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RATIO, PRODUCT AND SINGLE MOMENTS OF ORDER STATISTICS FROM INVERSE WEIBULL DISTRIBUTION

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ABSTRACT In this paper the ratio and product moments of two order statistics of different orders from Inverse Weibull Distribution have been obtained. Further the moments of single order statistics of different orders are also obtained.

1. INTRODUCTION

Let X_1, X_2, -----,X_n be a random sample of size n from a Inverse Weibull Distribution having probability density function:

$$f(x) = \frac{m}{\theta}\frac{1}{x^{m+1}}\ \exp\left(-\frac{1}{\theta x^m}\right) \quad ,x>0 \quad m,\theta>0 \qquad (1.1)$$

= 0 otherwise

and the corresponding distribution function:

$$F(x) = \exp\left(-\frac{1}{\theta x^m}\right) \quad ,x>0 \quad m,\theta>0 \qquad (1.2)$$

Here m is the shape parameter. At m=1, it reduces to the Inverse Exponential distribution, where as at m= 2, it is the Inverse Rayleigh distribution.

Let $X_{1:n} \le X_{2:n} \le ---\le X_{n:n}$ be the order statistics, then the pdf of $X_{r:n}(1\le r\le n)$ is given by:

$$f_{r:n}(x) = C_{r:n}\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-r} f(x) \quad ,x\ge 0 \qquad (1.3)$$

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and the Joint pdf of $X_{r:n}$ & $X_{s:n} (1 \le r < s \le n)$ is given by:

$$f_{r,s:n}(x,y) = C_{r,s:n}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1} [1-F(y)]^{n-s} f(x) f(y) \quad ,0 \le x < y \le \infty \tag{1.4}$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$$ and

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$$

Let us denote

$$\mu_{r:n}^{j} = E\left(X_{r:n}^{j}\right)$$

and $$= \mu_{r,s:n}^{j,k} E\left(X_{r:n}^{j} . X_{s:n}^{k}\right)$$

For different order moments of order statistics, see Ali, M. A. & Khan, A. H. (1996), Harter, H. L. & Balakrishnan, N. (1998) and Balakrishnan, N. & Rao, C. R. (1998), Ahsanullah, M. & Nevzorov, V. B. (2001). Mohsan, M. (2001) obtained the expressions for single and joint moments of order statistics for Inverse Rayleigh distribution. Aleem, M. (2002) obtained the expressions for Ratio, Product and Inverse moments of order statistics for Inverse Weibull distribution. Here we have obtained single relations of $\mu_{r:n}^{j}$ and $\mu_{r,s:n}^{j,k}$ in different manner.

2. PRODUCT MOMENTS

Lemma 2.1: Let we have non- negative integers a, b and c, then

$$\beta_{j,k}(a,b,c) = \sum_{l=0}^{b} (-1)^l \binom{b}{l} \beta_{j,k}(a+l,0,b+c-l) \tag{2.1}$$

where

$$\beta_{j,k}(a,b,c) = \int_0^\infty \int_0^y (x)^{j-m-1} (y)^{k-m-1} [F(x)]^a [F(y)-F(x)]^b [F(y)]^c \, dx\, dy \tag{2.2}$$

Proof: In (2.2), we can write

$$[F(y)-F(x)]^{b}=\sum_{l=0}^{b}(-1)^{l}\binom{b}{l}[F(x)]^{l}[F(y)]^{b-l}$$

Then

$$\beta_{j,k}(a,b,c)=\sum_{l=0}^{b}(-1)^{l}\binom{b}{l}\int_{0}^{\infty}\int_{0}^{y}x^{j-m-1}\,y^{k-m-1}$$

$$[F(x)]^{a+l}[F(y)]^{b+c-l}\,dx\,dy$$

It can be written as

$$\beta_{j,k}(a,b,c)=\sum_{l=0}^{b}(-1)^{l}\binom{b}{l}\beta_{j,k}(a+l,0,b+c-l) \tag{2.3}$$

Which proves the Lemma (2.1).

Lemma 2.2: For the Inverse Weibull Distribution (1.1), we get

$$\beta_{j,k}(a,0,c)=\frac{\theta^{t_j+t_k}}{m^{2}(a)^{t_j}}\left[\frac{\Gamma(t_j+t_k)}{(c)^{t_j+t_k}}-\frac{\Gamma t_j\,\Gamma t_k}{(c)^{t_k}}I_p(t_j,t_k)\right] \tag{2.4}$$

where

$$t_j=1-\frac{j}{m},\; t_k=1-\frac{k}{m},\; p=\frac{a}{a+c}\quad \text{and}$$

$$I_p(A,B)=\frac{1}{\beta(A,B)}\int_{0}^{p}x^{A-1}(1-x)^{B-1}\,dx$$

(Incomplete Beta Function)

Proof: Using (1.2) in (2.2), we have

$$\beta_{j,k}(a,0,c)=\int_{0}^{\infty}y^{k-m-1}e^{-\frac{c}{\theta\,y^{m}}}\left[\int_{0}^{y}x^{j-m-1}e^{-\frac{a}{\theta\,x^{m}}}\,dx\right]dy \tag{2.5}$$

Put $w = \frac{a\ y^m}{x^m}$ in (2.5) and change the order of the integration to get the result as

$$= \frac{\theta^{t_j + t_k}}{m^2\ (a)^{t_j}} \left[\frac{\Gamma(t_j + t_k)}{(c)^{t_j + t_k}} - \frac{\Gamma t_j \Gamma t_k}{(c)^{t_k}}\ I_p(t_j, t_k) \right] \tag{2.6}$$

Pearson (1955) gives the values of $I_p(*,*)$. Hence the Lemma (2.2) is proved.

Theorem 2.1: For Inverse Weibull Distribution (1.1)

$$\mu_{r,s:n}^{j,k} = C_{r,s!n} \left(\frac{m}{\theta}\right)^2 \sum_{i=0}^{n-s} \sum_{l=0}^{s-r-1} (-1)^{i+l} \binom{n-s}{i} \binom{s-r-1}{l}$$

$$\beta_{j,k}(r+l, 0, n-r+i-l) \tag{2.7}$$

where $1 \le r < s \le n$.

<u>Proof:</u> For Inverse weibull Distribution (1.1) & (1.2)

$$f(x) = \frac{m}{\theta}\ x^{-m-1}\ F(x) \tag{2.8}$$

Now we have

$$\mu_{r,s:n}^{j,k} = C_{r,s:n} \int_0^\infty \int_0^y x^j\ y^k [F(x)]^{i-1} [F(y) - F(x)]^{j-i-1}$$

$$[1 - F(y)]^{n-s}\ f(x) f(x)\ dxdy$$

$$= C_{r,s:n} \sum_{i=0}^{n-s} (-1)^i \binom{n-s}{i} \int_0^\infty \int_0^y x^j\ y^k\ [F(x)]^{r-1}$$

$$[F(y) - F(x)]^{s-r-1} [F(y)]^{n-s+i}\ f(x) f(y)\ dxdy \tag{2.9}$$

Thus the theorem is proved in view of (2.8) and (2.2).

Theorem (2.2): For Inverse Weibull Distribution (1.1)

$$\mu_{r,s:n}^{j,k} = \frac{1}{\theta^{\frac{j}{m}+\frac{k}{m}}} C_{r,s:n} \sum_{i=0}^{n-s} \sum_{l=0}^{s-r-1} (-1)^{i+l} \binom{n-s}{i} \binom{s-r-1}{l}$$

$$\frac{1}{(r+l)^{1-\frac{j}{m}}} \left[\frac{\Gamma\left(2-\frac{j}{m}-\frac{k}{m}\right)}{(n-r+i-l)^{2-\frac{j}{m}-\frac{k}{m}}} - \frac{\Gamma\left(1-\frac{j}{m}\right)\Gamma\left(1-\frac{k}{m}\right)}{(n-r+i-l)^{1-\frac{k}{m}}} I_p\left(1-\frac{j}{m}, 1-\frac{k}{m}\right) \right] \quad (2.10)$$

where $p = \frac{r+l}{n+i}$.

Proof: Using (2.4) in (2.7), the result (2.10) is proved. Now (2.10) can be used to obtained the moments of product of two order statistics for all $1 \le r < s \le n$. For the ratio, let $k = -j$, then $\mu_{r,s:n}^{j,-j} = E\left(X_{r:n}/X_{s!n}\right) \ \forall j$.

3- Single Moments.

Lemma 3.1: Let we have non- negative integers a & b, then

$$\beta_j(a,b) = \sum_{l=0}^{b} (-1)^l \binom{b}{l} \beta_j(a+l,0) \quad (3.1)$$

where

$$\beta_j(a,b) = \int_0^\infty x^{j-m-1} [F(x)]^a [1-F(x)]^b \, dx \quad (3.2)$$

Proof: In (3.2), we can write

$$[1-F(x)]^b = \sum_{l=0}^{b} (-1)^l \binom{b}{l} [F(x)]^l$$

Then

$$\beta_j(a,b) = \sum_{l=0}^{b} (-1)^l \binom{b}{l} \int_0^\infty x^{j-m-1} [F(x)]^{a+l} \, dx$$

It can be written as

$$\beta_j(a,b) = \sum_{l=0}^{b}(-1)^l \binom{b}{l} \beta(a+l,0) \tag{3.3}$$

which proves the Lemma (3.1).

Lemma 3.2: For Inverse Weibull Distribution (1.1), we get

$$\beta(a,0) = \frac{\theta^{t_j}}{m\, a^{t_j}} \Gamma t_j \tag{3.4}$$

where $t_j = 1 - \frac{j}{m}$.

Proof: Using (1.2) in (3.2), we have

$$\beta_j(a,0) = \int_0^{\infty} x^{j-m-1}\, e^{-\frac{a}{\theta\, x^m}}\, dx$$

$$\beta_j(a,0) = \frac{\theta^{t_j}}{m(a)^{t_j}} \Gamma t_j . \tag{3.5}$$

where $t_j = 1 - \frac{j}{m}$

Hence the Lemma (3.2) is proved.

Theorem 3.1: For Inverse Weibull Distribution (1.1)

$$\mu_{r:n}^{j} = \frac{C_{r:n}}{\theta^{\frac{j}{m}}} \sum_{l=0}^{n-r}(-1)^l \binom{n-r}{l} \frac{\Gamma\left(1-\frac{j}{m}\right)}{(r+l)^{1-\frac{j}{m}}} \tag{3.6}$$

where $1 \le r \le n$ and $j < m$.

Proof: For Inverse Weibull Distribution (1.1), by using (2.8) we get

$$\mu_{r:n}^{j} = C_{r:n} \frac{m}{\theta} \int_0^{\infty} x^{j-m-1} [F(x)]^r [1-F(x)]^{n-r}\, dx \tag{3.7}$$

Using (3.1) & (3.4) in (3.7), we get

$$\mu_{r:n}^{j} = \frac{C_{r:n}}{\theta^{\frac{j}{m}}} \sum_{l=0}^{n-r} (-1)^{l} \binom{n-r}{l} \frac{\Gamma(1-\frac{j}{m})}{(r+l)^{1-\frac{j}{m}}} \tag{3.8}$$

where $1 \leq r \leq n$ and $j < m$.

Which can be used to obtain moments of any order.

REFERENCES

1. Ahsanulla, M. & Nevzorov, V. B. (2001), " Ordered Random Variavles", Nova Science Publishers, Inc. New York.
2. Aleem, M. (2002), "Classical Properties, Characterization, Ratio Product and Inverse of Order Statistics from Inverse Weibull Distribution" (Submitted for pub. in PJS).
3. Ali, M. A. & Khan, A. H. (1996), " Ratio and Inverse moments of Order Statistics From Weibull and Exponential Distributions" JPASS, V. 4(1) pp. 1-8.
4. Akrishnan, N. & Rao C. R. (1998) "Order statistics: An introduction", In Handbook of Statistics — 16: Order Statistics: Theory & Methods, North-Holland, Amsterdam, pp. 3—24.

4- Harter, H. L. & Balakrishnan, N. (1998) "Order statistics: A historical perspective", In Handbook of Statistics — 16: Order Statistics: Theory & Methods, North-Holland, Amsterdam, pp. 25-64.

5- Mohsin, M. (2001), "Some distributional Properties of Lower Record Statistics from Inverse Rayleigh Distribution." An Unpublished M. Phil. Thesis, University of Lahore, Pak.

6- Pearson, K. (1955), " Tables of the Incomplete Beta Functions" Cambridge Unversity Press.