ISSN 1684-8403
J. Stat.
Vol. 10, No.1 (2003) PP 27-37

# RELATIVE EFFICIENCIES OF MAXIMUM LIKELIHOOD, MINIMUM CHI-SQUARE, AND MINIMUM B ESTIMATORS.

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**ABSTRACT** Debate on the choice between maximum likelihood estimator and minimum chi-square estimator is still on. Rao's school of thought advocates for the maximum likelihood estimator and Berkson's school of thought for the minimum chi-square estimator. In this paper a numerical study has been made by computing the relative efficiency of these estimators together with that of minimum B estimator for the Exponential parameter. The study reveals that for few choices of 'A' and 'B', the constants in Statistic B introduced by Khan and Baily (1987), the minimum B estimator is more efficient than the other two. Also maximum likelihood estimator is more efficient than the minimum chi-square estimator.

## 1. INTRODUCTION

Khan and Baily(1987) have shown that if X is Poisson random variable with mean μ then $X^{2/3}$ is symmetric about μ and $E(X+A)^{2/3} = \mu^{2/3} + O(\mu^{-1/3})$ and $Var(X+A)^{2/3} = 4/9(\mu^{1/3}) + O(\mu^{-2/3})$, where A is some constant. Following Fisher's (1922) approach they proposed a new $X^2$-type test statistic of goodness of fit as:

$$B = \frac{9}{4}\sum_{r=1}^{K}\frac{((o_r + A)^{\frac{2}{3}} - (Nf_r[\theta] + B)^{\frac{2}{3}})^2}{(Nf_r[\theta])^{\frac{1}{3}}} \qquad (1.1)$$

Where $o_r$ is assumed to be distributed as multinomial with $E(o_r) = Nf_r$, $f_r[\theta]$ is the probability that the observation fall in the rth cell, and A, B are some constants.

Khan (1988a) has shown that the estimator of θ, say $\hat{\theta}_B$ obtained by minimizing statistic B has variance as:

Var ($\hat{\theta}_B$) = Var (mle) +O ($N^{-2}$) which reduces to Var (mle) for the choices of A=B=0.0 and A= (3/2) B, where Var (mle) is the variance of maximum likelihood estimator of θ obtained by Rao (1962).

In the estimation theory there are differences of opinions on the efficiencies of minimum chi-square estimator and the maximum likelihood estimator. Berkson (1980) went for the minimum chi-square estimate while

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Robert and Rao (1993) advocated for maximum likelihood estimate. A numerical study has been carried out to investigate the relative efficiencies of minimum B estimate, minimum chi-square estimate and the maximum likelihood estimate of the parameter of exponential distribution.

## 2. SUCCESSIVE APPROXIMATION OF MLE, MCSE, AND MBE

### Maximum Likelihood Estimate (MLE)

Let a random sample $X_1, X_2 \ldots X_N$ be taken from a distribution $f(x, \theta)$ and categorized in K cells. Let $N_r$ and $N f_r[\theta]$ be the observed and expected frequencies respectively of the observation falling in the rth cell, where $f_r[\theta]$ is the probability of the observation falling in the rth cell. The log likelihood function for the sample observations is defined as:

$$\ln L(x,\theta) = \sum_{r=1}^{K} N_r \ln f_r[\theta]$$

The MLE say $\hat{\theta}_M$ of θ is the solution of $\frac{\partial \ln L(x,\theta)}{\partial \theta} = 0$. Expressing

$\frac{\partial \ln L(x,\hat{\theta}_M)}{\partial \hat{\theta}_M}$ in Taylor's series about $\theta_0$ up to first two terms we get

$$\frac{\partial \ln L(x,\hat{\theta}_M)}{\partial \hat{\theta}_M} = 0 = \frac{\partial \ln L(x,\theta_0)}{\partial \theta_0} + (\hat{\theta}_M - \theta_0)\frac{\partial^2 \ln L(x,\theta_0)}{\partial \theta_0^2}$$

This gives

$$\hat{\theta}_M = \theta_0 - \left(\frac{\partial \ln L(x,\theta_0)}{\partial \theta_0}\right) / \frac{\partial^2 \ln L(x,\theta_0)}{\partial \theta_0^2} \qquad (2.1)$$

The MLE of θ is obtained by successive approximation using (2.1) which give the first approximation $\hat{\theta}_M$ as:

$$\hat{\theta}_M = \theta_0 - \sum_{r=1}^{K} \left(\frac{N_r f_r'[\theta_0]}{f_r[\theta_0]}\right) / \sum_{r=1}^{K} \left(-\frac{N_r f_r'[\theta_0]^2}{f_r[\theta_0]^2} + \frac{N_r f_r^{(2)}[\theta_0]}{f_r[\theta_0]}\right) \qquad (2.2)$$

Where $\theta_0$ is the initial value.

### Minimum Chi-square Estimate (MCSE)

The chi-square statistic of goodness of fit is defined as

$$X^2 = \sum_{r=1}^{K} \frac{(N_r - Nf_r[\theta])^2}{Nf_r[\theta]}$$

The MCSE say $\hat{\theta}_C$ of θ is the solution of $\frac{\partial X^2}{\partial \theta} = 0$. Following the above procedure the MCSE is obtained by successive approximation through

$$\hat{\theta}_C = \theta_0 - (\frac{\partial X^2)}{\partial \theta_0}) / \frac{\partial^2 X^2}{\partial \theta_0^2}$$

The first approximation is therefore

$$\hat{\theta}_C = \theta_0 - \sum_{r=1}^{K} \frac{p_r^2 f_r'[\theta_0]}{f_r[\theta_0]^2} / \sum_{r=1}^{K} (-\frac{2p_r^2 f_r'[\theta_0]^2}{f_r[\theta_0]^3} + \frac{p_r^2 f_r^{(2)}[\theta_0]}{f_r[\theta_0]^2}) \quad (2.3)$$

where $p_r = \frac{N_r}{N}$.

Minimum-B Estimate (MBE)

By minimizing the B-statistic of goodness of fit defined in (1.1) can be written as

$$B = \frac{9}{4} \sum_{r=1}^{K} \frac{((N_r + A)^{\frac{2}{3}} - (Nf_r[\theta] + B)^{\frac{2}{3}})^2}{(Nf_r[\theta])^{\frac{1}{3}}}$$

and using the same procedure as above, the MBE is obtained by successive approximation through

$$\hat{\theta}_B = \theta_0 - (\frac{\partial B)}{\partial \theta_0}) / \frac{\partial^2 B}{\partial \theta_0^2} \quad (2.4)$$

So the first approximation of MBE of θ is obtained from

$$\hat{\theta}_B = \theta_0 - \sum_{r=1}^{K} (\frac{(A1 - (B1 + f_r[\theta_0])^{\frac{2}{3}})^2}{f_r[\theta]^{\frac{1}{3}}}) /$$

$$\sum_{r=1}^{K} (\frac{8 f_r'[\theta_0]^2}{9 f_r[\theta_0]^{\frac{1}{3}} (B1 + f_r[\theta_0])^{\frac{2}{3}}} + \frac{4(A1 - (B1 + f_r[\theta_0])^{\frac{2}{3}}) f_r'[\theta_0]^2}{9 f_r[\theta_0]^{\frac{1}{3}} (B1 + f_r[\theta_0])^{\frac{4}{3}}}$$

$$+ \frac{8(A1 - (B1 + f_r[\theta_0])^{\frac{2}{3}}) f_r'[\theta_0]^2}{9 f_r[\theta_0]^{\frac{4}{3}} (B1 + f_r[\theta_0]^{\frac{1}{3}}} + \frac{4(A1 - (B1 + f_r[\theta_0])^{\frac{2}{3}})^2 f_r'[\theta_0]^2}{9 f_r[\theta_0]^{\frac{2}{3}}}$$

$$\frac{4(A1-(B1+f_r[\theta_0])^{\frac{2}{3}})f_r^{(2)}[\theta_0]}{3f_r[\theta_0]^{\frac{1}{3}}(B1+f_r[\theta_0])^{\frac{1}{3}}}-\frac{(A1-(B1+f_r[\theta_0])^{\frac{2}{3}})^2 f_r^{(2)}[\theta_0]}{3f_r[\theta_0]^{\frac{4}{3}}})$$

(2.5)

Where $A1 = p_r^{\frac{2}{3}} + \frac{2A}{3Np_r^{\frac{1}{3}}} - \frac{A^2}{9N^2 p_r^{\frac{4}{3}}}$ $B1 = \frac{B}{N}$

**2.1 MLE, MCSE and MBE of the parameter of Exponential distribution.**

The probability density function of exponential random variable X with parameter $\theta$ is defined as:

$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta} \quad x \geq 0 \quad \theta > 0$$

And

$$f_r[\theta] = P(l_r < x < u_r) = \int_{l_r}^{u_r} \frac{e^{\frac{-x}{\theta}}}{\theta} dx \qquad (2.1.1)$$

Where $l_r$ and $u_r$ are the lower and upper class boundaries respectively of rth cell. The class interval is determined by adjusting the extreme classes. The extreme classes are taken as <c1θ and >c2θ where c1 and c2 are some constants. The class interval is then (c2-c1) θ/ (K-2). In this study the extreme classed are made with c1=0.5 and c2=5.0.

The first two derivations of (2.1.1) with respect to $\theta$, are

$$f_r'[\theta] = \frac{e^{\frac{-l_r}{\theta}} l_r - e^{\frac{-u_r}{\theta}} u_r}{\theta^2}$$

$$f_r^{(2)}[\theta] = \frac{e^{\frac{-l_r}{\theta}}(l_r - 2\theta)l_r - e^{\frac{-u_r}{\theta}}(u_r - 2\theta)u_r}{\theta^4}$$

The MLE, MCSE and MBE of θ are obtained from (2.2), (2.3) and (2.5) respectively with initial value $\theta_0 = \frac{\sum_{r=1}^{K} N_r m_r}{N}$ where $m_r$ is the class mark of rth class.

**3.1 Numerical Study**

The study is based on the following choices of N-the sample size, NS-the number of samples and K-the number of classes.

N=20, 30,50,100,500,1000.

NS=25000, 50000,100000 for N<100, N<1000 and N=1000 respectively.
K= 5, 10, 15, and 20
In case of Exponential distribution random samples are generated through

$$x = -\theta \ln(1 - F(x))$$

Where $F(x)$ -the distribution function of exponential distribution is generated by subroutine ran2 of FORTRAN 7.

For each sample MLE, MCSE, and MBE are computed using (2.2), (2.3) and (2.5) respectively. Then, mean, absolute percent bias, variance, and mean square error of the estimates are obtained. Berkson (1980) calculates the efficiency using ratio of Mean Square Errors whereas Robert and Rao (1993) use ratio of variances. The efficiency of MBE over MLE and MCSE is calculated for both of these criteria.

Notations:

APBB = $|\theta - E(\hat{\theta}_B)|/\theta \times 100$ MSEBM = MSE ($\hat{\theta}_M$) / MSE ($\hat{\theta}_B$) ×100

VAREBM = VAR ($\hat{\theta}_M$) / VAR ($\hat{\theta}_B$) ×100

APBC = $|\theta - E(\hat{\theta}_C)|/\theta \times 100$ MSEBC = MSE ($\hat{\theta}_C$) / MSE ($\hat{\theta}_B$) ×100

VAREBC = VAR ($\hat{\theta}_C$) / VAR ($\hat{\theta}_B$) ×100

APBM = $|\theta - E(\hat{\theta}_M)|/\theta \times 100$ MSEMC = MSE ($\hat{\theta}_C$) / MSE ($\hat{\theta}_M$) ×100

VAREMC = VAR ($\hat{\theta}_C$) / VAR ($\hat{\theta}_M$) ×100

# 4. CONCLUSION

Table 3.1 gives the mean square error and variance efficiencies of MLE over MCSE together with absolute percent bias for few choices of θ. The study of the table reveals that what ever is the sample size or the number of cells (K), MLE is more efficient than MCSE whether we use MSE efficiency or variance efficiency. The absolute percent bias in MLE for small samples is also less than that of MCSE. Table 3.2 shows the relative efficiency of MBE over MLE and MCSE for
θ = 0.5, 1.0,2.0, and 5.0 together with the absolute percent bias. The study of the table concludes that for A = 0.0 and B = 0.0, MBE is slightly more efficient than MLE for small samples but it is far more efficient than MCSE.

Table 3.1 Relative Efficiency of MLE over MCSE extreme classes: ( < .5 θ),( > 5.0 θ)

| θ = .5 K = 5 | | | | |
|---|---|---|---|---|
| N | MSEMC | APB(M) | APB(C) | VAREMC |
| 20 | 140.7 | .64 | 9.42 | 122.5 |
| 30 | 129.2 | .95 | 7.95 | 113.5 |
| 50 | 119.0 | .52 | 4.72 | 109.5 |
| 100 | 108.2 | .31 | 2.68 | 101.5 |
| 500 | 105.0 | .13 | .87 | 101.8 |
| 1000 | 101.8 | .13 | .27 | 101.3 |

| θ = 1.0 K = 5 | | | | |
|---|---|---|---|---|
| N | MSEMC | APB(M) | APB(C) | VAREMC |
| 20 | 133.8 | .27 | 9.12 | 119.3 |
| 30 | 134.3 | .37 | 7.96 | 117.8 |
| 50 | 122.0 | .33 | 5.07 | 111.5 |
| 100 | 110.0 | .56 | 2.57 | 103.9 |
| 500 | 101.2 | .17 | .54 | 100.2 |
| 1000 | 102.9 | .05 | .43 | 101.1 |

| θ = 1.5 K = 5 | | | | |
|---|---|---|---|---|
| N | MSEMC | APB(M) | APB(C) | VAREMC |
| 20 | 139.4 | 1.51 | 10.68 | 119.3 |
| 30 | 130.4 | .22 | 7.39 | 116.3 |
| 50 | 125.0 | .97 | 6.16 | 108.7 |
| 100 | 112.1 | .47 | 2.77 | 105.4 |
| 500 | 104.0 | .10 | .64 | 102.2 |
| 1000 | 102.1 | .05 | .44 | 100.5 |

| θ = 2.0 K = 5 | | | | |
|---|---|---|---|---|
| N | MSEMC | APB(M) | APB(C) | VAREMC |
| 20 | 138.2 | .71 | 9.41 | 122.4 |
| 30 | 132.7 | .00 | 7.42 | 118.4 |
| 50 | 119.6 | .32 | 5.37 | 107.2 |
| 100 | 113.1 | .46 | 2.77 | 106.4 |
| 500 | 103.4 | .00 | .76 | 101.0 |
| 1000 | 100.5 | .11 | .28 | 100.0 |

| θ = 5.0 K = 5 | | | | |
|---|---|---|---|---|
| N | MSEMC | APB(M) | APB(C) | VAREMC |
| 20 | 138.5 | .91 | 9.68 | 122.4 |
| 30 | 130.1 | .18 | 7.45 | 115.1 |
| 50 | 121.3 | .01 | 5.31 | 108.5 |
| 100 | 114.1 | .30 | 3.59 | 103.3 |
| 500 | 104.1 | .15 | .89 | 100.7 |
| 1000 | 101.9 | .09 | .47 | 100.1 |

| θ = .5 K = 10 | | | | |
|---|---|---|---|---|
| N | MSEMC | APB(M) | APB(C) | VAREMC |
| 20 | 216.3 | .11 | 18.43 | 142.1 |
| 30 | 191.2 | .73 | 15.26 | 125.9 |
| 50 | 170.0 | .14 | 10.72 | 115.6 |
| 100 | 138.2 | .40 | 5.96 | 104.6 |
| 500 | 119.3 | .18 | 1.72 | 104.1 |
| 1000 | 103.1 | .12 | .67 | 99.2 |

| θ = 1.0 K = 10 | | | | |
|---|---|---|---|---|
| N | MSEMC | APB(M) | APB(C) | VAREMC |
| 20 | 196.8 | .86 | 17.92 | 138.0 |
| 30 | 199.6 | .57 | 16.12 | 129.6 |
| 50 | 179.9 | .16 | 11.05 | 117.4 |
| 100 | 143.8 | .38 | 6.00 | 106.8 |
| 500 | 111.5 | .09 | 1.45 | 101.8 |
| 1000 | 107.1 | .02 | .81 | 100.6 |

| θ = 1.5 K = 10 | | | | |
|---|---|---|---|---|
| N | MSEMC | APB(M) | APB(C) | VAREMC |
| 20 | 221.4 | .87 | 19.92 | 146.1 |
| 30 | 189.1 | .20 | 14.98 | 125.4 |
| 50 | 181.6 | 1.20 | 11.87 | 116.2 |
| 100 | 148.3 | .32 | 6.18 | 112.0 |
| 500 | 113.3 | .03 | 1.45 | 103.0 |
| 1000 | 107.8 | .07 | .86 | 100.8 |

| θ = 2.0 K = 10 | | | | | θ = 5.0 K = 10 | | | | |
|---|---|---|---|---|---|---|---|---|---|
| N | MSEMC | APB(M) | APB(C) | VAREMC | N | MSEMC | APB(M) | APB(C) | VAREMC |
| 20 | 219.1 | .55 | 19.34 | 143.8 | 20 | 226.6 | .89 | 19.53 | 151.5 |
| 30 | 193.4 | .06 | 14.86 | 128.6 | 30 | 185.1 | .43 | 14.70 | 125.7 |
| 50 | 178.9 | .33 | 11.29 | 116.3 | 50 | 168.3 | .16 | 11.02 | 111.1 |
| 100 | 148.2 | .20 | 6.33 | 110.1 | 100 | 147.3 | .35 | 6.80 | 104.3 |
| 500 | 111.1 | .03 | 1.52 | 100.4 | 500 | 119.2 | .21 | 1.73 | 103.8 |
| 1000 | 104.3 | .10 | .69 | 100.1 | 1000 | 107.9 | .09 | .90 | 99.8 |

## Table 3.1 (Continued)

| θ = .5 K = 15 | | | | | θ = 1.0 K = 15 | | | | |
|---|---|---|---|---|---|---|---|---|---|
| N | MSEMC | APB(M) | APB(C) | VAREMC | N | MSEMC | APB(M) | APB(C) | VAREMC |
| 20 | 301.6 | .09 | 25.25 | 163.2 | 20 | 276.1 | .74 | 24.89 | 154.5 |
| 30 | 271.3 | .77 | 21.56 | 141.5 | 30 | 271.5 | .54 | 22.01 | 140.3 |
| 50 | 226.3 | .14 | 14.99 | 116.1 | 50 | 251.0 | .17 | 15.49 | 127.9 |
| 100 | 189.3 | .37 | 9.00 | 114.3 | 100 | 200.3 | .33 | 9.06 | 113.7 |
| 500 | 138.7 | .17 | 2.49 | 105.7 | 500 | 126.6 | .12 | 2.23 | 103.7 |
| 1000 | 112.9 | .12 | 1.12 | 100.9 | 1000 | 115.5 | .00 | 1.23 | 100.6 |

| θ = 1.5 K = 15 | | | | | θ = 2.0 K = 15 | | | | |
|---|---|---|---|---|---|---|---|---|---|
| N | MSEMC | APB(M) | APB(C) | VAREMC | N | MSEMC | APB(M) | APB(C) | VAREMC |
| 20 | 295.2 | 1.09 | 26.64 | 159.3 | 20 | 278.2 | .50 | 25.47 | 153.9 |
| 30 | 272.0 | .31 | 21.10 | 141.7 | 30 | 270.7 | .06 | 20.84 | 142.6 |
| 50 | 250.5 | 1.11 | 16.52 | 121.1 | 50 | 249.9 | .35 | 15.77 | 126.2 |
| 100 | 198.7 | .30 | 9.15 | 117.3 | 100 | 192.8 | .18 | 9.14 | 113.8 |
| 500 | 129.8 | .02 | 2.28 | 104.0 | 500 | 125.6 | .02 | 2.31 | 100.3 |
| 1000 | 116.7 | .07 | 1.28 | 101.4 | 1000 | 113.2 | .11 | 1.10 | 101.6 |

| θ = 5.0 K = 15 | | | | | θ = .5 K = 20 | | | | |
|---|---|---|---|---|---|---|---|---|---|
| N | MSEMC | APB(M) | APB(C) | VAREMC | N | MSEMC | APB(M) | APB(C) | VAREMC |
| 20 | 286.4 | .30 | 25.68 | 159.4 | 20 | 387.7 | .23 | 30.83 | 175.9 |
| 30 | 268.5 | .40 | 20.74 | 138.8 | 30 | 348.3 | .66 | 26.27 | 150.0 |
| 50 | 246.6 | .09 | 15.88 | 123.4 | 50 | 299.0 | .27 | 18.68 | 129.3 |
| 100 | 202.5 | .33 | 9.79 | 113.3 | 100 | 259.1 | .34 | 11.71 | 118.8 |
| 500 | 137.1 | .19 | 2.55 | 106.1 | 500 | 160.0 | .18 | 3.25 | 105.8 |
| 1000 | 117.8 | .10 | 1.30 | 101.3 | 1000 | 120.6 | .12 | 1.49 | 100.0 |

| θ = 1.0 K = 20 | | | | | θ = 1.5 K = 20 | | | | |
|---|---|---|---|---|---|---|---|---|---|
| N | MSEMC | APB(M) | APB(C) | VAREMC | N | MSEMC | APB(M) | APB(C) | VAREMC |
| 20 | 356.9 | .53 | 30.61 | 170.3 | 20 | 372.5 | 1.24 | 32.34 | 169.9 |
| 30 | 350.4 | .60 | 26.88 | 155.0 | 30 | 334.7 | .21 | 25.58 | 151.7 |
| 50 | 305.8 | .23 | 19.09 | 136.2 | 50 | 314.9 | 1.13 | 20.34 | 129.3 |
| 100 | 265.5 | .34 | 11.64 | 116.8 | 100 | 266.2 | .29 | 11.87 | 120.8 |
| 500 | 149.1 | .12 | 3.02 | 105.2 | 500 | 151.5 | .01 | 3.14 | 104.3 |
| 1000 | 125.8 | .01 | 1.62 | 100.9 | 1000 | 129.0 | .07 | 1.67 | 102.4 |

| θ = 2.0 K = 20 | | | | | θ = 5.0 K = 20 | | | | |
|---|---|---|---|---|---|---|---|---|---|
| N | MSEMC | APB(M) | APB(C) | VAREMC | N | MSEMC | APB(M) | APB(C) | VAREMC |
| 20 | 376.6 | .82 | 31.99 | 169.2 | 20 | 374.4 | .95 | 31.50 | 178.3 |
| 30 | 350.5 | .03 | 25.65 | 154.1 | 30 | 325.6 | .49 | 25.11 | 149.9 |
| 50 | 338.1 | .43 | 19.77 | 136.8 | 50 | 320.1 | .07 | 19.58 | 132.3 |
| 100 | 251.0 | .14 | 11.86 | 116.6 | 100 | 255.4 | .32 | 12.19 | 116.7 |
| 500 | 149.7 | .01 | 3.12 | 103.6 | 500 | 160.4 | .19 | 3.31 | 104.6 |
| 1000 | 124.7 | .10 | 1.51 | 102.1 | 1000 | 130.8 | .10 | 1.74 | 100.5 |

Table: 3.2 Relative Efficiency of MBE over MLE and MCSE

extreme classes: ( < .5 θ),( > 5.0 θ)

θ = .5 K = 5 A = 0 B = 0

| N | MSEBM | MSEBC | VAREBM | VAREBC | APB(M) | APB(B) | APB(C) | MSEMC | VAREMC |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 102.7 | 146.9 | 102.5 | 126.4 | 1.1 | .4 | 9.9 | 143.0 | 123.3 |
| 30 | 102.0 | 135.3 | 101.8 | 117.4 | 1.0 | .4 | 8.1 | 132.6 | 115.4 |
| 50 | 100.3 | 116.7 | 100.6 | 108.4 | .5 | 1.0 | 4.6 | 116.3 | 107.7 |
| 100 | 99.5 | 108.5 | 99.6 | 101.1 | .2 | .4 | 2.8 | 109.1 | 101.5 |
| 500 | 99.3 | 104.6 | 99.2 | 101.1 | .1 | .1 | .9 | 105.3 | 101.9 |
| 1000 | 99.2 | 98.6 | 99.3 | 98.3 | .1 | .2 | .3 | 99.4 | 99.0 |

θ = 1.0 K = 5 A = 0 B = 0

| N | MSEBM | MSEBC | VAREBM | VAREBC | APB(M) | APB(B) | APB(C) | MSEMC | VAREMC |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 102.7 | 145.5 | 102.5 | 127.0 | 1.3 | .6 | 10.1 | 141.6 | 123.9 |
| 30 | 102.0 | 135.7 | 101.9 | 119.2 | .4 | .1 | 7.7 | 133.1 | 117.0 |
| 50 | 100.9 | 121.7 | 101.1 | 110.3 | .4 | .8 | 5.0 | 120.6 | 109.0 |
| 100 | 99.6 | 110.0 | 99.8 | 103.5 | .3 | .5 | 2.8 | 110.5 | 103.8 |
| 500 | 100.0 | 100.3 | 100.0 | 99.2 | .2 | .2 | .5 | 100.3 | 99.2 |
| 1000 | 100.6 | 104.1 | 100.7 | 102.1 | .1 | .1 | .5 | 103.4 | 101.4 |

θ = 2.0 K = 5 A = 0 B = 0

| N | MSEBM | MSEBC | VAREBM | VAREBC | APB(M) | APB(B) | APB(C) | MSEMC | VAREMC |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 102.7 | 145.3 | 102.4 | 126.1 | 1.5 | .7 | 10.0 | 141.5 | 123.1 |
| 30 | 102.0 | 135.1 | 101.8 | 116.2 | .8 | .3 | 8.3 | 132.4 | 114.1 |
| 50 | 100.9 | 121.3 | 100.9 | 109.5 | .2 | .2 | 5.3 | 120.2 | 108.5 |
| 100 | 99.3 | 109.9 | 99.6 | 104.3 | .6 | .8 | 2.6 | 110.7 | 104.7 |
| 500 | 99.8 | 103.4 | 99.8 | 100.9 | .0 | .0 | .8 | 103.6 | 101.0 |
| 1000 | 100.2 | 100.7 | 100.3 | 100.0 | .1 | .1 | .3 | 100.6 | 99.7 |

θ = .5 K = 10 A = 0 B = 0

| N | MSEBM | MSEBC | VAREBM | VAREBC | APB(M) | APB(B) | APB(C) | MSEMC | VAREMC |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 104.7 | 229.7 | 104.8 | 148.5 | .3 | .8 | 19.0 | 219.4 | 141.6 |
| 30 | 104.1 | 196.8 | 104.0 | 128.2 | .7 | .2 | 15.6 | 188.9 | 123.2 |
| 50 | 101.7 | 171.4 | 102.4 | 118.6 | .5 | 1.2 | 10.2 | 168.5 | 115.8 |
| 100 | 99.9 | 146.1 | 100.3 | 107.5 | .2 | .7 | 6.3 | 146.2 | 107.1 |
| 500 | 99.6 | 118.2 | 99.6 | 103.9 | .2 | .2 | 1.7 | 118.7 | 104.3 |
| 1000 | 100.2 | 105.5 | 100.2 | 101.7 | .1 | .1 | .7 | 105.3 | 101.5 |

θ = 1.0 K = 10 A = 0 B = 0

| N | MSEBM | MSEBC | VAREBM | VAREBC | APB(M) | APB(B) | APB(C) | MSEMC | VAREMC |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 104.6 | 225.7 | 104.5 | 148.7 | .7 | .4 | 19.9 | 215.9 | 142.3 |
| 30 | 103.7 | 200.6 | 103.9 | 131.6 | .1 | .9 | 15.3 | 193.5 | 126.6 |
| 50 | 101.9 | 181.9 | 102.5 | 120.4 | .2 | 1.0 | 11.0 | 178.5 | 117.5 |
| 100 | 99.9 | 146.9 | 100.3 | 108.0 | .2 | .7 | 6.2 | 147.0 | 107.6 |
| 500 | 99.5 | 109.3 | 99.6 | 100.4 | .1 | .1 | 1.4 | 109.8 | 100.8 |
| 1000 | 99.9 | 108.0 | 99.9 | 101.8 | .0 | .0 | .8 | 108.1 | 101.8 |

Table: 3.2 (Continued)

θ = 2.0 K = 10 A = 0 B = 0

| N | MSEBM | MSEBC | VAREBM | VAREBC | APB(M) | APB(B) | APB(C) | MSEMC | VAREMC |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 105.0 | 238.1 | 104.5 | 151.0 | 1.5 | .4 | 20.6 | 226.8 | 144.4 |
| 30 | 103.4 | 199.6 | 103.5 | 132.8 | .2 | .8 | 15.1 | 193.2 | 128.3 |
| 50 | 102.8 | 185.0 | 102.9 | 122.5 | .4 | .4 | 11.3 | 179.9 | 119.1 |
| 100 | 100.4 | 147.9 | 101.0 | 110.4 | .3 | .8 | 6.3 | 147.3 | 109.3 |
| 500 | 99.6 | 110.7 | 99.6 | 100.5 | .0 | .0 | 1.5 | 111.1 | 100.9 |
| 1000 | 99.9 | 106.1 | 99.9 | 101.6 | .1 | .1 | .7 | 106.2 | 101.8 |

θ = 5.0 K = 10 A = 0 B = 0

| N | MSEBM | MSEBC | VAREBM | VAREBC | APB(M) | APB(B) | APB(C) | MSEMC | VAREMC |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 103.9 | 211.3 | 104.4 | 147.3 | .6 | 1.6 | 18.0 | 203.3 | 141.1 |
| 30 | 103.3 | 190.4 | 103.9 | 129.5 | .7 | 1.6 | 14.5 | 184.3 | 124.6 |
| 50 | 102.2 | 182.3 | 102.2 | 117.9 | .4 | .4 | 11.3 | 178.4 | 115.3 |
| 100 | 100.0 | 146.0 | 100.0 | 103.8 | .3 | .2 | 6.7 | 146.0 | 103.7 |
| 500 | 99.7 | 119.5 | 99.8 | 103.5 | .2 | .2 | 1.8 | 119.9 | 103.7 |
| 1000 | 99.9 | 107.7 | 99.9 | 100.1 | .1 | .1 | .9 | 107.8 | 100.2 |

θ = .5 K = 15 A = 0 B = 0

| N | MSEBM | MSEBC | VAREBM | VAREBC | APB(M) | APB(B) | APB(C) | MSEMC | VAREMC |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 104.3 | 317.2 | 104.5 | 168.8 | .1 | 1.0 | 25.8 | 304.2 | 161.6 |
| 30 | 105.2 | 272.9 | 105.1 | 140.2 | .7 | .4 | 21.7 | 259.4 | 133.4 |
| 50 | 102.8 | 231.5 | 103.8 | 122.3 | .6 | 1.5 | 14.5 | 225.2 | 117.8 |
| 100 | 100.5 | 201.0 | 101.2 | 114.0 | .2 | .8 | 9.2 | 199.9 | 112.6 |
| 500 | 99.4 | 135.8 | 99.4 | 104.9 | .2 | .1 | 2.5 | 136.6 | 105.6 |
| 1000 | 99.8 | 112.8 | 100.1 | 101.6 | .1 | .1 | 1.1 | 113.0 | 101.5 |

θ = 1.0 K = 15 A = 0 B = 0

| N | MSEBM | MSEBC | VAREBM | VAREBC | APB(M) | APB(B) | APB(C) | MSEMC | VAREMC |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 104.9 | 290.3 | 105.0 | 156.7 | .4 | .6 | 27.0 | 276.6 | 149.3 |
| 30 | 105.2 | 285.8 | 105.6 | 150.7 | .1 | 1.1 | 21.2 | 271.7 | 142.7 |
| 50 | 103.5 | 255.6 | 104.2 | 130.9 | .2 | 1.1 | 15.6 | 246.9 | 125.6 |
| 100 | 100.3 | 207.2 | 101.1 | 116.2 | .2 | .9 | 9.3 | 206.5 | 115.0 |
| 500 | 99.3 | 122.6 | 99.4 | 101.7 | .1 | .2 | 2.2 | 123.5 | 102.3 |
| 1000 | 99.8 | 115.8 | 99.8 | 101.5 | .0 | .0 | 1.2 | 116.1 | 101.7 |

θ = 2.0 K = 15 A = 0 B = 0

| N | MSEBM | MSEBC | VAREBM | VAREBC | APB(M) | APB(B) | APB(C) | MSEMC | VAREMC |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 105.0 | 302.5 | 104.6 | 160.7 | 1.5 | .4 | 27.1 | 288.1 | 153.6 |
| 30 | 104.6 | 288.4 | 104.9 | 148.7 | .1 | 1.0 | 21.6 | 275.7 | 141.8 |
| 50 | 103.7 | 251.1 | 103.7 | 131.4 | .4 | .6 | 15.6 | 242.2 | 126.6 |
| 100 | 101.1 | 192.7 | 102.0 | 111.9 | .3 | 1.0 | 9.1 | 190.7 | 109.7 |
| 500 | 99.1 | 123.0 | 99.1 | 98.6 | .0 | .1 | 2.3 | 124.2 | 99.5 |
| 1000 | 99.9 | 113.8 | 99.9 | 102.2 | .1 | .1 | 1.1 | 113.9 | 102.2 |

Table: 3.3 (Continued)

θ = 5.0 K = 15 A = 0 B = 0

| N | MSEBM | MSEBC | VAREBM | VAREBC | APB(M) | APB(B) | APB(C) | MSEMC | VAREMC |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 104.0 | 291.0 | 104.6 | 163.9 | .9 | 1.8 | 24.9 | 279.7 | 156.7 |
| 30 | 104.4 | 270.6 | 105.1 | 142.7 | .6 | 1.5 | 20.8 | 259.1 | 135.7 |
| 50 | 103.2 | 261.1 | 103.4 | 131.6 | .2 | .7 | 15.9 | 253.1 | 127.3 |
| 100 | 100.6 | 200.0 | 100.8 | 110.0 | .3 | .4 | 9.7 | 198.7 | 109.2 |
| 500 | 99.6 | 137.4 | 99.4 | 103.1 | .2 | .1 | 2.5 | 137.9 | 103.7 |
| 1000 | 99.8 | 117.7 | 99.6 | 100.5 | .1 | .1 | 1.3 | 117.9 | 100.8 |

θ = .5 K = 20 A = 0 B = 0

| N | MSEBM | MSEBC | VAREBM | VAREBC | APB(M) | APB(B) | APB(C) | MSEMC | VAREMC |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 104.1 | 407.7 | 104.1 | 181.9 | .4 | .5 | 31.4 | 391.8 | 174.8 |
| 30 | 105.1 | 364.9 | 104.9 | 155.4 | .9 | .1 | 26.8 | 347.2 | 148.2 |
| 50 | 103.9 | 311.1 | 105.2 | 133.8 | .6 | 1.6 | 18.3 | 299.5 | 127.2 |
| 100 | 101.0 | 270.7 | 101.9 | 122.7 | .2 | .9 | 12.0 | 268.0 | 120.4 |
| 500 | 99.4 | 157.6 | 99.4 | 105.2 | .2 | .0 | 3.3 | 158.5 | 105.9 |
| 1000 | 99.3 | 120.6 | 99.4 | 99.3 | .1 | .1 | 1.5 | 121.5 | 99.9 |

$\theta = 1.0$ K = 20 A = 0 B = 0

| N | MSEBM | MSEBC | VAREBM | VAREBC | APB(M) | APB(B) | APB(C) | MSEMC | VAREMC |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 104.4 | 363.8 | 104.4 | 172.2 | .1 | .5 | 32.3 | 348.5 | 164.9 |
| 30 | 105.4 | 364.7 | 105.6 | 160.6 | .2 | .7 | 26.5 | 346.0 | 152.2 |
| 50 | 104.0 | 330.7 | 104.9 | 142.7 | .4 | 1.4 | 19.2 | 318.1 | 136.0 |
| 100 | 101.4 | 266.1 | 102.3 | 121.3 | .1 | .9 | 11.9 | 262.5 | 118.7 |
| 500 | 99.0 | 145.7 | 99.4 | 103.5 | .2 | .3 | 3.0 | 147.1 | 104.2 |
| 1000 | 99.6 | 126.5 | 99.7 | 100.6 | .0 | .0 | 1.6 | 127.0 | 100.9 |

$\theta = 2.0$ K = 20 A = 0 B = 0

| N | MSEBM | MSEBC | VAREBM | VAREBC | APB(M) | APB(B) | APB(C) | MSEMC | VAREMC |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 104.0 | 387.7 | 103.7 | 175.2 | 1.4 | .6 | 33.0 | 372.7 | 169.0 |
| 30 | 104.4 | 355.9 | 104.7 | 158.7 | .0 | 1.0 | 25.8 | 340.9 | 151.5 |
| 50 | 104.4 | 325.8 | 104.4 | 137.4 | .5 | .6 | 19.5 | 312.2 | 131.6 |
| 100 | 101.6 | 260.2 | 102.8 | 121.3 | .3 | 1.1 | 11.8 | 256.2 | 118.1 |
| 500 | 99.2 | 149.8 | 99.3 | 103.1 | .0 | .1 | 3.1 | 151.0 | 103.8 |
| 1000 | 99.8 | 124.4 | 99.8 | 102.3 | .1 | .1 | 1.5 | 124.7 | 102.5 |

$\theta = 5.0$ K = 20 A = 0 B = 0

| N | MSEBM | MSEBC | VAREBM | VAREBC | APB(M) | APB(B) | APB(C) | MSEMC | VAREMC |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 104.0 | 381.1 | 104.3 | 176.7 | .5 | 1.2 | 30.9 | 366.4 | 169.5 |
| 30 | 104.5 | 341.3 | 105.1 | 157.5 | .6 | 1.5 | 25.2 | 326.5 | 149.8 |
| 50 | 103.7 | 342.1 | 104.0 | 140.1 | .3 | .8 | 19.6 | 329.7 | 134.7 |
| 100 | 101.6 | 258.0 | 101.9 | 117.8 | .2 | .6 | 12.2 | 253.8 | 115.6 |
| 500 | 99.5 | 157.2 | 99.3 | 103.3 | .2 | .1 | 3.3 | 158.0 | 104.0 |
| 1000 | 99.7 | 133.5 | 99.4 | 102.9 | .1 | .1 | 1.7 | 133.9 | 103.6 |

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