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EXACT AND LIMITING DISTRIBUTIONS OF THE RECORD VALUES

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ABSTRACT: In this paper alternative technique to derive the exact distribution of the upper (lower) record values of independent and identically distributed random variables is given. The limiting behaviors of the ratio of the various functions of upper record values are studied;

1. INTRODUCTION

Suppose we consider a sequence of products, which may fail under stress. We are interested to determine the minimum failure stress of the products sequentially. We test the first product until it fails with stress less than X_1 then we record its failure stress otherwise we consider the next product. In general we will record the failure stress X_m of the mth product if $X_m < \min(X_1, ---, X_{m-1})$. The recorded failure stresses are the lower record values of the sequence $\{X_n,\ n \geq 1\}$ are the same as the upper record values of the sequence $\{-X_{n,}\ n \geq 1\}$.

Let X_{ij} be the highest water level of a river on the jth day of the ith location. Suppose we are interested to study at each location the local maximum values of X_{ij}, and then the local maxima are the upper record values.

Chandler (1952) introduced record values, record times and inter record times. Feller (1966) gave some examples of record values with respect to gambling problems. Suppose that $X_1, X_2, \ldots$ is a sequence of independent and identically distributed random variables with distribution function F (x). Let $Y_n = \max\{X_1, \ldots, X_n\}$ for n ≥ 1. We say X_j is upper record value of , $\{X_n, n \geq 1\}$ if Y_j, Y_{j-1}, $j > 1$. By definition X_1 is an upper Record Value. Thus the upper Record Values in the sequence $\{X_n, n \geq 1\}$ are the successive maxima. Unless mentioned otherwise we will call the upper record values as record values. The indices

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at which the record values occur are given by the record times (L (n), $n \geq 0$), where

$$L(n) = \min\{j \mid j > L(n-1),\ X_j > X_{L(n-1)},\ n \geq 1\} \text{ and } L(0) = 1.$$

The record times of the sequence $\{X_n,\ n \geq 1\}$ is the same as for the sequence $F(X_n)$, $n \geq 1$. Since F (X) has uniform distribution, it follows that the distribution function of L (n), $n \geq 0$ does not depend on F. The limiting distributions of inter record time and record values are given by Ahsanullah (1995), Aleem (2000), Ahsanullah and Nevzorov (2001).

2. THE EXACT DISTRIBUTION OF THE RECORD VALUES

Many properties of the record value sequence can be expressed in terms of the functions

$R(x) = -\ln \bar{F}(x)$, $0 < \bar{F}(x) < 1$ and $\bar{F}(x) = 1 - F(x)$. If we define

$F_n(x)$ as the distribution function of $X_{L(n)}$, for $n \geq 0$, then we hav

$$F_0(x) = P[X_{L(0)} \leq x] = F(x) \tag{2.1}$$

and

$$F_1(x) = P[X_{L(1)} \leq x] \tag{2.2}$$

$$= \int_{-\infty}^{x}\int_{-\infty}^{y} \sum_{i=1}^{\infty} (F(u))^{i-1}\, df(u)\, df(y)$$

$$= \int_{-\infty}^{x}\int_{-\infty}^{y} \frac{dF(u)}{\bar{F}(u)}\, dF(y)$$

$$= \int_{-\infty}^{x} R(y)\, dF(y)$$

If F (x) has a density f (x), then $f_1(x)$, the probability density function (pdf) of $X_{L(1)}$ is

$$f_1(x) = R(x)\ f(x)\ , \qquad -\infty < x < \infty. \qquad (2.3)$$

In general it can be shown that

$$F_n(x) = \int_{-\infty}^{x} \frac{R^n(u)}{\Gamma(n+1)}\, dF(u) \qquad -\infty < x < \infty\ , \qquad (2.4)$$

And the corresponding pdf $f_n(x)$ of $X_{L(n)}$ is

$$f_n(x) = \frac{R^n(x)}{\Gamma(n+1)}\ f(x)\ , \qquad -\infty < x < \infty\ . \qquad (2.5)$$

The joint pdf $f(x_0, x_1, \dots, x_n)$ **of the n+1 record values** $X_{L(0)}, X_{L(1)}, \dots, X_{L(n)}$ is given by

$$f(x_0, x_1, \dots, x_n) = r(x_0)\ r(x_1) \dots r(x_{n-1}) \quad f(x)\ , \qquad (2.6)$$

$$\text{for } -\infty < x_0 < x_1 < \dots < x_n < \infty$$

$$= 0\ ,\ \text{otherwise}\ ,$$

where $r(x) = \dfrac{dR(x)}{dx} = f(x) / \overline{F}(x)$.

The function r (x) is known as hazard rate. The joint pdf of $x_{L(i)}, x_{L(j)}$ **is**

$$f_{ij}(x_i, x_j) = \frac{(R(x_i))^{j}}{\Gamma(i+1)}\; r(x_i).\; \frac{(r(x_j)-R(x_i))^{j-i-1}}{\Gamma(j-1)} \tag{2.7}$$

for $-\infty < x_i < x_j < \infty$

$= 0$, otherwise

In particular for $i = 0$ and $j = n$, we have

$$f_{0,n}(x_0, x_n) = r(x_0)\,\frac{(R(x_n) - R(x_0))^{n-1}}{\Gamma(n)}\; f(x_n) \tag{2.8}$$

for $-\infty < x_0 < x_n < \infty$

$= 0$, otherwise

The conditional pdf of $x_{L(j)} \mid x_{L(i)} = x_i$ is

$$f(x_j \mid x_{L(i)} = x_i) = \frac{f_{ij}(x_i, x_j)}{f_i(x_i)} \tag{2.9}$$

$$= \frac{(R(x_j) - R(x_i))^{j-i-1}}{\Gamma(j-i)} \cdot \frac{f(x_j)}{\bar{F}(x_i)},$$

for $-\infty < x_i < x_j < \infty$

$= 0$, otherwise.

3. ALTERNATIVE TECHNIQUE TO DERIVE THE EXACT DISTRIBUTIONS OF UPPER (LOWER) RECORD VALUES

Case: 1

Many properties of the upper record values sequence can be expressed in terms of the function R (x), where $R(x) = -\ln \bar{F}(x)$, $0 < \bar{F}(x) < 1$ and $\bar{F}(x) = 1 - F(x)$.

Here " ln " is used for natural logarithm. The joint pdf $f(x_1, x_2 \ldots\ldots x_n)$ of **the n record values** $X_{u(1)}, X_{u(2)}, X_{u(n)}$ is given by :

$$f(x_1, x_2, x_n) = r(x_1) \cdot r(x_2) r(x_{n-1}) \cdot f(x_n)$$

$$\text{for } -\infty < x_1 < x_2 < x_2 < \ldots\ldots < x_{n-1} < x_n < \infty$$

Where

$$r(x) = \frac{d R(x)}{d x} = \frac{f(x)}{1 - F(x)}, \quad 0 < F(x) < 1.$$

The function r (x) is known as hazard rate. The exact distribution of the nth and joint distribution of ith and jth upper record values are given by Ahsanullah (1988, 1995). We give alternative proof of these results. The marginal pdf of $X_{u(n)}$ is **given by:**

$$f_n(x) = \int_{-\infty}^{x_n} \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} r(x_1)\ r(x_2)\ r(x_2) \ldots\ldots r(x_{n-1})\ r(x_{n-1})\ f(x_n) \quad (3.1)$$

$$d x_1\, d x_2 \ldots\ldots d x_{n-1}$$

Since $R(x) = \int_{-\infty}^{x} r(w)\ dw.$

And

$$\int_{-\infty}^{x} R(w)\ r(w)\ dw = \frac{[R(w)]^2}{2} \Big|_{-\infty}^{x}$$

$$= \frac{[R(x)]^2}{2} \quad (3.2)$$

Using (3.2) in (3.1) , we get :

$$f_n(x_n) = \frac{[R(x_n)]^{n-1}}{(n-1)!} \quad f(x_n) \qquad -\infty < x_n < \infty. \quad (3.3)$$

and the cdf $F_n(x_n)$ of $X_{u(n)}$ is

$$F_n(x_n) = \int_{-\infty}^{x_n} \frac{[R(Xn)]^{n-1}}{(n-1)!} \, dF(x_n) \quad , \quad -\infty < xn < \infty .$$

(3 .3) is identical to the result (.) in Ahsanullah (1995)

The joint pdf of $X_{u(i)}$ and $X_{u(j)}$ is given as:

$$f_{ij}(x_i, x_j) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} . \int_{x_1}^{x_j}\int_{x_{j-2}}^{x_j} r(x_1) \; r(x_{j-1}) f(x_j) \tag{3.4}$$

$$dx_{j-1} \; dx_{i+1} \, . \, dx_1 \, dx_{i-1}$$

Since, for r > 0,

$$\int_x^y [R(y)-R(w)]^{\gamma-1} \; \gamma(w) \, dw = \frac{[R(y)-R(w)]^{\gamma}}{\gamma} \Big|_x^y$$

$$= \frac{[R(y)-R(x)]^{\gamma}}{\gamma} \tag{3.5}$$

Using (3 .2) and (3 .5) in (3 .4), we get:

$$f_{ij}(x_i, x_j) = \frac{[R(x_i)]^{i-1}}{(i-1)!} \; r(x_i) \; \frac{[R(x_j)-R(x_i)]^{g-i-1}}{(j-i-1)!} \; f(x_j) \tag{3.6}$$

$$\text{for } i < j \text{ and } -\infty < xi < xj < \infty$$

(3 .6) is identical to the result (?) in Ahsanullah (1995),

<u>Case: 2</u>

Many properties of the lower record values sequence can be expressed in terms of the function H (x), where H (x) = - ln F (x), 0 < F (x) < 1, here " ln " is used for the natural logarithm.

The joint pdf $f_{(1),(2),(m)}(x_1, x_2, x_n) = h(x_1) . h(x_2) \; h(x_{m-1}) \; f(x_m)$. **The function h (x) is known as hazard rate. The distribution of the mth, joint distribution of rth and sth lower record values as given by Ahsanullah (1988, 1995). We give an alternative proof of these results.**

The marginal pdf of $X_{u(m)} = x_m$ is given by

$$f_{(m)}(x) = \int_{x_m}^{\infty}\int_{x_1}^{\infty}\int_{x_2}^{\infty} h(x_1).h(x_2)\ h(x_{m-1})\ .\ f(x_m)$$

$$d x_1\ d x_2\ d x_{m-1} \qquad (3.8)$$

Since

$$-H(x) = \int_{x}^{\infty} h(u)\ du \qquad (3.9)$$

And

$$\int_{x}^{\infty} H(w)\ h(w)\ dw = \frac{-[H(w)]^2}{2}\Big|_{x}^{\infty} \qquad (3.10)$$

$$= \frac{[H(x)]^2}{2}$$

Using (3 .9) and (3 .10) in (3. 8), we get:

$$f_{(m)}(x_m) = \frac{[H(x_m)]^{m-1}}{(m-1)!}\ .\ f(x_m) \qquad (3.11)$$

$$\text{for } -\infty < x_m < \infty$$

And the cdf $F_{(m)}(x\ m)$ of $X_{2(m)}$ is

$$F_{(m)}(x\ m) = \int_{-\infty}^{x_m} \frac{[H(u)]^{m-1}}{(m-1)!}\ dF(u)$$

(3.11) is identical to the result ( ) in Ahsanullah (1995)

The joint pdf of $X_{L(r)}$ and $X_{L(s)}$ is given as:

$$f_{(r),(s)}(x_r, x_s) = \int_{x_r}^{\infty}\int_{x_3}^{\infty}\int_{x_2}^{\infty}\int_{x_s}^{x_r}\int_{x_s}^{x_{s-3}}\int_{x_s}^{x_{s-2}} h(x_1) - - - - - h(x_{s-1})\ f(x_s)$$

$$dx_{s-1}\ dx_{s-2}\ \ dx_{r+1}\ dx_1\ dx_2\ \ dx_{s-1} \qquad (3.12)$$

Since, for $\gamma > 0$,

$$\int_{y}^{x} [H(y)-H(w)]^{\gamma-1} h(w)\, dw = \frac{[H(y)-H(w)]^{\gamma}}{\gamma} \Big|_{y}^{x} \qquad (3.13)$$

$$= \frac{[H(y) - H(x)]^{\gamma}}{\gamma}$$

Using (3 .9), (3 .10) and (3 .13) in (3 .12), we get:

$$f_{(r),(s)}(x_r, x_s) = \frac{[H(x_r)]^{r-1}}{(r-1)!} h(x_r). \frac{[H(x_s)-H(x_r)]^{s-r-1}}{(s-r-1)!} f(x_s) \qquad (3.14)$$

$$\text{for } s > r \text{ and } -\infty < x_s < x_r < \infty$$

(3 .14) is identical to the result (.) in Ahsanullah (1995).

4. LIMITING DISTRIBUTIONS OF THE RATIOS OF THE UPPER RECORD VALUES

Let $X_{u(1)}, X_{u(2)}, X_{u(n)}$ **are the upper record values with a common distribution function F (x) as:**

$$F(x) = P[R(x_i) \leq x]$$

<u>**Case: 1**</u>

Consider now the ratios :

$$Zi = \frac{j R[x_{u(i)}]}{R[x_{u(j)}]} , \quad i < j \text{ and } i = 1,2,j-1. \qquad (4.1)$$

Using (4.1) in (3.6) and integrating for marginal pdf $f_1(z)$ **of z, we have:**

$$f_1(z) = \frac{\Gamma(j)}{(j)^{i} (i-1)!(j-i-1)!} z^{i-1} \left(1 - \frac{z}{j}\right)^{j-i-1} \qquad (4.2)$$

Where 0 < z < ∞, i < j.
Then the density of z in (4.2), for large j, coverage's to :

$$\lim_{j \to \infty} f_1(z) = \frac{z^{i-1} e^{-z}}{(i-1)!} = \Phi_i(z) \qquad (4.3)$$

Case: 2

Now for the same random variables as considered in (4 .1), let us define :

$$P = \log\left[\frac{Ru(j)}{jRu(i)}\right] \tag{4.4}$$

, i < j , i = 1, 2, -----, j –1

Using (4.4) in (3.6) and integrating for marginal pdf $f_2(p)$ **of p, we have:**

$$f_2(p) = \frac{\Gamma(j)}{(j)^i (i-1)!\,(j-i-1)!}\,(e^{-p})^i \left(1 - \frac{e^{-p}}{j}\right)^{j-i-1} \tag{4.5}$$

where $-\infty \leq p \leq \infty$

The density of P in (4.5), for large j, coverage's to :

$$\lim_{j\to\infty} f_2(p) = \frac{1}{(i-1)!}\,(e^{-p})^i \exp(-e^{-p}) \tag{4.6}$$

We observed that for large j, i = 1 , $f_2(p) \to$ **type I extreme value (Gumbel) distribution.**

Case: 3

Now for the same random variables as considered in (4 .1) , let us define :

$$w = \frac{j.Ru(i)}{Ru(i) + Ru(j)} \quad , i < j \tag{4.7}$$

Using (4.7) in (3.6) and integrating for marginal pdf $f_3(w)$ **of w, we have:**

$$f_3(w) = \frac{\Gamma(j)}{(j)^i (i-1)!\,(j-i-1)!}\,w^{i-1}\left(1 - \frac{2w}{j}\right)^{j-i-1}\left(1 - \frac{w}{j}\right)^{-j}$$

where $0 \leq w \leq \infty$ **(4.8)**

Then the density of z in (4.8), for large j, coverage's to :

$$\lim_{j\to\infty} f_3(z) = \frac{1}{(i-1)!} z^{i-1} e^{-z} = \Phi_i(z) \tag{4.9}$$

Case : 4)

Now for the same r.v's as considered in (4.1) let us define

$$Y = \log\left[\frac{Ru(i) + Ru(j)}{jRu(i)}\right] \quad , \ i < j \tag{4.10}$$

Using (4.10) in (3.6) and integrating for marginal pdf $f_4(y)$ of y, we have:

$$f_4(y) = \frac{\Gamma j}{(j)^i (i-1)!(j-i-1)!} (e^{-Y})^i \left(1 - \frac{2e^{-Y}}{j}\right)^{j-i-1} \left(1 - \frac{e^{-y}}{j}\right)^{-j} \tag{4.11}$$

where $-\infty \leq y \leq \infty$

The density of y in (4.11), for large j, coverage's to:

$$\lim_{j\to\infty} f4(y) = \frac{1}{(i-1)!} (e^{-Y})^i e^{-e^{-Y}} \tag{4.12}$$

We observed that for large j , i = 1, $f_4(y) \to$ **type I extreme value (Gumble) distribution .**