} \binom{[\alpha/p]}{i} \frac{C_{r-1}^{(k)}}{C_{r-1}^{(k+\frac{i}{c})}} \tag{2.3}$$

where $[\alpha/p]$ represent the integer part of α/p and

$$C_{r-1}^{(k+\frac{i}{c})}=\prod_{j=1}^{r}\gamma_j^{(k+\frac{i}{c})},\quad \gamma_j^{(k+\frac{i}{c})}=k+\frac{i}{c}+(n-j)(m+1) \tag{2.4}$$

Proof: From (2.2), we have

$$E[X^{\alpha}(r,n,m,k)]=E\left[\frac{1}{a}\left(\prod_{j=1}^{r}B_j^{1/c}-b\right)\right]^{\alpha/p}=\left(\frac{-b}{a}\right)^{\alpha/p}\sum_{i=0}^{\alpha/P}(-1)^i\frac{1}{b^i}\binom{\alpha/p}{i}\prod_{j=1}^{r}\frac{c\gamma_j}{c\gamma_j+i}$$

and hence the result.

Remark 2.1: At $p=1$ in (2.3), we get

$$E[X^{\alpha}(r,n,m,k)]=(-1)^{\alpha}\left(\frac{b}{a}\right)^{\alpha}\sum_{i=0}^{\alpha}(-1)^i\frac{1}{b^i}\binom{\alpha}{i}\frac{C_{r-1}^{(k)}}{C_{r-1}^{(k+\frac{i}{c})}} \tag{2.5}$$

as obtained by Khan et al. (2008).

2.2 Relations for Product Moments:

Theorem 2.2: For the distribution given in (2.1),

$$E[X^{\alpha}(r,n,m,k)X^{\beta}(s,n,m,k)]$$

$$=\left(\frac{-b}{a}\right)^{[\frac{\alpha+\beta}{p}]}\sum_{u=0}^{[\alpha/p]}\sum_{v=0}^{[\beta/p]}(-1)^{u+v}\frac{1}{b^{u+v}}\binom{[\alpha/p]}{u}\binom{[\beta/p]}{v}\frac{C_{s-1}^{(k)}\,C_{r-1}^{(k+\frac{v}{c})}}{C_{s-1}^{(k+\frac{v}{c})}\,C_{r-1}^{(k+\frac{u+v}{c})}} \tag{2.6}$$

where $[\alpha/p]$ and $[\beta/p]$ are the integer parts of α/p and β/p, respectively.

Proof: We have from (2.2)

$$E[X^{\alpha}(r,n,m,k)X^{\beta}(s,n,m,k)]$$

$$=\left(\frac{-b}{a}\right)^{[\frac{\alpha+\beta}{p}]}\sum_{u=0}^{[\alpha/p]}\sum_{v=0}^{[\beta/p]}(-1)^{u+v}\binom{[\alpha/p]}{u}\binom{[\beta/p]}{v}\frac{1}{b^{u+v}}\prod_{j=1}^{r}\frac{c\gamma_j}{c\gamma_j+(u+v)}\prod_{j=r+1}^{s}\frac{c\gamma_j}{c\gamma_j+v}$$

$$=\left(\frac{-b}{a}\right)^{[\frac{\alpha+\beta}{p}]}\sum_{u=0}^{[\alpha/p]}\sum_{v=0}^{[\beta/p]}(-1)^{u+v}\binom{[\alpha/p]}{u}\binom{[\beta/p]}{v}\frac{1}{b^{u+v}}\frac{\prod_{j=1}^{r}\gamma_j^{(k)}}{\prod_{j=1}^{r}\gamma_j^{(k+\frac{u+v}{c})}}\frac{\prod_{j=r+1}^{s}\gamma_j^{(k)}}{\prod_{j=r+1}^{s}\gamma_j^{(k+\frac{v}{c})}}$$

and hence the result.

Remark 2.2: At $\beta=0$, (2.6) reduces to single moments as given in (2.3).

Remark 2.3: At $p=1$ in (2.6), we get

$$E[X^{\alpha}(r,n,m,k)X^{\beta}(s,n,m,k)]$$

$$=(-1)^{\alpha+\beta}\left(\frac{b}{a}\right)^{\alpha+\beta}\sum_{u=0}^{\alpha}\sum_{v=0}^{\beta}(-1)^{u+v}\frac{1}{b^{u+v}}\binom{\alpha}{u}\binom{\beta}{v}\frac{C_{s-1}^{(k)}\,C_{r-1}^{(k+\frac{v}{c})}}{C_{s-1}^{(k+\frac{v}{c})}\,C_{r-1}^{(k+\frac{u+v}{c})}} \tag{2.7}$$

as obtained by Khan et al. (2008).

3. Illustrated Examples

This family, apart from the distributions considered by Khan et al. (2008) at $p=1$ also include Burr and Weibull distributions.

3.1 Single Moments

a. *Burr distribution*

$\bar{F}(x)=[\theta x^{p}+1]^{-\mu},\quad 0<x<\infty$ where $p=1/\xi>0$ and ξ in an integer.

At $a=\theta$, $b=1$ and $c=-\mu$ in (2.1), we get

$$E[X^{\alpha}(r,n,m,k)]=\left(\frac{-1}{\theta}\right)^{[\alpha/p]}\sum_{i=0}^{[\alpha/p]}(-1)^{i}\binom{[\alpha/p]}{i}\frac{C_{r-1}^{(k)}}{C_{r-1}^{(k-\frac{i}{\mu})}}$$

b. Weibull distribution

$\bar{F}(x) = [a x^p + b]^c, \quad p > 0,\ 0 < x < \infty$. Let $a = \frac{-\lambda}{c}$, $b = 1$, then we have

$\underset{c \to \infty}{Lim} \bar{F}(x) = e^{-\lambda x^p}$ by an application of result (Athar et al., 2009),

$$\sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{1}{\gamma_{r-u}^{(k)}} = \frac{(m+1)^{r-1} (r-1)!}{\prod_{j=1}^{r} \gamma_j^{(k)}}$$

We have

$$E[X^{\alpha}(r,n,m,k)] = \frac{(-1)^{[\alpha/p]} (\lambda)^{-[\alpha/p]}}{c'^{[\alpha/p]} (m+1)^{r-1} (r-1)!} \times \sum_{i=0}^{[\alpha/p]} (-1)^i \binom{[\alpha/p]}{i} C_{r-1}^{(k)} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{1}{\gamma_{r-u}^{(k+i c')}}$$

At $c' = \frac{1}{c} = 0$, above expression is of the form $\frac{0}{0}$ as $\sum_{i=0}^{[\alpha/p]} (-1)^i \binom{[\alpha/p]}{i} = 0$.

Therefore applying L' Hospital rule and using the result (Ruiz, 1996)

$$\sum_{u=0}^{n} (-1)^u \binom{n}{u} (x-u)^n = n! \qquad (3.1)$$

we have,

$$E[X^{\alpha}(r,n,m,k)] = \frac{(\lambda)^{-[\alpha/p]} [\alpha/p]!\, C_{r-1}^{(k)}}{(m+1)^{r-1} (r-1)!} \times \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{1}{\left[\gamma_{r-u}^{(k)}\right]^{[\alpha/p]+1}}, \quad m \neq -1$$

as obtained by Kamps (1995).

3.2 Product moments

a. Burr distribution

$\bar{F}(x) = [\theta x^p + 1]^{-\mu}, \quad 0 < x < \infty$ where $p = 1/\xi > 0$ and ξ in an integer.

Here $a = \theta,\ b = 1$ and $c = -\mu$,

From (2.6), we have

$$E[X^{\alpha}(r,n,m,k)X^{\beta}(s,n,m,k)]$$

$$= \left(\frac{-1}{\theta}\right)^{[\frac{\alpha+\beta}{p}]} \sum_{u=0}^{[\alpha/p]} \sum_{v=0}^{[\beta/p]} (-1)^{u+v} \frac{1}{b^{u+v}} \binom{[\alpha/p]}{u} \binom{[\beta/p]}{v} \frac{\prod_{j=1}^{r} \gamma_j^{(k)}}{\prod_{j=1}^{r} \gamma_j^{(k-\frac{u+v}{\mu})}} \frac{\prod_{j=r+1}^{s} \gamma_j^{(k)}}{\prod_{j=r+1}^{s} \gamma_j^{(k-\frac{v}{\mu})}}$$

b. Weibull distribution

$\bar{F}(x) = [ax^p + b]^c$, Here $a = -\dfrac{\lambda}{c},\ b = 1$ and $c' = \dfrac{1}{c} \to 0$, then

$$E[X^{\alpha}(r,n,m,k)X^{\beta}(s,n,m,k)]$$

$$= \left(\frac{1}{\lambda c'}\right)^{[\frac{\alpha+\beta}{p}]} C_{s-1}^{(k)} \sum_{u=0}^{[\alpha/p]} \sum_{v=0}^{[\beta/p]} (-1)^{u+v} \binom{[\alpha/p]}{u} \binom{[\beta/p]}{v} \frac{1}{\prod_{j=r+1}^{s} \gamma_j^{(k+vc')} \prod_{j=1}^{r} \gamma_j^{\{k+(u+v)c'\}}}$$

In view of the relation (Athar et al., 2009)

$$\sum_{i=0}^{s-r-1} (-1)^i \binom{s-r-1}{i} \frac{1}{\gamma_{s-i}^{(k)}} = \frac{(m+1)^{s-r-1}(s-r-1)!}{\prod_{j=r+1}^{s} \gamma_j^{(k)}}$$

$$E[X^{\alpha}(r,n,m,k)X^{\beta}(s,n,m,k)]$$

$$=\left(\frac{1}{\lambda}\right)^{[\frac{\alpha+\beta}{p}]} C_{s-1}^{(k)} \sum_{u=0}^{[\alpha/p]} \sum_{v=0}^{[\beta/p]} (-1)^{u+v} \binom{[\alpha/p]}{u}\binom{[\beta/p]}{v}$$

$$\times\left[\frac{1}{(m+1)^{s-r-1}(s-r-1)!}\sum_{i=0}^{s-r-1}(-1)^i\binom{s-r-1}{i}\frac{[\gamma_{s-i}^{(k+vc')}]^{-1}}{c'^{[\beta/p]}}\right]$$

$$\times\left[\frac{1}{(m+1)^{r-1}(r-1)!}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{[\gamma_{r-i}^{\{k+(u+v)c'\}}]^{-1}}{c'^{[\alpha/p]}}\right]$$

Taking the limit and using the relation (3.1), we get

$$E[X^{\alpha}(r,n,m,k)X^{\beta}(s,n,m,k)]=\left(\frac{1}{\lambda}\right)^{[\frac{\alpha+\beta}{p}]}\frac{C_{s-1}^{(k)}\ [\alpha/p]!\ [\beta/p]!}{(m+1)^{s-2}\ (s-r-1)!\ (r-1)!}$$

$$\times\left[\sum_{i=0}^{s-r-1}(-1)^i\binom{s-r-1}{i}\frac{1}{[\gamma_{s-i}^{(k)}]^{[\beta/p]+1}}\right]\left[\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{1}{[\gamma_{r-i}^{(k)}]^{[\alpha/p]+1}}\right],\quad m\neq -1$$

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