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Recurrence Relations for the Product, Ratio and Single Moments of Order Statistics from Truncated Inverse Weibull (IW) Distribution

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Abstract

In this paper we established some recurrence relations for the product, ratio and single moments of order Statistics from doubly truncated Inverse Weibull distribution.

Key words

Recurrence relations, order statistics, truncation, Inverse Weibull (IW) Distribution

1. Introduction

Order statistics arise naturally in may real life applications. The moments of order statistics have assumed considerable interest in recent years and have been tabulated quite extensively for several distributions. For an extensive survey, see for example, Arnold, Balakrishnan and Nagaraja (1992) and Balakrishnan and Sultan (1998). Many authors have established some recurrence relations for the single and product moments see for example, Ali and Khan (1998) and Khan, Yaqub & Parvez (1983),Aleem and Pasha (2003),and Aleem (2004).

A random variable X has a doubly truncated Inverse Weibull Distribution with pdf $f\ (x)$ is given by:

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Let

$$f(x)=\frac{1}{P-Q}\qquad\frac{m}{\theta\,x^{m+1}}\exp^{\left[\frac{-1}{\theta\,x^{m}}\right]}\qquad Q_1\le x\le P_1\qquad(1.1)$$

$$\theta,m>0$$

And the corresponding cdf F(x)

$$F(x)=\frac{1}{P-Q}\left[\exp^{\left[\frac{-1}{\theta\,x^{m}}\right]}-Q\right]\qquad Q_1\le x\le P_1\qquad(1.2)$$

Where Q and 1-P (0 < Q < P < 1) are respectively the properties of truncation on the left and right of the Inverse Weibull distribution, respectively.

With

$$Q_1=\left[\frac{-1}{\theta\log Q}\right]^{\frac{1}{m}}\qquad\text{and}\qquad P_1=\left[\frac{-1}{\theta\log P}\right]^{\frac{1}{m}}$$

If we put Q = 0, the distribution will be truncated to the right and for P = 1, it will be truncated to the left. Where as for Q = 0 and P = 1 i.e. $Q^*=0$, get the non-truncated distribution.

Let $h(x)=\dfrac{m}{\theta x^{m+1}}$ and then from (1.1) and (1.2) we have

$$f(x)=\left[Q^*+F(x)\right]h(x)\qquad(1.3)$$

Where

$$Q^*=\frac{Q}{P-Q}$$

If we put m = 1, it reduces to the truncated Inverse Exponential distribution. If we put m = 2, it reduces to the truncated Inverse Rayleigh distribution. The doubly truncated Inverse Weibull Distribution may be used to describe some phenomena that depend on time start from $t_0\neq 0$.

2 For Single Moments

Let $X_{1;n} \le X_{2;n} \le \cdots \le X_{n;n}$ be the order Statistics. The truncated pdf of $X_{r;n}$ (1≤r≤n) is given by

$$f_{r:n}(x) = C_{r:n}\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-s} f(x) \quad a \le x \le b \tag{2.1}$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$$

Theorem 2.1

For the truncated inverse Weibull distribution (1.3),
Let

$\mu_{r:n}^{(J)} = E\left(X_{r:n}^{J}\right)$ then

$$\mu_{r+1:n}^{(J-m)} = \frac{\theta(J-m)}{mr}\mu_{r:n}^{(J)} + \mu_{r:n}^{(J-m)} - \frac{nQ^*}{r}\left[\mu_{r:n-1}^{(J-m)} - \mu_{r-1:n-1}^{(J-m)}\right] \tag{2.2}$$

Proof:

Using the pdf of $X_{r:n}$ from (2.1) with (1.3), we have

$$\mu_{r:n}^{(J)} = \frac{m}{\theta}C_{r:n}\int_{a\le x\le b} x^{J-m-1}[F(x)]^{r}[1-F(x)]^{r-n}dx + \frac{mQ^*}{\theta}C_{r:n}\int_{a\le x\le b} x^{J-m-1}[F(x)]^{r-1}[1-F(x)]^{n-r}dx$$

upon integrating by parts treating x^{J-m-1} for integration and rest of the integrand for differentiation, we get

$$= \frac{nmQ^*}{\theta(J-m)}\Big[C_{r:n-1}\int_{a\le x\le b} x^{J-m}[F(x)]^{r-1}[1-F(x)]^{n-r-1}f(x)dx - C_{r-1:n-1}\int_{a\le x\le b} x^{J-m}[F(x)]^{r-2}[1-F(x)]^{n-r}$$

$$f(x)dx\Big] + \frac{mr}{\theta(J-m)}\left[C_{r+1:n}\int_{a\le x\le b} x^{J-m}[F(x)]^{r}[1-F(x)]^{n-r-1}f(x)dx - C_{r:n}\int_{a\le x\le b} x^{J-m}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)dx\right]$$

$$\mu_{r:n}^{(J)} = \frac{nmQ^*}{\theta(J-m)}\left[\mu_{r:n-1}^{(J-m)} - \mu_{r-1:n-1}^{(J-m)}\right] + \frac{rm}{\theta(J-m)}\left[\mu_{r+1:n}^{(J-m)} - \mu_{r:n}^{(J-m)}\right]$$

Rewriting the above equation, we get

$$\mu_{r+1:n}^{(J-m)} = \frac{\theta(J-m)}{mr}\mu_{r:n}^{(J)} + \mu_{r:n}^{(J-m)} - \frac{nQ^*}{r}\left[\mu_{r:n-1}^{(J-m)} - \mu_{r-1:n-1}^{(J-m)}\right]$$

Hence the Theorem is proved.

Corollary 2.1

By repeated application of the recurrence relation in (2.2), we obtain for r ≥ 0 and J=0,1,2,…

$$\mu_{r+1:n}^{J-m} = \frac{\theta(J-m)}{m}\sum_{p=0}^{n-1}(r-p)^{-1}\left[\mu_{r-p:n}^{J} - nQ^{*}\left\{\mu_{r-p+1:n-1}^{J-m} - \mu_{r-p:n-1}^{J-m}\right\}\right]$$

3. For Product Moments

Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the Order Statistics. the doubly truncated joint pdf of $X_{r:n}$

And $X_{s:n}$ (1 ≤ r < s ≤ n) is given by

$$f_{r,s:n}(x,y) = C_{r,s:n}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} f(x)f(y) \quad a \le x < y \le b \qquad (3.1)$$

where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$$

Theorem 3.1

For truncated inverse Weibull distribution (1.3), let $\mu_{r,s:n}^{(J,k)} = E\left[x_{r:n}^{J}.x_{s:n}^{K}\right]$, then

$$\mu_{r+1,s:n}^{(J-m,k)} = \frac{\theta(J-m)}{mr}\mu_{r,s:n}^{(J,s)} + \mu_{r,s:n}^{(J-m,s)} - \frac{nQ^{*}}{r}\left[\mu_{r,s-1:n-1}^{(J-m,k)} - \mu_{r-1,s-1:n-1}^{(J-m,k)}\right] \qquad (3.2)$$

Proof:

Using the joint pdf of $X_{r:n}$ and $X_{s:n}$ from (3.1) with (1.3), we have

$$\mu_{r,s:n}^{(J,k)} = \frac{m}{\theta}C_{r,s:n}Q^{*}\iint_{a \le x < y \le b} x^{J-m-1}y^{k}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} f(y)dxdy - \frac{m}{\theta}C_{r,s:n}$$

$$\iint_{a \le x < y \le b} x^{J-m-1}y^{k}[F(x)]^{r}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} f(y)dxdy$$

Upon Integrating R.H.S with respect to X by parts taking X^{J-m-1} for integration and rest of the integrand for differentiation, we get.

$$=\frac{mn}{\theta(J-m)}Q^*\Bigg[C_{r,s-1:n-1}\iint_{a\le x<y\le b} x^{J-m}y^k[F(x)]^{r-1}[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s}f(x)f(y)dxdy-$$

$$C_{r-1,s-1:n-1}\iint_{a\le x<y\le b} x^{J-m}y^k[F(x)]^{r-2}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y)dxdy\Bigg]+\frac{mr}{\theta(J-m)}\Bigg[C_{r+1,s:n}$$

$$\iint_{a\le x<y\le b} x^{J-m}y^k[F(x)]^{r}[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s}f(x)f(y)dxdy-C_{r,s:n}\iint_{a\le x<y\le b} x^{J-m}y^k[F(x)]^{r-1}$$

$$[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y)dxdy\Bigg]$$

$$\mu_{r,s:n}^{(J,k)}=\frac{mnQ^*}{\theta(J-m)}\left[\mu_{r,s-1,n-1}^{(J-m,k)}-\mu_{r+1,s-1:n-1}^{(J-m,k)}\right]+\frac{rm}{\theta(J-m)}\left[\mu_{r+1,s:n}^{(J-m,k)}-\mu_{r,s:n}^{(J-m,k)}\right]$$

Rewriting the above equation, we get

$$\mu_{r+1,s:n}^{(J-m,k)}=\frac{\theta(J-m)}{mr}\mu_{r,s:n}^{(J,s)}+\mu_{r,s:n}^{(J-m,s)}-\frac{nQ^*}{r}\left[\mu_{r,s-1:n-1}^{(J-m,k)}-\mu_{r-1,s-1:n-1}^{(J-m,k)}\right]$$

Hence the theorem is proved.

Corollary 3.1

By repeated application of the recurrence relation in (3.2) we obtain for $0 \le r < s \le n$ and $J,k = 0,1,2,\ldots$

$$\mu_{r+1,s:n}^{(J-m,k)}=\frac{\theta(J-m)}{m}\sum_{p=0}^{n-1}(r-p)^{-1}\left[\mu_{r-p,s:n}^{(J,k)}-nQ^*\left\{\mu_{r-p+1,s-1:n-1}^{(J-m)}-\mu_{r-p,s-1:n-1}^{(J-m,k)}\right\}\right]$$

Conclusion and Remarks

The recurrence relations for the single and product moments of order statistics are established for truncated IW distribution. These relations may be used to compute the single, product and ratio moments in a simple recursive manner for any sample size. Some remarks that connect our results with relevant literature results are listed below:

1. Setting m=1 in (2.2), (2.3), (3.1) & (3.2) we get the corresponding recurrence relations for the single & product moments of order statistics in the case of IE distribution (Aleem 2003).

2. Putting m=2 in (2.2), (2.3), (3.1) & (3.2) we get the corresponding results for IR distribution (Mohsin 2001)
3. If Q=0 & P=1 then (2.2), (2.3), (3.1) & (3.2) reduces to the non - truncated case (Aleem 2003, 2005).

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