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On Probability Density Function of the Quotient of Generalized Order Statistics from the Weibull Distribution

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Abstract

The probability density function of $Z = \frac{X_{(i,n,m,k)}}{Y_{(j,n,m,k)}}$ where $X_{(i,n,m,k)}$ and $Y_{(j,n,m,k)}$, $(i < j)$ are ith and jth generalized order statistics from Weibull distribution by using Mellin transform technique, has been developed.

Keywords

Generalized order statistics, Mellin transform technique, Weibull distribution

1. Introduction

The distribution of the quotient of two random variables is widely used in many areas of statistical analysis especially in the problem of selection and ranking rules. They are also used in life testing and reliability problems. The distribution of the ratio of order statistics for some continuous distributions has been discussed by Malik (1982).

Let $X_{1,n,m,k} \le X_{2,n,m,k} \le --- \le X_{n,n,m,k}$ denote the generalized order statistics of a sample of size n from the pdf $f(x)$ and cdf $F(x)$ over the range $(-\infty,\infty)$, then the joint distribution of the ith and jth generalized order statistics is given by (Kamps, 1995):

$$f_{n,m,k}(x_i, y_j) = \frac{C_j}{(i-1)!\,(j-i-1)!}\left[1-F(x_i)\right]^m \left[g_m\ F(x_i)\right]^{i-1}$$
$$\left[g_m\left\{F(y_j)\right\} - g_m\left\{F(x_i)\right\}\right]^{j-i-1}\left[1-F(y_j)\right]^{\gamma_j-1} \quad f(x_i)\, f(y_j) \qquad (1.1)$$

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where $C_j = \prod_{i=1}^{j} \gamma_i$, $\gamma_r = k + (n-r)(m+1)$, $-\infty < x_i < y_j < \infty,\ 0 < i < j \le n,\ k \ge 1$ and m is a real number.

$$g_m(x) = \begin{cases} \frac{1}{(m+1)}\left[1-(1-x)^{m+1}\right], & m \ne -1 \\ -\log(1-x), & m = -1, \quad x\varepsilon(0,1) \end{cases}$$

In this paper we use the Mellin transform technique to find the distribution of the quotient of any two generalized order statistics from the Weibull distribution. We have also given specific results and its applications.

2. The Mellin Transform

Epstien (1948) was the first to suggest a systematic approach to the study of products and quotients of independent random variables by using a Mellin transform technique. Fox (1957) extended this integral transform to the two-dimensional case in order to evaluate products and ratios of random variables X , Y with pdf $f(x, y)$, which is non-negative in the first quadrant and zero elsewhere. The Mellin transform of $f(x, y)$ is defined as:

$$M(s_1, s_2) = \int_0^\infty \int_0^\infty x^{s_1-1} y^{s_2-1} f(x, y)\, dxdy \tag{2.1}$$

where s_1 and s_2 are complex variables. Under suitable conditions discussed by Fox (1957), it possesses the inverse

$$f(x, y) = \frac{1}{(2\pi i)^2} \int_{h-i\infty}^{h+i\infty} \int_{k-i\infty}^{k+i\infty} M(s_1, s_2) x^{-s_1} y^{-s_2} ds_1 ds_2$$

with the paths of integration being two lines parallel to the imaginary axis and to the right of the origin in the Argand plane.

Extensions when $f(x, y)$ is positive in all four quadrants are stated in Fox (1957) but are not required in this paper. However, we will be greatly interested in two particular cases of the above:

$$M(t/u) = M(t,) \tag{2.2}$$

$$M(t/z) = M(t, -t+2) \tag{2.3}$$

They correspond to the Mellin transforms for the pdf of the products $U = XY$ and the pdf of the ratio $Z = \frac{X}{Y}$.

We readily see then that the Mellin transform provides us with a powerful tool in reaching our objective. It enables to find the distribution of the product and quotient of jointly distributed variates when a mere transformation is either awkward or futile. It also displays its strength in simplifying the treatment of probability density functions of order statistics. A problem arises however when the inverse of the Mellin transform cannot be found among the formulae in Erdelyi et al. (1953), whereupon we must revert to an appropriate transformation.

3. Distribution of Quotient of Any Two Generalized Order Statistics from Weibull Distribution

Suppose $X_{(1,n,m,k),---,}X_{(n,n,m,k)},(k \geq 1, m \text{ is a real number})$, are n generalized order statistics from Weibull cumulative distribution:

$$F(x) = 1 - e^{-x^{\alpha}/\theta}, \text{ where } x, \alpha, \theta > 0. \tag{3.1}$$

We define ratio as:

$$Z = \frac{X_{(i,n,m,k)}}{Y_{(j,n,m,k)}}, \quad i = 1, 2, \ldots, n; \; j = 1, 2, \ldots, n \tag{3.2}$$

By using (3.1) and (3.2) in (1.1) we obtain the joint pdf of $X_{(i,n,m,k)}$ and $X_{(j,n,m,k)}$ for Weibull distribution given as :

$$f_{i,j,n,m,k}(x,y) = \frac{C_j \alpha^2}{(i-1)!(j-i-1)!(m+1)^{j-2}} (xy)^{\alpha-1} e^{-\frac{x^{\alpha}}{\theta}-\frac{y^{\alpha}}{\theta}} \left[1 - e^{-(m+1)\frac{x^{\alpha}}{\theta}}\right]^{i-1} \left[e^{-(m+1)\frac{x^{\alpha}}{\theta}} - e^{-(m+1)\frac{y^{\alpha}}{\theta}}\right]^{j-i-1} e^{-\frac{y^{\alpha}}{\theta}(\gamma_j - 1)} \tag{3.3}$$

where $0<x<y<\infty$, $0<i<j\le n$, $\alpha,\theta>0$, $k\ge 1$, m is a real number.

$C_j=\prod_{i=1}^{j}\gamma_i$ and $\gamma_i=k+(n-i)(m+1)$

The Mellin transform (2.1) of (3.3) is given by:

$$M\left(S_{1,}S_2\right)=\frac{\alpha^2 C_j}{\theta^2(i-1)!(j-i-1)!(m+1)^{j-2}}\sum_{r=0}^{j-i-1}\sum_{s=0}^{i-1}(-1)^{r+s}\binom{j-i-1}{r}\binom{i-1}{s}$$

$$\int_0^\infty\int_0^y x^{s_1+\alpha-2}y^{s_2+\alpha-2}e^{-\frac{x^\alpha}{\theta}\{(m+1)(j-i-r+s-1)+1\}}e^{-\frac{y^\alpha}{\theta}\{(m+1)r+\gamma_j\}}dxdy.$$

$$M(S_1,S_2)=\frac{\alpha^2 C_j}{\theta^2(i-1)!(j-i-1)!(m+1)^{j-2}}\sum_{r=0}^{j-i-1}\sum_{s=0}^{i-1}(-1)^{r+s}\binom{j-i-1}{r}\binom{i-1}{s}$$

$$\int_0^\infty y^{s_2+\alpha-2}e^{-\frac{y^\alpha}{\theta}\{(m+1)r+\gamma_j\}}.I.dy \qquad (3.4)$$

where

$t=\frac{x^\alpha}{\theta}\{(j-i-r+s-1)(m+1)+1\}$, $w=\frac{y^\alpha}{\theta}\{(j-i-r+s-1)(m+1)+1\}$

and

$$I=\frac{1}{\alpha}\left[\frac{\theta}{(j-i-r+s-1)(m+1)+1}\right]^{\frac{s_1-1}{\alpha}+1}\int_0^w t^{\frac{s_1-1}{\alpha}}e^{-t}dt$$

$$=\frac{1}{\alpha}\left[\frac{\theta}{(j-i-r+s-1)(m+1)+1}\right]^{\frac{s_1-1}{\alpha}+1}\gamma\left(\frac{s_1-1}{\alpha}+1,w\right) \qquad (3.5)$$

with $s_1>1-\alpha$ and $\gamma(a,x)$ is incomplete gamma function.

Putting (3.5) in (3.4) we get

$$M\left(S_1,S_2\right)=\frac{\alpha C_j}{\theta^2\left(i-1\right)!\left(j-i-1\right)!\left(m+1\right)^{j-2}}\times$$

$$\sum_{r=0}^{j-i-1}\sum_{s=0}^{i-1}\left(-1\right)^{r+s}\binom{j-i-1}{r}\binom{i-1}{s}\left[\frac{\theta}{\left(j-i-r+s-1\right)\left(m+1\right)+1}\right]^{\frac{s_1-1}{\alpha}+1}.J \tag{3.6}$$

where

$$J=\frac{1}{\alpha}\left[\frac{\theta}{\left(j-i-r+s-1\right)\left(m+1\right)+1}\right]^{\frac{s_1-1}{\alpha}+1}\int_0^\infty w^{\frac{s_2-1}{\alpha}}e^{-w\left[\frac{(m+1)r+\gamma_j}{(j-i-r+s-1)(m+1)+1}\right]}.\gamma\left(\frac{s_1-1}{\alpha},w\right)dw \tag{3.7}$$

By using Erdelyi et al. (1953), we get

$$J=\frac{\theta^{\frac{S_2-1}{\alpha}+1}\;\Gamma(\frac{s_1+s_2}{\alpha}+2)\;\left[\left(j-i-r+s-1\right)\left(m+1\right)-1\right]^{\frac{s_1-1}{\alpha}+1}}{\left[\gamma_j+\left(j-i+s-1\right)\left(m+1\right)+1\right]^{\frac{s_1+s_2-2}{\alpha}+2}\left(s_1+2-1\right)}\times$$

$${}_2F_1\left[1,\frac{s_1+s_2-2}{\alpha}+2,\frac{s_1-1}{\alpha}+2;\frac{\left(j-i-r+s-1\right)\left(m+1\right)+1}{\gamma_j+\left(j-i+s-1\right)\left(m+1\right)+1}\right] \tag{3.8}$$

With $s_1+s_2>2(1-\alpha)$ and $2F_1(a,b;c;z)$ is the usual hyper geometric function. From (3.6) and (3.8) on some simplification, we get

$$M_{(s_1,s_2)}=\frac{C_j}{(i-1)!(j-i-1)!(m+1)^{j-2}}\sum_{r=0}^{j-i-1}\sum_{s=0}^{i-1}(-1)^{r+s}\binom{j-i-1}{r}\binom{i-1}{s}$$

$$\left[\frac{\theta}{\gamma_j+(j-i+s-1)(m+1)+1}\right]^{\frac{s_1+s_2-2}{\alpha}}\frac{\Gamma(\frac{s_1+s_2-2}{\alpha})}{\left(\frac{s_1-1}{\alpha}+1\right)\left[\gamma_j+(j-i+s-1)(m+1)+1\right]^2} \tag{3.9}$$

$${}_2F_1\left[1,\frac{s_1+s_2-2}{\alpha}+2;\frac{s_1-1}{\alpha}+2;\frac{(j-i-r+s-1)(m+1)+1}{\gamma_j+(J-i+s-1)(m+1)+1}\right]$$

where $0 < i < j \le n, \quad s_1 + s_2 > 2(1-\alpha), \alpha, \theta > 0, K \ge 1$ and m is a real number. Now set $s_1 = t$ *and* $s_2 = -t + 2$ in (3.9), the Mellin transform (2.2) of the distribution of the quotient “Z” of the ith and jth generalized Oder Statistics is given as:

$$M_{(t/z)} = \frac{C_j}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{r=0}^{j-i-1}\sum_{s=0}^{i-1}(-1)^{r+s}\binom{j-i-1}{r}\binom{i-1}{s} \left[\gamma_j + r(m+1)\right]^{-2}\left[\frac{t-1}{\alpha}+1\right]^{-1} {}_2F_1\left[1,2;\frac{t-1}{\alpha}+2;\frac{(j-i-r+s-1)(m+1)+1}{\gamma_j + (J-i+s-1)(m+1)+1}\right] \tag{3.10}$$

By using Abramowitz and Stegun (1965), (3.10) can be written as:

$$M_{(t/z)} = \frac{C_j}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{r=0}^{j-i-1}\sum_{s=0}^{i-1}(-1)^{r+s}\binom{j-i-1}{r}\binom{i-1}{s} \left[\gamma_j + r(m+1)\right]^{-2}\left[\frac{t-1}{\alpha}+1\right]^{-1} {}_2F_1\left[2;\frac{t-1}{\alpha}+1;\frac{t-1}{\alpha}+2;\frac{(j-i-r+s-1)(m+1)+1}{-\gamma_j - r(m+1)}\right] \tag{3.11}$$

where $0 < i < j \le n, \quad \alpha, \theta > 0, K \ge 1$ and m is a real number. The inverse Mellin transform of (3.11) is given as:

$$g_{i,j,n,m,k}(z) = \frac{C_j \alpha}{(i-1)!(j-i-1)!(m+1)^{j-2}} \sum_{r=0}^{j-i-1}\sum_{s=0}^{i-1}(-1)^{r+s}\binom{j-i-1}{r}\binom{i-1}{s} Z^{\alpha-1} \left[\left\{\gamma_j + r(m+1)\right\} + \left\{(j-i-r+s-1)(m+1)+1\right\}Z^{\alpha}\right]^{-2} \tag{3.12}$$

where $0 < i < j \le n,\ \alpha > 0,\ 0 \le z \le \infty,\ k > 0$ and m is a real number (see Erdelyi et al., 1953).

4. Specific Results

a. If we put $i=1,\ j=n$ in (3.12), we get the pdf of the ratio of the extreme generalized Order Statistics from Weibull distribution as:

$$g_{1,n,n,m,k}(z)=\frac{C_n\alpha}{(n-2)!(m+1)^{n-2}}\sum_{r=0}^{n-2}(-1)^r\binom{n-2}{r}z^{\alpha-1}$$
$$\left[\{k+r(m+1)\}+\{(n-r+s-2)(m+1)+1\}z^{\alpha}\right]^{-2} \tag{4.1}$$

where $\alpha>0,\ 0\le z\le 1,\ k>0$ and m is a real number.

b. If we put $j=i+1$ in (3.12), we get the pdf of the ratio of the consecutive generalized Order Statistics as:

$$g_{i,i+1,n,m,k}(z)=\frac{C_{i+1}\alpha}{(i-1)!(m+1)^{i-1}}\sum_{s=0}^{i-1}(-1)^s\binom{i-1}{s}z^{\alpha-1}$$
$$\left[\{\gamma_{i+1}+r(m+1)\}+\{s(m+1)+1\}z^{\alpha}\right]^{-2} \tag{4.2}$$

where $0<i<n,\alpha>0,\ 0\le z\le 1, k>0$ and m is a real number.

c. If we take n to be odd and putting $n=j=2p+1,\ i=p+1$ in (3.12), we get the pdf of the ratio of peak to median generalized Order Statistics of a random sample of size $2p+1$ from the Weibull distribution as:

$$g_{p+1,2p+1,m,k}(z)=\frac{C_{2p+1}\alpha}{(p)!(p-1!(m+1)^{2p-1}}\sum_{r=0}^{p-1}\sum_{s=0}^{p}(-1)^{r+s}\binom{p-1}{r}\binom{p}{s}z^{\alpha-1}$$
$$\left[\{\gamma_{rp+1}+r(m+1)\}+\{(p-r+s-1)(m+1)+1\}z^{\alpha}\right]^{-2} \tag{4.3}$$

where $0\le z\le 1,\ p>1,\ k>0$ and m is a real number.

Remarks

a. By putting $m_1=\cdots\cdots=m_{n-1}=0$ and $k=1.$ in (3.12), it reduces to the density of the quotient of any two ordinary order statistics (Malik, 1982).

b. By putting $m_1=\cdots\cdots=m_{n-1}=0$ and $k=\alpha-n+1$ with $(n-1)<\alpha\in R(i.e\,\gamma_r=\alpha-r+1, 1\le r\le n-1)$ in (3.12), it reduces to the density of the quotient of any two order statistics with non-integral sample size.

c. By putting $m=(n-i+1)\alpha_i-(n-i)\alpha_{i+1}-1$, α_i is a positive real number, for $1\le i\le n-1$ ($i.e\,\gamma_r=(n-r+1)\alpha_r, 1\le r\le n-1$), and $k=\alpha_n$ in (3.12), it reduces to the density of the quotient of any two sequential order statistics.

5. Applications

The corresponding results for generalized order statistics (with $m_1=\cdots\cdots=m_{n-1}=m$ and k) for the Weibull distribution in (3.1) can easily be deduced as special case of the results in (3.12). The lth moment can be deduced from (3.12), given as:

$$\mu'_{(l,i,n,m,k)} = \frac{c_j\,\Gamma(1+l/\alpha)\;\Gamma(1-l/\alpha)}{(i-1)!\,(j-i-1)!\,(m+1)^{j-2}} \sum_{r=0}^{j-i-1}\sum_{s=0}^{i-1}(-1)^{r+s}\binom{j-i-1}{r}\binom{i-1}{s} \frac{\left[\gamma_j+r(m+1)\right]^{l/\alpha-1}}{\left[(j-i-r+s-1)(m+1)+1\right]^{l/\alpha+1}} \tag{4.4}$$

where $\mu'_{(l,r,n,m,k)}=E(Y^l_{(r,n,m,k)})$ is the l th moment of $Y_{(r,n,m,k)}$.

In particular:

a. With $n=9,\ i=1,\ j=9,\ m=0,\ k=1$ and for $l<\alpha$,(4.4) reduces to the lth moment of the ratio of the extreme generalized order statistics from Weibull distribution as:

$$\mu'_{(l,1,9,0,1)} = 72\,\Gamma(1+l/\alpha)\;\Gamma(1-l/\alpha)\sum_{r=0}^{7}(-1)^r\,\frac{\binom{7}{r}}{(r+1)(8-r)}\left(\frac{r+1}{8-r}\right)^{l/\alpha} \tag{4.5}$$

b. With $n=9,\ i=1,\ j=2,\ m=0,\ k=1$ and for $l<\alpha$, (4.4) reduces to the lth moment of the ratio of the consecutive generalized order statistics from Weibull distribution as:

$$\mu'_{(l,1,9,0,1)} = 9\,(8)^{\frac{l}{\alpha}}\ \Gamma(1+ l/\alpha)\ \Gamma(1- l/\alpha) \tag{4.6}$$

c. With $n = j,\ j = 5,\ m = 0,\ k = 1$ and for $l < \alpha$, (4.4) reduces to the *l*th moment of the ratio of peak to median generalized order statistics from Weibull distribution as:

$$\mu'_{(l,5,9,0,1)} = 3\,\Gamma(1+ l/\alpha)\ \Gamma(1- l/\alpha) \sum_{R=0}^{3}\sum_{S=0}^{4}(-1)^{r+s}\binom{3}{r}\binom{4}{s}\frac{(1+r)^{\frac{l}{\alpha}-1}}{(4-r+s)^{\frac{l}{\alpha}+1}} \tag{4.7}$$

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