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CLASSICAL PROPERTIES, RATIO, PRODUCT AND INVERSE MOMENTS OF ORDER STATISTICS AND CHARACTERIZATION FROM INVERSE WEIBULL DISTRIBUTION

M.Aleem[1]

ABSTRACT: In this paper besides studying the classical properties, ratio and product moments of two order statistics of different orders from Inverse Weibull distribution have been obtained. Further, moments and inverse moments of single order statistics of different orders are obtained.
A characterization of the Inverse Weibull distribution based on the distribution $Y = Max(y_1, y_2, ..., y_n)$ for $X_i, (i = 1,2,...,n)$ is also given.

1.INTRODUCTION

The probability dessity function of inverse Weibull distribution is obtained by using $X = \frac{1}{Z}$, where z is weibull random variable with parameters m and θ, as.

$$f(x) = \frac{m}{\theta}\frac{1}{x^{m+1}}\exp\left(-\frac{1}{\theta\, x^{m}}\right) \qquad , x > 0,\ m,\ \theta > 0. \qquad (1.1)$$

$= 0$ other wise .

where $f(x) \to 0 \ as \ x \to 0 \ and \ f(x) \to 0 \ as \ x \to \infty.$

and the corresponding distribution function is:

$$F(x) = \exp\left(-\frac{1}{\theta\, x^{m}}\right) \qquad , x > 0,\ m, \theta > 0. \qquad (1.2)$$

Here m is the shape parameter. At m=1, it reduces to the inverse exponential distribution, whereas at m=2 , it is the Inverse Rayleigh distribution. Inverse Rayligh distribution was first considered by Voda(1972), Mukerjee and Saren(1984), Gharraph(1993), Mukerjee and Maiti(1996). For different distributional properties of ordered variables see

[1] Associate Professor Deptt. of Statistics Islamia University Bahawalpur, Pakistan
E-mail: draleemiub@hotmail.com

Balakrishman & Rao (1998), Harter and Balakrishnana(1998) and Ahsanullah and Nevzorov (2001), Recently order statistics and lower record statistics of Inverse Rayleigh distribution have been studied by Mohsin (2001). But none paid any attention to Inverse Weibull distribution.
The rth moment of Inverse Weibull distribution is

$$E(X)^r = \frac{1}{\frac{r}{\theta m}}\Gamma\left(1-\frac{r}{n}\right) \quad , r < m \qquad (1.3)$$

In particular, we have

$$E(X) = \frac{1}{\frac{1}{\theta m}}\Gamma\left(1-\frac{r}{n}\right) \qquad (1.4)$$

$$Var(X) = \frac{1}{\frac{2}{\theta m}}\left[\Gamma\left(1-\frac{2}{m}\right)-\left(\Gamma\left(1-\frac{1}{m}\right)\right)^2\right] \qquad (1.5)$$

It is observed that the moments of order r ≥ m does not exists.
The rth negative moment is

$$E(X)^{-r} = \theta^{\frac{r+2}{m}-1}\,\Gamma\left(\frac{r+2}{m}\right) \qquad (1.6)$$

The p-th percentile is

$$\xi_p = \left(\frac{1}{\theta\, Ln\, p^{-1}}\right)^{\frac{1}{m}} \qquad (1.7)$$

Table 1.1 values of $\theta(\xi_p)^m$ for some selected values of P.

P	0.01	0.05	0.10	0.25	0.50	0.75	0.95	0.99
$\theta(\xi_p)^m$	0.217147	0.333808	0.434294	0.721347	1.442695	3.476059	19.495725	99.49926

$\zeta_{0.5}$ gives the median of the distribution.

$$Mode = \left[\frac{m}{\theta(m+1)}\right]^{\frac{1}{m}} \qquad (1.8)$$

$$H.M = \frac{1}{\theta^{\frac{1}{m}}\Gamma\left(1+\frac{1}{m}\right)} \tag{1.9}$$

2.Order Statistics for Inverse Weibull Distribution

Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ **be the order statistics, then the pdf of** $X_{r:n}(1 \le r \le n)$ **form Inverse Weibull distribution is given by:**

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}\frac{m}{\theta x^{m+1}}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i e^{-\frac{1}{\theta x^m}(r+1)}$$

, x≥0, m, θ>0

(2.1)

and the joint pdf of $X_{r:n}$ **and** $X_{s:n}$ $(1 \le r \le s \le n)$ **is given by**

$$f_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-n-r)!(n-s)!}\frac{m^2}{\theta^2 (xy)^{m+1}}\sum_{i=0}^{s-r-1}\sum_{j=0}^{n-s}(-1)^{i+1}$$

$$\binom{s-r-1}{i}\binom{n-s}{j}e^{-\frac{1}{\theta x^m}(r+1)}e^{-\frac{1}{\theta y^m}(s-r-i-j)}$$ **,0 ≤ x ≤ y ≤ ∞, m, θ > 0.**

(2.2)

where $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$ and $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$

<u>Theorem 2.1:</u> For Inverse Weibull distribution (1.1), Let

$$\mu_{r:n}^{(k-m)} = E\left(X_{r:n}^{k-m}\right), then$$

$$\mu_{r:n}^{(k-m)} = \frac{C_{r:n}\Gamma\left(2-\frac{k}{m}\right)}{\theta^{\frac{k}{m}-1}}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i (a_j)^{\frac{k}{m}-2} \tag{2.3}$$

<u>Proof:</u> Using the pdf of $X_{r:n}$ from (2.1) we have

$$\mu_{r:n}^{(k-m)} = C_{r:n}\frac{m}{\theta}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i\int_0^{\infty} x^{k-2m-1}e^{-\frac{a_i}{\theta x^m}}dx$$

where $a_i = r + 1$
and

$$\mu_{r:n}^{(k-m)} = \frac{C_{r:n}\Gamma\left(2-\frac{k}{m}\right)}{\theta^{\frac{k}{m}-1}}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i(a_j)^{\frac{k}{m}-2} \tag{2.4}$$

which can be used to obtain moments and Inverse Moments of any order. If we set k – m =1 in (2.4), we get the mean of the rth order statistics of Inverse Weibull distribution.

<u>Theorem 2.2:</u> For Inverse Weibull distribution (1.1), Let

$$\mu_{r,s:n}^{(k,L-m)} = E\left(X_{r:n}^{k}.X_{s:n}^{L-m}\right), then$$

$$\mu_{r,s:n}^{(k,L-m)} = \frac{C_{r,s:n}}{\theta^{\frac{k+L}{m}-1}}\sum_{i=0}^{s-r-1}\sum_{j=0}^{n-s}(-1)^{i+j}\binom{s-n-r}{i}\binom{m-s}{j}(a_i)^{\frac{k}{m}-1}$$
$$\left[\Gamma\left(3-\frac{k}{m}-\frac{L}{m}\right)b_{ij}^{\frac{k}{m}+\frac{L}{m}-3} - b_{ij}^{\frac{L}{m}-2}\Gamma\left(1-\frac{k}{m}\right)\Gamma\left(2-\frac{L}{m}\right)I_p\left(1-\frac{k}{m},2-\frac{L}{m}\right)\right] \tag{2.5}$$

where $a_i = r + i, b_{ij} = s - r - i + j, P = \dfrac{a_i}{a_i + b_{ij}}$

and $I_p(a,b) = \dfrac{1}{B(a,b)}\int_0^1 x^{a-1}(1-x)^{b-1}dx$ is incomplete beta function.

<u>Proof:</u> Using the joint pdf of X_{rn} and X_{sn} form (2.2) we have

$$\mu_{r,s:n}^{(k,L-m)} = C_{r,s:n}\frac{m^2}{\theta^2}\sum_{i=0}^{s-r-1}\sum_{j=0}^{n-s}(-1)^{i+j}\binom{s-r-1}{i}\binom{n-s}{j}$$

$$\int_0^\infty y^{L-2m-1} e^{-\frac{b_{ij}}{\theta\, y^m}} I(y)dy \tag{2.6}$$

where $$I(y) = \int_0^y x^{k-m-1} e^{-\frac{a_i}{\theta\, x^m}} dx \tag{2.7}$$

and $$I(y) = \frac{(a_i)^{\frac{k}{m}-1}}{m} y^{k-m}\left[1 - \int_0^{a_i} w^{-\frac{k}{m}} e^{-\frac{w}{\theta\, y^m}} dw\right] \tag{2.8}$$

Putting I(y) from (2.8) in (2.6) and integrating the resultant expression we get.

$$\mu_{r,s:n}^{(k,L-m)} = \frac{C_{r,s:n}}{\theta^{\frac{k+L}{m}-1}} \sum_{i=0}^{s-r-1}\sum_{j=0}^{n-s}(-1)^{i+j}\binom{s-r-1}{i}\binom{n-s}{j}(a_j)^{\frac{k}{m}-1}\left[\Gamma\left(3-\frac{k}{m}-\frac{L}{m}\right)b_{ij}^{\frac{k}{m}+\frac{L}{m}-3} - b_{ij}^{\frac{L}{m}-2}\,\Gamma\left(1-\frac{k}{m}\right)\Gamma\left(2-\frac{L}{m}\right)I_p\left(1-\frac{k}{m},2-\frac{L}{m}\right)\right] \tag{2.9}$$

where $a_i = r+i, b_{ij} = s-r-i+j, p = \dfrac{a_i}{a_i+b_{ij}}$

and $I_p(a,b) = \dfrac{1}{\beta(a,b)}\int_0^1 x^{a-1}(1-x)^{b-1}dx$ **is an incomplete Beta function.**

This completes the proof of the Theorem. One may refer Pearson(1955) for the values of $I_p(.,.)$

Now (2.9) can be used to obtain the moments of ratio (when L< m) of two order statistics as well as product moments (L > m) for all 1 ≤ r ≤ s ≤ n. For thw ratio, Let L – m = - k then

$$\mu_{r,s:n}^{k,-k} = E\left(X_{r:n}\big/X_{s:n}\right)^k \;\forall k.$$

For finding $\mu_{r,s:n}^{(k,L)}$ replace L-m by L in (2.5).

Theorem 2.3: For Inverse Weibull distribution (1.1)

$$\mu_{r,s:n}^{k,L-m} = \frac{n!}{(r-1)!(n-r-1)!\theta^{\frac{k+L}{2}-1}} \sum_{j=0}^{n-r-1} (-1)^j \binom{n-n-r}{j} (a_i)^{\frac{k}{m}-1} \left[\Gamma\left(3-\frac{k}{m}-\frac{L}{m}\right) b_j^{\frac{k+L}{m}-3} - b_j^{\frac{L}{m}-2} \right.$$

$$\left. \Gamma\left(1-\frac{k}{m}\right)\Gamma\left(2-\frac{L}{m}\right) I_p\left(1-\frac{k}{m}, 2-\frac{L}{m}\right)\right] \qquad \textbf{(2.10)}$$

where $a_i = r, b_j = j+1, p = \frac{a_i}{a_i + b_j}$

and $I_p(a,b) = \frac{1}{B(a,b)} \int_0^1 x^{a-1}(1-x)^{b-1} dx$ is incomplete ¾ta function.

Proof: The results is easily proved by putting s = r + 1 in (2.5)

Theorem 2.4: Let $X_i (i = 1,2,...,n)$ be i.i.d random variable. Then $Max(X_1, X_2,..., Xn)$ has Inverse Weibull distribution if and only the common distribution of X_i's is an Inverse Weibull distribution.

Proof: Let $X_i (i = 1,2,...,n)$ be i.i.d random variable each with Inverse Weibull distribution (1.1) and let Y= $Max(X_1, X_2,..., Xn)$ then

$$P(Y < y) = P[Max(X_1, X_2,..., Xn) < y]$$

$$= P\left(\prod_{i=1}^{n} X_i < Y\right)$$

$$= \prod_{i=1}^{n} P(X_i < y) = [P(X_i < y)]^n \qquad \textbf{(2.11)}$$

Now using (1.1), we get

$$P(X_i < y) = \frac{m}{\theta} \int_0^y \frac{1}{x^{m+1}} e^{-\frac{1}{\theta x^m}} dx$$

and

$$P(X_i \le y) = e^{-\frac{1}{\theta\, y^m}} \quad (2.12)$$

substituting (2.12) in (2.11) we get

$$P(Y < y) = \exp\left[-\frac{1}{(\theta\, n^{-1})y^m}\right]$$

This implies that Y has the same Inverse Weibull distribution as X_i's with the difference that the parameter θ is replaced by (θn^{-1}).

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