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CHARACTERIZATION OF IONS WITH DIFFERENT CHARGE STATES EMITTED FROM LASER PRODUCED PLASMA OF ALUMINUM, NICKEL AND TANTALUM

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ABSTRACT: Metallic targets of Al, Ni and Ta were laser ablated by Thum-Jaeger et al. (2000) using Nd:YAG laser (λ = 1.06 μm, τ = 5 ns, E = 200 mJ, and $I = 5\times10^{10}$ W/cm^2). Ions up to charge state q = 4, emitted from the laser produced plasma (LPP) of these metallic targets are found to distribute angularly in the form of a cone, and each charge state follows the cosine power-law: $F = F_o\cos^n\theta$. The value of exponent n of $\cos^n\theta$ distribution function increases with the increase in charge state of ions. It is found that exponent n of each charge state exhibits an excellent correlation with the room temperature Debye-Waller thermal parameter B, which is proportional to the mean-square amplitude of the atomic vibrations $<u^2>$ of target metals. All above is also valid in the case of LPP ions flux comprising all the four ionization states.

Keywords: Nd:YAG laser, Metals, LPP ions, Ionization states, Angular distribution.

1. INTRODUCTION

One of the most important characteristics of laser produced plasma (LPP) is the angular distribution of LPP ions. Its knowledge is very important to understand the plasma expansion and the physical processes taking place in it. Numerous experiments relating to the angular distribution of LPP ions of different mono- and bi-atomic targets has been done in the past [1−12]. One can infer from these experiments that (i) the maximum flux of LPP ions emission takes place close to the target normal, and (ii) the higher the ion velocity the narrower is the distribution width. In most cases, angular distribution of LPP ions have been studied using cosine power-law: $F= F_o\cos^n\theta$, where F is the flux of LPP particles emitted in a direction making an angle θ with the target-normal and F_o is the maximum flux along the

target normal, i.e. for $\theta=0^o$. The value of exponent n of $\cos^n\theta$ distribution function reflects on the distribution width of LPP ions. A number of attempts have been made to correlate the exponent n with the atomic mass [1−8], the sublimation energy [9−11], and the room temperature Debye-Waller thermal parameter B, which is proportional to the mean-square amplitude of atomic vibrations $<u^2>$ [12], of the target metals.

Recently, Thum-Jaeger et al. [5] used Nd:YAG laser Q-switch pulses(λ = 1.06μm, τ = 5ns, E = 200 mJ, and $I = 5\times10^{10}$ W/cm^2), which were incident at a fixed angle of – 45° onto flat, rotating targets inside a vacuum chamber to ablate Al, Ni, and Ta metals. They obtained time-of-flight (TOF) spectra for LPP ions emitted per unit time per unit solid angle as a function of flight time at angles of emission in the range –17.5° to 60°. Nearly 90% of the plasma ions were emitted along a direction normal to the target, and consisted of singly, doubly, triply, and quadruply ionized states. These ions carried with them major portion of the plasma thermal energy. Moreover, along this direction, the ions of higher ionization states always had higher values of temperature than those for lower ionization states.

In the present work, we shall investigate, using the wealth of data obtained by Thum-Jaeger et al. [5], whether the angular distribution of LPP ions emitted follows the cosine power law or not? If yes, it will be further examined to what extent the value of exponent n of $\cos^n\theta$ distribution function of each ionization state as well as of cumulative flux of LPP ions correlates with the atomic mass, the sublimation energy, and the room temperature Debye-Waller Thermal parameter B of target metals.

2. DATA ANALYSIS

2.1. INDIVIDUAL IONIZATION STATES

Reference to Fig. 1(a) – (c) shows the angular distribution of LPP ions up to charge state +4 for (a) Al, (b) Ni, and (c) Ta. The data points depict the maximum number of ions per unit solid angle in the TOF spectra obtained

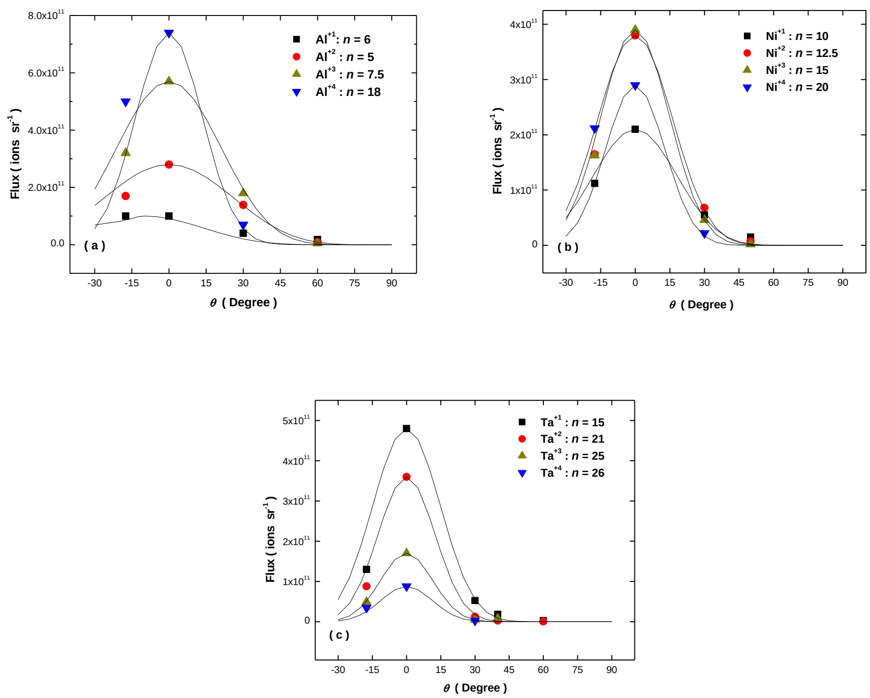

Fig. 1: Angular distribution of LPP ions flux *F* of (a) Al, (b) Ni, and (c) Ta for different charge states. The curve fitted to the data points for a given charge state is a graphical manifestation of cosine power-law: $F = F_0\cos^n\theta$.

Table 1: Parameters characterizing angular distribution of LPP ions for various charge states emitted from three metallic targets.

Charge State	F_o (ions sr^{-1})	*n*
Al^{+1}	1.00×10^{11}	6
Al^{+2}	2.80×10^{11}	5
Al^{+3}	5.70×10^{11}	7.5
Al^{+4}	7.40×10^{11}	18
Ni^{+1}	2.10×10^{11}	10
Ni^{+2}	3.80×10^{11}	12.5
Ni^{+3}	3.90×10^{11}	15
Ni^{+4}	2.80×10^{11}	20
Ta^{+1}	4.80×10^{11}	15
Ta^{+2}	3.60×10^{11}	21
Ta^{+3}	1.77×10^{11}	25
Ta^{+4}	0.9×10^{11}	26

by Thum-Jaeger et al. [5] as a function of angle *θ* relative to the target-normal for ions of different ionization states for each target. The curve fitted to data points pertaining to a given metal and a given ionization state

is a graphical manifestation of the cosine power-law: $F = F_o \cos^n \theta$. The values of exponent n and F_o for individual ionization states used to accomplish agreement with experiment have been listed in Table 1.The points in Fig. 2 denote the values of exponent n (Table 1) as a function of ionization state q, which were derived from Fig. 1. The value of exponent n increases linearly with the increase in charge state of ions emitted from a given metal target. The lines passing through the data points were obtained by least-squares fitting, and are encompassed by the mathematical expressions:

Al: $n = -0.50 + 3.85\, q$ **(1)**

Ni: $n = 6.25 + 3.25\, q$ **(2)**

Ta: $n = 12.5\,0 + 3.70q$ **(3)**

with the linear correlation coefficient r = 0.828, 0.982, and 0.957, respectively.

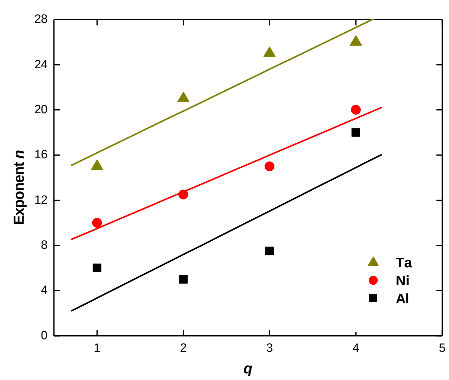

Fig. 2: Relation between the exponent n of $\cos^n\theta$ distribution function and the charge state (q) of three metallic targets.

Table 2: Values of square-root of atomic mass, sublimation energy, room temperature *B*-factor, and values of exponent *n* for various charge states of three metallic targets.

Elements	$M^{1/2}$ ($amu^{1/2}$)	*SE* (kJ / mol)	*B* (nm^2)	*n*			
				q^{+1}	q^{+2}	q^{+3}	q^{+4}
Al	5.194	293.4	0.0086	6	5	7.5	18
Ni	7.662	370.4	0.0037	10	12.5	15	20
Ta	13.45	743	0.0032	15	21	25	26

In Fig. 3, the values of exponent *n* for LPP ions of Al, Ni, and Ta (Table 1) have been plotted as a function of the square-root of atomic mass ($M^{1/2}$) for individual ionization states (q^{+1}, q^{+2}, q^{+3}, q^{+4}). The linear least-squares fit to the data points for various ionization states is represented by:

q^{+1}: $n = 1.13 + 1.05\, M^{1/2}$ (4)

q^{+2}: $n = -3.24 + 1.85\, M^{1/2}$ (5)

q^{+3}: $n = -2.14 + 2.05\, M^{1/2}$ (6)

q^{+4}: $n = 12.7 + 0.98\, M^{1/2}$ (7)

with the linear correlation coefficient r = 0.986, 0.982, 0.989, and 0.998, respectively.

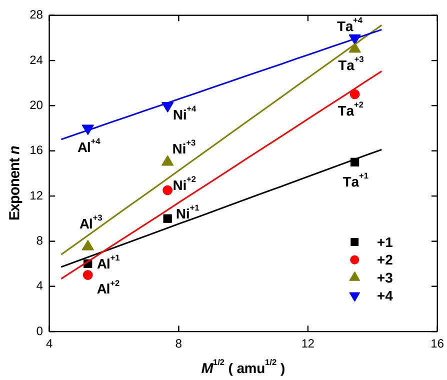

Fig. 3: Relation between the exponent *n* of $\cos^n\theta$ distribution function and the square-root of atomic mass ($M^{1/2}$) for different charge states of three metallic targets.

Referring to Fig. 4, the points denote the values of exponent *n* for individual ionization states as a function of the sublimation energy of the target metals at room temperature. The values of sublimation energy were taken from [13]. The straight lines drawn through these data points for various ionization states by least-squares fit method are given by:

q^{+1}: $n = 1.93 + 17.9\times10^{-3} SE$ (8)

q^{+2}: $n = -1.95 + 31.5 \times10^{-3} SE$ (9)

q^{+3}: $n = -0.60 + 35.1 \times10^{-3} SE$ (10)

q^{+4}: $n = 13.20 + 17.3 \times10^{-3} SE$ (11)

with the linear correlation coefficient r = 0.955, 0.947, 0.961, and 0.997, respectively.

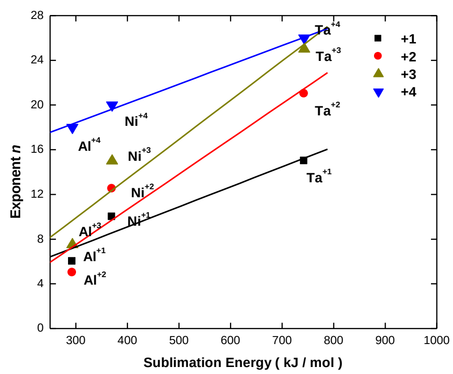

Fig. 4: Relation between the exponent n of $\cos^n\theta$ distribution function and the sublimation energy for different charge states of three metallic targets.

Finally, the exponent n of LPP ions has been plotted as a function of the room temperature Debye-Waller thermal parameter B for individual ionization states (q^{+1}, q^{+2}, q^{+3}, q^{+4}) in Fig. 5.

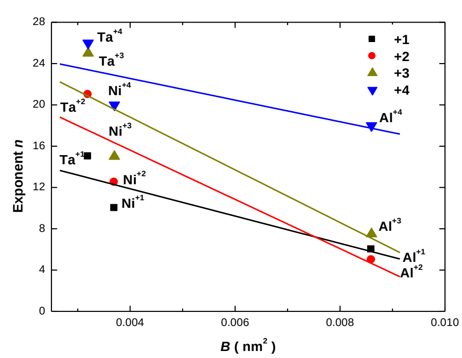

Fig. 5: Dependence of the exponent n of $\cos^n\theta$ distribution function for various charge states on the room temperature Debye-Waller thermal parameter B of three metallic targets.

The linear least-squares fit to the data points for various ionization states is represented by:

q^{+1}: $$n = 17.17 - 1323\,B \quad (12)$$

q^{+2}: $$n = 25.15 - 2385\,B \quad (13)$$

q^{+3}: $$n = 29.01 - 2551\,B \quad (14)$$

q^{+4}: $n = 26.74 - 1048\,B$ (15)

with the linear correlation coefficient r = −0.875, −0.889, −0.867, and −0.751, respectively. The B-values were taken from [14].

2.2. CUMULATIVE IONIZATION STATES

Figure 6 shows the angular distribution of LPP ions flux comprising all the four ionization sates for Al, Ni, and Ta. The data points represent the values of cumulative flux of LPP ions measured at various angles ranging from −17.5˚ to 60˚, and the curve fitted to the data points for a given target is a graphical manifestation of the cosine-power law : $F = F_o \cos^n\theta$. The values of exponent n and constant F_o used to accomplish agreement with experimental data of various targets have been listed in Table 3.

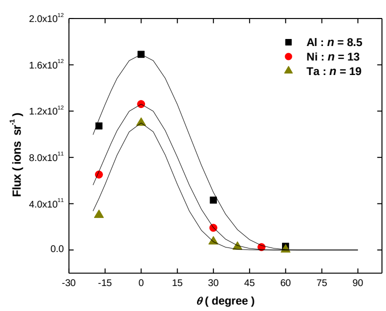

Fig. 6: Angular distribution of the cumulative flux F of LPP ions emitted from Al, Ni, and Ta.

Table 3: Parameters characterizing angular distribution of cumulative flux of LPP ions for three metals.

Elements	F_o(ions sr^{-1})	n
Al	1.69×10^{12}	8.5
Ni	1.26×10^{12}	13
Ta	1.09×10^{12}	19

Like individual ionization states, the values of exponent n of $\cos^n\theta$ distribution function obtained from Fig. 6 (Table 3) for cumulative ionization states of LPP ions have been plotted as a function of (a) the square-root of atomic mass, (b) sublimation energy, and (c) room temperature Debye-Waller thermal parameter B. The lines drawn through the data points in Fig. 7(a) – (c) using least-squares fit method are given by:

$$n = 2.72 + 1.23\, M^{1/2} \tag{16}$$

$$n = 3.63 + 2.11\times10^{-2} SE \tag{17}$$

$$n = 21.4 - 1530\, B \tag{18}$$

with the linear correlation coefficient r = 0.989, 0.961, and 0.867, respectively . A good linear dependence of exponent n on all the three parameters, i.e. $M^{1/2}$, *SE*, and *B*, is thus evident.

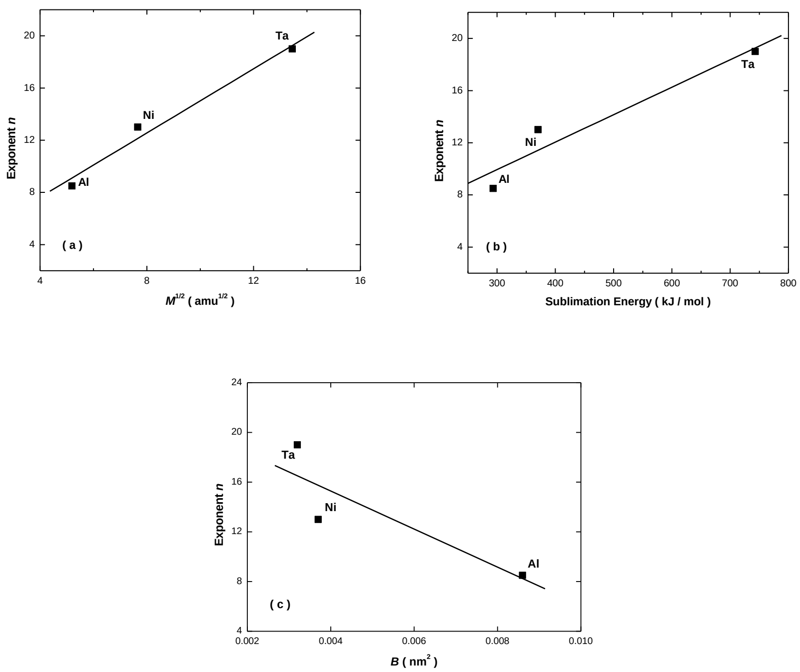

Fig. 7: Relation between the exponent n of $\cos^n\theta$ distribution function of cumulative flux F of LPP ions and (a) the square-root of atomic mass ($M^{1/2}$), (b) the sublimation energy, and (c) the room temperature Debye-Waller thermal parameter *B*.

Since the sublimation energy of metals has been shown to be a function of room temperature *B*-factor [12], and so is true in case of their atomic mass (Fig. 8), i.e.

$$M^{1/2} = 14.53 - 1115\, B \tag{19}$$

with the linear correlation coefficient $r = -0.785$, one may rightly consider the room temperature *B*-factor as the most fundamental determining factor for angular distribution of LPP ions of individual charge states or in toto.

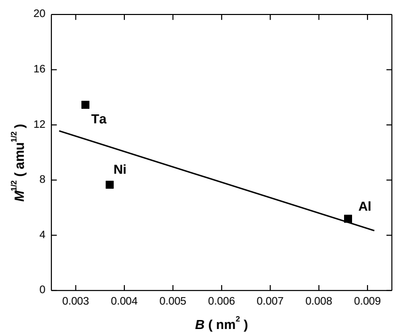

Fig. 8: The square-root of atomic mass as a function of the room temperature *B*-factor.

3. CONCLUSIONS

From the foregoing analysis of the laser-ablation data obtained by Thum-Jaeger et al. [5] with Al, Ni and Ta target metals, we conclude as under:

1. **The angular distribution of LPP ions flux F of each charge state emitted from a metallic target follows the cosine power-law: $F = F_0\cos^n\theta$.**
2. **The value of exponent n of $\cos^n\theta$ distribution function increases with the increase in the charge state of LPP ions.**
3. **Exponent n of each charge state exhibits a good linear dependence upon the room temperature Debye-Waller thermal parameter B of metal targets.**
4. **The above is also valid for angular distribution of cumulative LPP ions flux comprising all the four ionization states.**

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EXTERNAL PATH LENGTH OF RANDOM *m*-ORIENTED RECURSIVE

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ABSTRACT: The random recursive tree is a combinatorial structure used to model a variety of applications such as contagion, chain letters, philology, etc. In this paper, we determine the expectation and variance of X_n the external path length in a random *m*-oriented recursive tree of size *n*.

Keywords: Random recursive trees, path length.

1. INTRODUCTION

The analysis of the length of paths in tree families has received a lot of attention, see, e.g., [1,3,6-8], often due to their importance in the analysis of algorithms. In [3,6,7] the total path length is investigated in random recursive trees. However, up to now there is no result about the external path length of random m-oriented recursive trees. Here we obtain the expectation and variance of the external path length in random *m*-oriented recursive trees.

By a recursive tree we mean a labeled rooted tree such that each path from the root to any node of the tree is labeled with an increasing sequence of labels.

A survey of applications and results on recursive trees is given in [10]. These trees are used, e.g., to model chain letters and pyramid schemes [5], and as a simplified growth model of the World Wide Web [2].

Fig. 1: Two different plane-oriented trees.

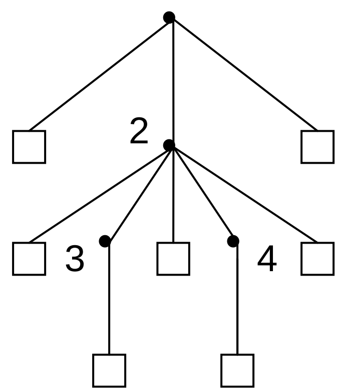

Fig. 2: An extended plane-oriented recursive tree.

In working with recursive trees it is to convenient to consider an extension of these trees obtained by adding a different type of node called external at each possible insertion position.

Orientation in the plane was not taken into account in the definition of recursive trees. The two labeled trees in Fig. 2 are only two drawings of the same recursive tree. If different orientations are taken to represent different trees, we arrive at a definition of a plane-oriented re-cursive tree. In such a tree, if a node has outdegree d; there are d children under it, with d+1 external nodes.

Figure 2 shows one of the plane-oriented recursive trees of Fig. 1 after it has been extended; the external nodes are shown as squares in Fig. 2.

In this paper we consider m-oriented recursive trees the generalization of random plane-oriented recursive trees where if a node has outdegree *d*; there are *d* children under it, with (*m*-1)*d*+1 external nodes, (this generalization first defined in [9]). If *d*(*i*) denotes the outdegree of the *i*th node then the total number of external nodes in an extended *m*-oriented recursive tree of size *n* is

$$\begin{aligned}\sum_{i=1}^{n}((m-1)d(i)+1) &= (m-1)\sum_{i=1}^{n}d(i)+n \\ &= (m-1)(n-1)+n \\ &= m(n-1)+1.\end{aligned}$$

A random *m*-oriented recursive tree of size n is constructed as follows. One starts from a root node holding the label 1; at stage *i* (*i*=2,3, . . . ,*n*) a new node holding label *i* (the *i*th node) is attached to any previous node j of outdegree *d*(*j*) of the already grown tree T_{i-1} of size *i*-1 with probability $\frac{(m-1)d(j)+1}{m(i-2)+1}$ (the number of remaining external nodes for the node *j*, (*m*-1)*d*(*j*)+1, is divided by *m*(*i*-2)+1, the number of all external nodes). This function implies that the higher outdegree nodes possess a higher attraction for new neighbors.

As the tree grows by the progressive insertion of nodes, two cumulative random variables may serve as measures of the overall cost of construction of the tree, or the cost of later processing of the whole tree if each internal or external node is to be accessed equally often.

Let be the depth of *j*th node in a random m-oriented recursive tree of size n. The first cumulative random variable is the internal path length $I_n = \sum_{j=1}^{n} D_j$. Suppose the external nodes are indexed by 1, 2, . . . , *m*(*n*-1)+1, and be the depth of the *j*th external node. The second cumulative random variable is $X_n = \sum_{j=1}^{m(n-1)+1} x_j$. This random variable is called the external path length. By the proof of Theorem 1, the relation

$$X_n = mI_n + n \tag{1}$$

and then $X_n = m\sum_{j=1}^{n} D_j + n$ can be deduced. The strong dependence between the random variables D_j makes it difficult to compute the exact distribution of X_n. In Section 2, we compute the expectation and variance of X_n.

In the following, the term random tree without qualification will refer to a random m-oriented recursive tree.

2. EXPERIMENTAL PROCEDURES

The total path length in random recursive trees has investigated in [3]. Expectation and variance of the external path length of plane-oriented recursive trees are derived in [8]. In this section, the following results for D_n, will be used (see [4]):

$$E[D_n] = \sum_{j=1}^{n-1} \frac{1}{m(j-1)+1} \quad \text{and} \quad Var[D_n] = \sum_{j=1}^{n-1} \frac{m(j-1)}{(m(j-1)+1)^2}.$$

THEOREM 1:

$$E[X_n] = \sum_{j=1}^{n} \frac{m(n-1)+1}{m(j-1)+1},$$

and

$$Var[X_n] = \sum_{j=1}^{n} \frac{m^3(n-j)(j-1)(m(n-1)+1)}{(m(j-1)+1)^2(mj+1)}.$$

***Proof*: Observe that a tree T_n of size *n* is obtained algorithmically from a tree T_{n-1} of size *n*-1 by inserting the *n*th node at level D_n. The *n*th node may replace any of the $m(n-2)+1$ external nodes of T_{n-1} with probability**

$1/(m(n-2)+1)$. **The new node gives the tree *m*+1 new external nodes, but one of the external nodes of T_{n-1} is lost in the process. The net gain in the external path length is therefore $mD_n + (D_n + 1) - D_n = mD_n + 1$, i.e., $X_n = X_{n-1} + mD_n + 1$.**

Let $\Im_n$ denote the sigma field generated by tree T_n. When the shape of the tree T_{n-1} is available, the levels $x_1, \ldots, x_{m(n-2)+1}$ of the external nodes are completely determined. Thus D_n may assume any of the values $x_1, \ldots, x_{m(n-2)+1}$ with equal probability $1/(m(n-2)+1)$. We can now formulate a conditional expectation,

$$E[X_n \mid \Im_{n-1}] = \frac{1}{m(n-2)+1} \sum_{j=1}^{m(n-2)+1} (X_{n-1} + mx_j + 1)$$

$$= X_{n-1} + 1 + \frac{m}{m(n-2)+1} \sum_{j=1}^{m(n-2)+1} x_j.$$

But the remaining sum is the external path length of T_{n-1}, i.e.,

$$E[X_n \mid \Im_{n-1}] = \frac{m(n-1)+1}{m(n-2)+1} X_{n-1} + 1.$$

Taking expectations of the last relation we get the following recurrence on expected external path length

$$E[X_n] = \frac{m(n-1)+1}{m(n-2)+1} E[X_{n-1}] + 1,$$

which can be easily solved under the initial condition $E[X_1] = 1$ to yield the first required result.

To compute the variance of X_n we formulate a recurrence for

$$Q_n := \frac{Var[X_n]}{(mn+1)(m(n-1)+1)}$$

as follows. Let $Z_n = \frac{X_n - E[X_n]}{m(n-1)+1}$**. Replace** X_n **by** $X_{n-1} + mD_n + 1$ **in the definition of** Z_n **and write**

$$Z_n = \frac{X_{n-1} + mD_n + 1 - E[X_{n-1} + mD_n + 1]}{m(n-1)+1}$$

$$= \frac{m(n-2)+1}{m(n-1)+1} Z_{n-1} + \frac{m}{m(n-1)+1}(D_n - E[D_n]).$$

Upon squaring the latter relation and taking expectations we get

$$E[Z_n^2] = \left(\frac{m(n-2)+1}{m(n-1)+1}\right)^2 E[Z_{n-1}^2] + \left(\frac{m}{m(n-1)+1}\right)^2 Var[D_n]$$

$$+ \frac{2m(m(n-2)+1)}{(m(n-1)+1)^2} E[Z_{n-1}(D_n - E[D_n])]. \qquad \textbf{(2)}$$

Since the component $E[Z_{n-1}E[D_n]]$ **is zero, in the last term we need only to find** $E[Z_{n-1}D_n]$**. For the required term we compute**

$$E[Z_{n-1}D_n] = E[E[Z_{n-1}D_n \mid \mathfrak{I}_{n-1}]] = E[Z_{n-1}E[D_n \mid \mathfrak{I}_{n-1}]].$$

But according to the algorithmic development,

$$E[D_n \mid \mathfrak{I}_{n-1}] = \sum_{j=1}^{m(n-2)+1} \frac{x_j}{m(n-2)+1} = \frac{X_{n-1}}{m(n-2)+1}.$$

So,

$$E[Z_{n-1}D_n] = E[Z_{n-1}^2].$$

Plugging this relation into (2) we arrive at the recurrence

$$E[Z_n^2] = \frac{(mn+1)(m(n-2)+1)}{(m(n-1)+1)^2} E[Z_{n-1}^2] + \frac{m^2}{(m(n-1)+1)^2} Var[D_n]. \qquad (3)$$

The substitution Q_n **linearizes the recurrence (3) into the simple recurrence**

$$Q_n = Q_{n-1} + \frac{m^2}{(m(n-1)+1)(mn+1)} Var[D_n].$$

By the relation for the variance of D_n**, the solution to the last recurrence gives**

$$Q_n = \sum_{j=3}^{n} \frac{m^2}{(m(j-1)+1)(mj+1)} \sum_{k=1}^{j-1} \frac{m(k-1)}{(m(k-1)+1)^2}.$$

Expanding $1/(m(j-1)+1)(mj+1)$ **by partial fractions and collapsing the resulting telescopic sums, we have**

$$Q_n = \sum_{j=2}^{n-1} \frac{m^3(n-j)(j-1)}{(m(j-1)+1)^2(mj+1)(mn+1)}.$$

So by definition of Q_n**, the proof is complete.**

REMARK: By (1) the expectation of internal path length I_n **is**

$$E[I_n] = \frac{1}{m} \sum_{j=1}^{n} \frac{m(n-1)+1}{m(j-1)+1} - \frac{n}{m}.$$

So the average external path length is asymptotically *m* times as much as the average internal path length $E[I_n] = \frac{n}{m} \ln n$ **.**

The standard deviation of the external path length of a random *m*-oriented recursive tree is relatively small compared to the mean value, since $Var[X_n] = (mn+1)(m(n-1)+1)Q_n$ **and** $Q_n = O(1)$ **then** $\sqrt{Var[X_n]} = O(n)$**, while** $E[X_n] = n \ln n$**, as** $n \to \infty$**. From an application of Chebychev's inequality we can conclude that** $\frac{X_n}{n \ln n} \to 1$**, in probability.**

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TECHNICAL NOTES ON THE CHLORIDE ION ENHANCEMENT RATIO

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ABSTRACT: The present work shows that the chloride ions enhancement ratio is of the order of 3 to 7, which is 2 to 3 times greater than that produced by oxygen during irradiation. Adding sodium chloride to food patient increases the amount of water, and the amount of chloride ions inside the tumor cells. This process provides excellent route for hydroxyl radical production during irradiation. The present work suggests that chloride ions must add to patient food at the highest concentration allowed. Adjustment of patient food is highly recommended during radiotherapy procedure.

Keywords: Reaction kinetics, Chloride ions and radiation therapy.

1. INTRODUCTION

The aim of radiotherapy procedures is to increase damage to tumor area with a minimum risk to normal surrounding tissue. This can be achieved by shielding to normal tissue or by using a precise collimation of the radiation beam to fit only the area of interest [1,2]. A high radiation dose is to be delivered, thus a computerized collimator is frequently used to shape the radiation beam to the required area. Hypothetically same result can be achieved but with lower radiation dose by using proper radio-sensitizer which can enhance radiation damage to tumor cells. Many radio-sensitizers have been studied and shown to be of varying effect. Among the natural occurring radio-sensitizers is the oxygen and it was shown that the oxygen enhancement ratio (OER) is of the order of 2 to 4 [3]. The present work flash a spot light on another natural radio-sensitizer which appear to be stronger than the sensitization of oxygen.

2. MATERIAL AND METHOD

Oxidation damage is the main reason of radiation damage in mammalian cells. The free radicals formed by irradiation near to the critical target of the cell reacts rapidly with the available oxygen to form peroxyradical molecules which is an irreversible reaction leads to damage fixation.

$$R^{\cdot} + O_2 \rightarrow RO_2^{\cdot} \quad (1)$$

Some of the oxygen we breath (1-3 %) converted to oxygen radical molecules (superoxide $O_2^{\cdot-}$).

If oxygen present at high tension, then many superoxide molecules can react with hydrogen ions H^+ formed by radiolysis of water molecules inside the cells. This reaction leads to form hydrogen peroxide molecules, which is a very strong oxidative agent to the cell membrane.

$$O_2^{\cdot-} + O_2^{\cdot-} + 2H^+ \rightarrow H_2O_2 + O_2 \quad (2)$$

The rate of the above reaction is faster at acidic pH value [4]. This equation is therefore radiation dependent. Oxidation and damaging process started when hydrogen peroxide inter a chain reactions to form free radicals. Hydrogen peroxide reacts slowly with superoxide to form hydroxyl radical molecule.

$$H_2O_2 + O_2^{\cdot-} \rightarrow OH^{\cdot} + OH^- + O_2 \quad (3)$$

Hydrogen peroxide can also penetrate through cell membrane and catalyze intercellular metallic ions such as Fe^{+2}, Cu^+, and other transition ions and form free radicals

$$H_2O_2 + Fe^{2+} \rightarrow OH^{\cdot} + OH^- + Fe^{3+} \quad (4)$$

$$Cu^+ + H_2O_2 \rightarrow Cu^{2+} + OH^{\cdot} + OH^- \quad (5)$$

Copper ion reacts with H_2O_2 to make $OH^{\cdot}$ with much greater rate constant than do Fe^{+2}.

Chloride ion role started when hydrogen peroxide reacts with chloride ions to form hypochlorous acid.

$$H_2O_2 + Cl^- \rightarrow HOCl + OH^- \quad (6)$$

This reaction is radiation dependent since the hydrogen peroxide produced in large amount by irradiation. In fact, hypochlorous acid is also formed naturally in biological system by combining chlorine molecules with water molecules.

$$Cl_2 + H_2O \leftrightarrow HOCl + HCl \quad \textbf{(7)}$$

Hypochlorous acid then undergoes a reaction with superoxide molecules to form free radicals.

$$HOCl + O_2^{\cdot -} \rightarrow OH^{\cdot} + O_2 + Cl^{-} \quad \textbf{(8)}$$

This reaction is more dominate at high oxygen tension.

The hypochlorous acid can also react with intercellular metallic ions inside the cells and form more free radicals.

$$HOCl + Fe^{2+} \rightarrow OH^{\cdot} + Fe^{3+} + Cl^{-} \quad \textbf{(9)}$$

3. DISCUSSION

The rate of reaction (8) is seven orders of magnitude faster than the rate of reaction (3), and the rate of reaction (9) is about three orders of magnitude faster than the rate of reaction (4). Reactions (8) and (9) are called "kinetic amplification reactions" [5]. The amount of hydroxyl radicals generated via reaction (8) and (9) is three to seven times greater in magnitude than that of reaction (3) and (4). The chain reactions of chloride ion can very much enhance the radiation damage during radiotherapy procedure. The chloride ion enhancement ratio is therefore more effective than that for oxygen.

Both healthy and cancerous cells makes energy, it requires sugar and oxygen to store energy in the form of ATP molecules. Cancerous cells require to burns more sugar molecules for more nourishment and since oxygen is very limited inside this type of cells, an anaerobic, oxygen free, process takes place. Every time the cell takes in a single molecule of sugar from the bloodstream, it expels two positively charged potassium ions. At the same time, it absorbs three positively charged sodium ions. As the anaerobic energy production continues, more sodium absorbed. The more sodium they contain, the more water they absorb to dilute the sodium, and the more hydroxyl radical produced by irradiation. Cancer cell bloat

because they absorb sodium, which the diet provides. This is one of the outcomes from adding more sodium chloride to patient food. The main advantage comes from the availability of chloride ions in environment of the cells which provide a strong route for hydroxyl radical production.

The present work suggests that chloride ions must add to patient food at the highest concentration allowed during the treatment of cancer. Careful adjustment for patient food is highly recommended.

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WEAK AND STRONG CONVERGENCE OF ISHIKAWA ITERATES FOR NONSELF ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

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ABSTRACT: **In this paper, we define and study convergence of an Ishikawa type iteration scheme with a new and weaker control condition for two nonself asymptotically quasi- nonexpansive mappings on a uniformly convex Banach space. Our results improve and extend the corresponding results of Khan, Takahashi [4], Rhoades [6], Tan, Xu [9], Chidume, Ofoedu, Zegeye [14], Wang [15], and Qihou [18].**

2010 Mathematics Subject Classification: 47H09, 47H10

Keywords: Ishikawa type iteration, nonself asymptotically quasi-nonexpansive mapping, common fixed point, weak and strong convergence.

1. INTRODUCTION AND PRELIMINARIES

Let C **be a nonempty subset of a real Banach space** E**. Recall that a mapping** $T:C\to C$ **is : (i) uniformly** L**-Lipschitzian if for some** $L>0$**,** $\left\|T^n x-T^n y\right\|\le L\left\|x-y\right\|$ **for all** $x,y\in C$ **and for all** $n\ge 1$ **(ii)** $(\lambda-\alpha)-$ **uniform Lipschitz if there exists** $\alpha>0$ **such that** $\left\|T^n x-T^n y\right\|\le \lambda\left\|x-y\right\|^{\alpha}$ **for all** $x,y\in C$ **and for all** $n\ge 1$ **(iii) uniformly continuous if for some** $\varepsilon>0$**, there exists a** $\delta>0$ **such that** $\left\|T^n x-T^n y\right\|<\varepsilon$ **whenever** $\left\|x-y\right\|<\delta$ **for all** $x,y\in C$ **and** $n\ge 1$**, or, equivalently,** T **is uniformly continuous if and only if** $\left\|T^n x_n-T^n y_n\right\|\to 0$ **whenever** $\left\|x_n-y_n\right\|\to 0$ **as** $n\to\infty$**; (iv)**

asymptotically nonexpansive if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1,$ we have $\|T^n x - T^n y\| \le k_n \|x - y\|$ for all $x, y \in C$ and for all $n \ge 1.$

It is obvious that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) but the converse implications do not hold, in general.

Asymptotically nonexpansive mappings, ever since their introduction by Goebel and Kirk [1] in 1972, remained under study of many authors. Goebel and Kirk [1] proved: If C is a nonempty bounded closed convex subset of a uniformly convex Banach space E and $T: C \to C$ is an asymptotically nonexpansive mapping, then T has a fixed point. In recent years, iterative techniques for approximating fixed points of asymptotically nonexpansive and nonexpansive mappings have been studied by many authors(see, e.g., [2-9]).

Finding common fixed points of a finite family $\{T_j : j = 1,2,3,...,n\}$ of mappings acting on a Hilbert space is a problem that often arises in applied mathematics. Probably the most important case is the one where each mapping T_j is the metric projection onto some closed convex set C_j, under the assumption that intersection of all involved sets C_j is nonempty. In fact, many algorithms for solving 'convex feasibility problem' connected to metric projections may be generalized to different classes of more general mappings having a nonempty set of common fixed points; for more details, see [10]. In 2001, Khan and Takahashi [4] introduced and studied the following modified Ishikawa iterative scheme of two asymptotically nonexpansive self mappings S, T on a convex set C:

$$\begin{cases} x_1 \in C, \\ y_n = \beta_n T^n x_n + (1-\beta_n) x_n, \\ x_{n+1} = \alpha_n S^n y_n + (1-\alpha_n) x_n, \quad n \ge 1, \end{cases} \tag{1}$$

where $0 < \delta \le \alpha_n, \beta_n \le 1-\delta$ for some $\delta \in (0, \frac{1}{2})$. If $T = I$ in (1), then it becomes modified Mann iteration [7].

It is remarked that weak convergence result of Khan and Takahashi [4] does not apply to L^p spaces with $p \ne 2$ because none of these spaces satisfies the Opial's property. Some useful results in this direction have been recently obtained in [11].

For nonself nonexpansive mappings, some authors (see, e.g.,[12-13]) have studied strong and weak convergence in Hilbert spaces or uniformly convex Banach spaces.

The concept of nonself asymptotically nonexpansive mappings has been introduced by Chidume, Ofoedu and Zegeye[14], in 2003, as a generalization of asymptotically nonexpansive self mappings as follows:

A subset C of E is said to be a retract of E if there exists a continuous mapping $P: E \to C$ such that $Px = x$ for all $x \in C$. Recall that $P: E \to E$ is a retraction if $P^2 = P$. Let $P: E \to C$ be a nonexpansive retraction of E onto C. A nonself mapping $T: C \to E$ is asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \le k_n \left\|x-y\right\|$ for all $x, y \in C$ and for all $n \ge 1$.

Using the iteration process:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \quad n \ge 1, \end{cases} \qquad \textbf{(2)}$$

Chidume et al. [14] obtained some convergence theorems for nonself asymptotically nonexpansive mappings in uniformly convex Banach spaces.

For approximating common fixed points of two nonself asymptotically nonexpansive mappings $S, T: C \to E$, Wang [15] introduced the following iteration scheme:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P(\alpha_n S(PS)^{n-1} y_n + (1-\alpha_n)x_n), \\ y_n = P(\beta_n T(PT)^{n-1} x_n + (1-\beta_n)x_n), \ n \geq 1, \end{cases} \tag{3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$.

In case $T = I$, the iteration scheme (3) reduces to (2). For self mappings S and T, P becomes identity mapping, so (3) collapses into (1).

Define set of fixed points of T by $F(T) = \{x \in C : Tx = x\}$. If $F(T) \neq \phi$, then T is asymptotically quasi-nonexpansive [5,10,15] if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, we have $\|T^n x - p\| \leq k_n \|x - p\|$ for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

From the above definitions, it is clear that an asymptotically nonexpansive mapping must be uniformly L-Lipschitzian(or $(\lambda - \alpha)-$ uniform Lipschitz or uniformly continuous) as well as asymptotically quasi-nonexpansive but the converse statement is not generally true.

Qihou [18-20] has constructed fixed points of asymptotically quasi-nonexpansive self mappings through Mann and Ishikawa iterations. Fukhar-ud-din and Khan[16] have also studied common fixed points of these mappings by constructing the Ishikawa type iteration scheme with errors and generalized the results appearing in [18-20]. Recently, Khan et al. [17] have investigated Ishikawa type iterations via an n-step iteration scheme for a finite family of asymptotically quasi-nonexpansive self mappings.

A nonself mapping $T : C \to E$ is said to be asymptotically quasi-nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T(PT)^{n-1} x - q\| \leq k_n \|x - q\|$ for all $x \in C$, $q \in F(T)$ and for all $n \geq 1$.

A Banach space E is uniformly convex if for each $r \in (0,2]$, the modulus of convexity of E, given by

$$\delta(r) = \inf\left\{1-\frac{1}{2}\|x+y\| : \|x\|\le 1, \|y\|\le 1, \|x-y\|\ge r\right\},$$

satisfies the inequality $\delta(r) > 0$. For a sequence, the symbol $\to$ (resp. $\rightharpoonup$) denotes norm (resp. weak) convergence. The space E is said to satisfy : (i) *Opial's property* [21] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|$ for all $y \in E$ with $y \neq x$ (ii) *Kadec-Klee property [9]* if for every sequence $\{x_n\}$ in E, $x_n \xrightarrow{\text{weakly}} x$ and $\|x_n\| \to \|x\|$ together imply $x_n \to x$ as $n \to \infty$.

A mapping $T : C \to E$ is demiclosed at $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \xrightarrow{\text{weakly}} x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

Let $S = \{x \in E : \|x\| = 1\}$ and let E^* be the dual of E, that is, the space of all continuous linear functionals f on E. The norm of E is : (iii) *Gâteaux differentiable [9]* if

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in S and (iv) *Fréchet differentiable [9]* if for each x in S, the above limit is attained uniformly for $y \in S$. In the case of *Fréchet differentiable norm*, it has been obtained in [9] that

$$\langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 \le \frac{1}{2}\|x+h\|^2 \le \langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|h\|) \qquad (*)$$

for all x, h in E, where J is the normalized duality map from E to E^* defined by

$$J(x) = \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\right\},$$

$\langle .,. \rangle$ is the duality pairing between E and E^* and b is a function defined

on $[0,\infty)$ such that $\lim_{t\downarrow 0}\dfrac{b(t)}{t}=0.$

The iterative construction of fixed points hings on the parametric sequences used in the process. Indeed, a survey of the literature about approximation of fixed points of some nonlinear mappings through convergence of the iterative schemes reflects that conditions on the iteration parametric sequences play a vital role to establish the convergence results (see,e.g., [2,10,18]).

For the parametric sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0,1],$ the condition

$$(C1) \qquad \varepsilon \le \alpha_n, \beta_n \le 1-\varepsilon \text{ for some } \varepsilon > 0$$

has been used by Chidume et al. [14], Khan and Takahashi [4], Rhoades[6] and Wang [15].

We introduce a new and weaker condition on the parametric sequences $\{\alpha_n\}$ and $\{\beta_n\}$ as follows:

$$(C2) \qquad \liminf_{n\to\infty} \alpha_n > 0, \sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty \text{ and } \beta_n \in [\delta, 1-\delta]$$

for some $\delta \in (0,\frac{1}{2}).$

Clearly, $(C1)$ implies $(C2).$

We establish weak and strong convergence of the iterative scheme (1.3) under the new control condition $(C2)$ for two nonself asymptotically quasi-nonexpansive mappings in a uniformly convex Banach space equipped with several boundary conditions. Khan and Takahashi [11], Qihou [18-20] and Rhoades [6] have studied iteration sequences for asymptotically nonexpansive(quasi-nonexpansive) self mappings while Chidume at el. [14] and Wang [15] have established iteration results in the context of nonself asymptotically nonexpansive mappings. Our results will thus generalize and improve the corresponding recent results in [3-8,14-16,18-20] for nonself asymptotically quasi-nonexpansive mappings.

2. PREPARATORY LEMMAS

We need the following useful known lemmas in the next two sections.

Lemma 2.1 [14]: ***Let*** C ***be a nonempty closed convex subset of a uniformly convex Banach space*** E ***and let*** $T: C \to E$ ***be a nonself asymptotically nonexpansive mapping with a sequence*** $\{k_n\}$ ***and*** $k_n \to 1$ ***as*** $n \to \infty$***. Then*** $I - T$ ***is demiclosed at*** 0.

Lemma 2.2 [3]: ***Let*** $\{r_n\}$ ***and*** $\{s_n\}$ ***be two nonnegative real sequences such that***

$r_{n+1} \leq (1+s_n)r_n$ for all $n \geq 1$. ***If*** $\sum_{n=1}^{\infty} s_n < \infty$, ***then*** $\lim_{n\to\infty} r_n$ ***exists.***

Lemma 2.3 [22]: ***Let*** E ***be a reflexive Banach space such that*** E^* ***has the Kadec-Klee property. Let*** $\{x_n\}$ ***be a bounded sequence in*** E ***and*** $x^*, y^* \in \omega_w(x_n)$***(weak w-limit set of*** $\{x_n\}$***). Suppose*** $\lim_{n\to\infty} \|tx_n + (1-t)x^* - y^*\|$ ***exists for all*** $t \in [0,1]$***. Then*** $x^* = y^*$.

Lemma 2.4 [23]: ***Let*** $p > 1$ ***and*** $r > 0$ ***be two fixed real numbers. Then a Banach space*** E ***is uniformly convex if and only if there is a continuous strictly increasing convex function*** $g: [0,\infty) \to 0,\infty)$ ***with*** $g(0) = 0$ ***such that, for all*** $x, y \in B_r[0] = \{x \in E : \|x\| \leq r\}$,

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - \pi_p(\lambda)g(\|x-y\|)$$

where $\pi_p(\lambda) = \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$ ***for all*** $\lambda \in [0,1]$.

Lemma 2.5 [24]: ***Let*** C ***be a nonempty bounded closed convex subset of a uniformly convex Banach space*** E***. Then there is a strictly increasing and continuous convex function*** $g: [0,\infty) \to 0,\infty)$ ***with*** $g(0) = 0$ ***such that, for every Lipschitzian continuous mapping*** $T: C \to E$ ***and for all*** $x, y \in C$ ***and*** $t \in [0,1]$***, the following inequality holds:***

$$\|T(tx + (1-t)y) - (tTx + (1-t)Ty\| \leq Lg^{-1}(\|x-y\| - L^{-1}\|Tx - Ty\|),$$

where $L \geq 1$ ***is the Lipschitz constant of*** T.

We now set out to establish some useful lemmas for the development of our convergence results.

Lemma 2.6.: ***Let C be a nonempty bounded closed convex subset of a normed space E. Let $S,T:C\to E$ be uniformly continuous. Define sequence $\{x_n\}$ as in (3). If***

$$\lim_{n\to\infty}\left\|x_n-S(PS)^{n-1}x_n\right\|=0=\lim_{n\to\infty}\left\|x_n-T(PT)^{n-1}x_n\right\|,$$

then

$$\lim_{n\to\infty}\left\|x_n-Sx_n\right\|=0=\lim_{n\to\infty}\left\|x_n-Tx_n\right\|.$$

Proof: Set $c_n=\left\|x_n-S(PS)^{n-1}x_n\right\|$ **and** $d_n=\left\|x_n-T(PT)^{n-1}x_n\right\|.$

Since $\left\|x_n-y_n\right\|\le\beta_n\left\|x_n-S(PS)^{n-1}x_n\right\|\to 0$ **as** $n\to\infty$, **therefore we have the following estimate:**

$$\begin{aligned}\left\|x_n-x_{n+1}\right\|&=\left\|x_n-P\left(\alpha_nS(PS)^{n-1}y_n+(1-\alpha_n)x_n\right)\right\|\\&=\le\left\|x_n-\left(\alpha_nS(PS)^{n-1}y_n+(1-\alpha_n)x_n\right)\right\|\\&=\alpha_n\left\|x_n-S(PS)^{n-1}y_n\right\|\\&\le\left\|x_n-S(PS)^{n-1}x_n\right\|+\left\|S(PS)^{n-1}x_n-S(PS)^{n-1}y_n\right\|\\&=c_n+\left\|S(PS)^{n-1}x_n-S(PS)^{n-1}y_n\right\|.\end{aligned}$$

This together with the uniform continuity of S gives that $\lim_{n\to\infty}\left\|x_n-x_{n+1}\right\|=0.$

Note that

$$\begin{aligned}\left\|x_n-Sx_n\right\|\le\left\|x_n-x_{n+1}\right\|+\left\|x_{n+1}-S(PS)^n x_{n+1}\right\|\\+\left\|S(PS)^n x_{n+1}-S(PS)^n x_n\right\|+\left\|S(S(PS)^{n-1})x_n-Sx_n\right\|.\end{aligned}$$

Apply $\limsup$ **on both sides of the above inequality and use the definition of uniform continuity of S to get**

$$\limsup_{n\to\infty}\left\|x_n-Sx_n\right\|\le 0.$$

Hence $\lim_{n\to\infty}\|x_n - Sx_n\| = 0.$ **Similarly, we have** $\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$

That is, $$\lim_{n\to\infty}\|x_n - Sx_n\| = 0 = \lim_{n\to\infty}\|x_n - Tx_n\|.$$

Remark 2.1: Lemma 2.6 holds for the large class of uniformly continuous mappings while Lemma 3 in [4] and Lemma 3.3 in [14] work for the restricted class of uniformly $L-$ **lipschitzian mappings.**

Lemma 2.7: ***Let*** C ***be a nonempty closed convex subset of a normed space*** E ***and let*** $S,T : C \to E$ ***be nonself asymptotically quasi-nonexpansive mappings with sequences*** $\{s_n\},\{t_n\} \subset [1,\infty)$ ***such that*** $\sum_{n=1}^{\infty}(s_n - 1) < \infty, \sum_{n=1}^{\infty}(t_n - 1) < \infty,$ ***respectively. If the sequence*** $\{x_n\}$ ***is given by (3), then*** $\lim_{n\to\infty}\|x_n - q\|$ ***exists for all*** $q \in F(S)\cap F(T).$

Proof. Set $k_n = \max\{s_n, t_n\}.$ **Then** $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ **if and only if** $\sum_{n=1}^{\infty}(s_n - 1) < \infty$ **and** $\sum_{n=1}^{\infty}(t_n - 1) < \infty.$

Now for any $q \in F(S)\cap F(T),$ **we have**

$$\begin{aligned}
\|x_{n+1} - q\| &= \|P(\alpha_n S(PS)^{n-1} y_n + (1-\alpha_n)x_n) - q\| \\
&\le \|\alpha_n (S(PS)^{n-1} y_n - q) + (1-\alpha_n)(x_n - q)\| \\
&\le \alpha_n s_n \|y_n - q\| + (1-\alpha_n)\|x_n - q\| \\
&= \alpha_n s_n \|P(\beta_n T(PT)^{n-1} x_n + (1-\beta_n)x_n) - q\| + (1-\alpha_n)\|x_n - q\| \\
&\le \alpha_n s_n \|\beta_n (T(PT)^{n-1} x_n - q) + (1-\beta_n)(x_n - q)\| + (1-\alpha_n)\|x_n - q\| \\
&\le \alpha_n \beta_n s_n t_n \|x_n - q\| + \alpha_n (1-\beta_n) k_n \|x_n - q\| + (1-\alpha_n)\|x_n - q\| \\
&\le k_n^2 \|x_n - q\|.
\end{aligned}$$

By Lemma 2.2, $\lim_{n\to\infty}\|x_n - q\|$ **exists for all** $q \in F(S)\cap F(T)$ **as desired.**

Lemma 2.8. ***Let*** C ***be a nonempty closed convex subset of a uniformly convex Banach space*** $E.$ ***Let*** $S,T : C \to E$ ***be uniformly continuous and***

nonself asymptotically quasi-nonexpansive mappings with sequences $\{s_n\},\{t_n\}\subset[1,\infty)$ ***such that*** $\sum_{n=1}^{\infty}(s_n-1)<\infty,\sum_{n=1}^{\infty}(t_n-1)<\infty,$ ***respectively. Define sequence*** $\{x_n\}$ ***by (3), where*** $\{\alpha_n\}$ ***and*** $\{\beta_n\}$ ***are real sequences in*** $[0,1]$ ***and satisfy*** $(C2)$. ***Then there exists a subsequence*** $\{x_i\}$ ***of*** $\{x_n\}$ ***such that***

$$\lim_{i\to\infty}\|x_i-Sx_i\|=0=\lim_{i\to\infty}\|x_i-Tx_i\|.$$

Proof: Let $\{k_n\}$ **be as in the above proof. For any** $q\in F(S)\cap F(T)$**,** $\lim_{n\to\infty}\|x_n-q\|$ **exists as proved in Lemma 2.7. Consequently,** $\{x_n-q,y_n-q,T(PT)^{n-1}x_n-q,S(PS)^{n-1}y_n-q\}$ **is bounded. Therefore, we can obtain a closed ball** $B_r[0]$ **such that** $\{x_n-q,y_n-q,T(PT)^{n-1}x_n-q,T(PT)^{n-1}y_n-q\}\subset B_r[0]\cap C.$

Applying Lemma 2.4 to the scheme(1.3), we have

$$\begin{aligned}\|y_n-q\|^2&=\left\|P(\beta_nT(PT)^{n-1}x_n+(1-\beta_n)x_n)-q\right\|^2\\&\le\left\|\beta_nT(PT)^{n-1}x_n+(1-\beta_n)x_n-q\right\|^2\\&=\left\|\beta_n(T(PT)^{n-1}x_n-q)+(1-\beta_n)(x_n-q)\right\|^2\\&\le\beta_n\left\|T(PT)^{n-1}x_n-q\right\|^2+(1-\beta_n)\|x_n-q\|^2-\pi_2(\beta_n)g(d_n)\\&\le t_n^2\|x_n-q\|^2-\pi_2(\beta_n)g(e_n)\end{aligned}\tag{5}$$

where $e_n=\left\|x_n-T(PT)^{n-1}x_n\right\|$**.**

Using the scheme (3), Lemma 2.4 and the inequality (5), we infer that

$$\begin{aligned}\|x_{n+1}-q\|^2&=\left\|P(\alpha_nS(PS)^{n-1}y_n+(1-\alpha_n)x_n)-q\right\|^2\\&\le\left\|\alpha_nS(PS)^{n-1}y_n+(1-\alpha_n)x_n-q\right\|^2\end{aligned}$$

$$\leq \left\|\alpha_n(S(PS)^{n-1}y_n - q) + (1-\alpha_n)(x_n - q)\right\|^2$$

$$\leq \alpha_n \left\|S(PS)^{n-1}y_n - q\right\|^2 + (1-\alpha_n)\|x_n - q\|^2$$

$$-\pi_2(\alpha_n) g\left(\left\|S(PS)^{n-1}y_n - x_n\right\|\right)$$

$$\leq \alpha_n s_n^2 \|y_n - p\|^2 + (1-\alpha_n)\|x_n - q\|^2 - \pi_2(\alpha_n) g(h_n)$$

$$\leq \alpha_n s_n^2 t_n^2 \|x_n - q\|^2 - \alpha_n s_n^2 \pi_2(\beta_n) g(e_n)$$

$$+ (1-\alpha_n)\|x_n - q\|^2 - \pi_2(\alpha_n) g(h_n)$$

$$\leq k_n^4 \|x_n - q\|^2 - \alpha_n \pi_2(\beta_n) g(e_n) - \pi_2(\alpha_n) g(h_n)$$

$$\leq \|x_n - q\|^2 - \alpha_n \pi_2(\beta_n) g(e_n) - \pi_2(\alpha_n) g(h_n) + \left(k_n^4 - 1\right)Q$$

where $h_n = \left\|S(PS)^{n-1}y_n - x_n\right\|$ **and** Q **is a real number such that** $\|x_n - p\|^2 \leq Q.$

From the above estimate, we obtain the following two important inequalities:

$$\pi_2(\alpha_n) g(h_n) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \left(k_n^4 - 1\right)Q \qquad \textbf{(6)}$$

and

$$\alpha_n \pi_2(\beta_n) g(e_n) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \left(k_n^4 - 1\right)Q. \qquad \textbf{(7)}$$

Let m **be any positive integer. Summing up the terms from** 1 **to** m **on both sides in the inequality (6), we have**

$$\sum_{n=1}^{m} \pi_2(\alpha_n) g(h_n) \leq \|x_1 - q\|^2 - \|x_{m+1} - q\|^2 + Q\sum_{n=1}^{m}(k_n^4 - 1)$$

$$\leq \|x_1 - q\|^2 + Q\sum_{n=1}^{m}(k_n^4 - 1).$$

When $m \to \infty$ **in the above inequality, we get**

$$\sum_{n=1}^{\infty}\pi_2(\alpha_n)g(h_n)<\infty$$

and hence

$$\liminf_{n\to\infty} g(h_n)=0.$$

By the properties of g **, we have**

$$\liminf_{n\to\infty} h_n=0.$$

Since $\lim\inf_{n\to\infty}\alpha_n>0,$ **therefore we have** $\alpha_n>\alpha$ **for all** $n\geq n_0$ **and** $\alpha>0.$

By the above fact, the inequality (7) reduces to

$$\alpha\delta^2\sum_{n=n_0}^{\infty}g(e_n)\leq\left\|x_{n_0}-q\right\|^2+Q\sum_{n=n_0}^{\infty}(k_n^4-1)<\infty,$$

which further, implies that

$$\lim_{n\to\infty}e_n=0.$$

That is :

$$\lim_{n\to\infty}\left\|T(PT)^{n-1}x_n-x_n\right\|=0.$$

Observe that

$$\left\|x_n-S(PS)^{n-1}x_n\right\|\leq\left\|S(PS)^{n-1}x_n-S(PS)^{n-1}y_n\right\|+h_n$$

Operating $\liminf$ **on both sides of the above inequality, we get**

$$\liminf_{n\to\infty}\left\|x_n-S(PS)^{n-1}x_n\right\|=0.$$

Hence, there exists a subsequence $\{x_i\}$ **of** $\{x_n\}$ **such that**

$$\lim_{i\to\infty}\left\|x_i-S(PS)^{i-1}x_i\right\|=0=\lim_{i\to\infty}\left\|x_i-S(PS)^{i-1}x_i\right\|.$$

Finally, by Lemma 2.6, we get that

$$\lim_{i\to\infty}\left\|x_i-Sx_i\right\|=0=\lim_{i\to\infty}\left\|x_i-Tx_i\right\|.$$

3. NONSELF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

We prove a pair of lemmas in the context of asymptotically nonexpansive nonself mappings.

In this section, we obtain weak and strong convergence of the scheme (3) for a larger class of mappings, namely, nonself asymptotically quasi-nonexpansive mappings; for this we have to impose the additional hypothesis of "uniform continuity" on the mappings. We thus obtain the following generalization of Theorem 3.1(i).

Theorem 4.1. ***Let*** E ***be a uniformly convex Banach space satisfying Opial's property and*** C ***be a nonempty closed convex subset of*** E. ***Let*** $S,T:C\to E$ ***be uniformly continuous and asymptotically quasi-nonexpansive mappings with sequences*** $\{s_n\},\{t_n\}\subset[1,\infty)$ ***such that*** $\sum_{n=1}^{\infty}(s_n-1)<\infty, \sum_{n=1}^{\infty}(t_n-1)<\infty,$ ***respectively. Assume that*** $I-T$ ***and*** $I-S$ ***are demiclosed with respect to*** 0. ***Define sequence*** $\{x_n\}$ ***as in (3), where*** $\{\alpha_n\}$ ***and*** $\{\beta_n\}$ ***are real sequences in*** $[0,1]$ ***which satisfy*** $(C2)$. ***If*** $F(S)\cap F(T)\neq\phi,$ ***then*** $\{x_n\}$ ***converges weakly to a common fixed point of*** S ***and*** T.

Proof: Let $q\in F(S)\cap F(T)$. **Then** $\lim_{n\to\infty}\|x_n-q\|$ **exists as proved in Lemma 2.7: Let** $\{x_i\}$ **be the subsequence introduced in Lemma 2.8. Since** E **is reflexive, there exists a subsequence** $\{x_j\}$ **of** $\{x_i\}$ **converging weakly to some** $z_1\in C$. **As in Lemma 2.8,** $\lim_{i\to\infty}\|x_i-Sx_i\|=0=\lim_{i\to\infty}\|x_i-Tx_i\|$ **and** $I-S, I-T$ **are demiclosed at** $0,$ **so we obtain** $Sz_1=z_1$ **and** $Tz_1=z_1.$ **That is,** $z_1\in F(S)\cap F(T)$. **In order to show that** $\{x_i\}$ **converges weakly to** $z_1,$ **take another subsequence** $\{x_k\}$ **of** $\{x_i\}$ **converging weakly to some** $z_2\in C$. **Again in the same way, we can prove that** $z_2\in F(S)\cap F(T)$. **We can prove that** $z_1=z_2$ **on the basis of the Opial's property as in Theorem**

3.1(i).

Therefore, $\{x_i\}$ converges weakly to a common fixed point of S and T. The existence of the limit of the sequence $\{\|x_n - q\|\}$ implies that $\{x_n\}$ converges weakly to a a common fixed point of S and T.

Recall that a mapping $T: C \to E$ is (i) demi-compact if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to p \in C$ (ii) completely continuous or compact if $\{x_n\}$ is bounded in C implies that $\{Tx_n\}$ has a convergent subsequence in C.

Next we establish strong convergence results.

Theorem 4.2: ***Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $S,T: C \to E$ be uniformly continuous and nonself asymptotically quasi-nonexpansive mappings with sequences $\{s_n\},\{t_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(s_n - 1) < \infty, \sum_{n=1}^{\infty}(t_n - 1) < \infty$, respectively. Define sequence $\{x_n\}$ by (3), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$ which satisfy $(C2)$. If $F(S)\cap F(T) \neq \phi$ and either S or T is completely continuous, then $\{x_n\}$ converges strongly to some common fixed point of S and T.***

Proof: As proved in Lemma 2.8, there exists a subsequence $\{x_i\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}\|x_i - Sx_i\| = 0 = \lim_{i\to\infty}\|x_i - Tx_i\|. \quad \textbf{(10)}$$

Since $\{x_i\}$ is bounded and S is completely continuous, so $\{Sx_i\}$ has a convergent subsequence $\{Sx_j\}$. Suppose $Sx_j \to z \in C$.

Then $$\|x_j - z\| \le \|x_j - Sx_j\| + \|Sx_j - z\| \to 0.$$

Hence, $x_j \to z$. Now (10) assures that z is a common fixed point of S and

$T.$ As $\lim_{n\to\infty}\|x_n - q\|$ exists for all $q \in F(S) \cap F(T),$ so $x_n \to z.$

Theorem 4.3: ***Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $S, T : C \to E$ be uniformly continuous and nonself asymptotically quasi-nonexpansive mappings with sequences $\{s_n\}, \{t_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(s_n - 1) < \infty, \sum_{n=1}^{\infty}(t_n - 1) < \infty,$ respectively. Define sequence $\{x_n\}$ by (3), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$ which satisfy $(C2)$. If $F(S) \cap F(T) \neq \phi$ and either S or T is demi-compact, then $\{x_n\}$ converges strongly to some common fixed point of S and T.***

Proof: Suppose that S is demi-compact. Since $\{x_n\}$ is bounded and $\lim_{i\to\infty}\|x_i - Sx_i\| = 0$ for a subsequence $\{x_i\}$ of $\{x_n\}$, therefore there exists a subsequence $\{x_j\}$ of $\{x_i\}$ such that $\{x_j\}$ converges strongly to $z \in C.$ In view of (10), z is a common fixed point of S and $T.$ As $\lim_{n\to\infty}\|x_n - q\|$ exists for all $q \in F(S) \cap F(T),$ so $x_n \to z.$

Remark 4.1: Recall that (i) $(C1)$ is a special case of $(C2)$ (ii) nonself asymptotically nonexpansive mapping is nonself asymptotically quasi-nonexpansive (iii) $(L-\alpha)$ uniform Lipschitz mapping is uniformly continuous and (iv) for self mappings S and T, P becomes identity mapping.

In view of these facts, Theorems 3.3-3.4 of Wang [15], Theorem of Qihou[20], Theorems 2-3 of Rhoades [6], Theorem 2 of Khan and Takahashi [4] and Theorem 2.2 of Schu [8] are special cases of our strong convergence results (Theorems 4.2-4.3).

Remark 4.2. Note that $(C1)$ does not hold for the sequence

$\{\alpha_n\} = \left\{0,1,1-\frac{1}{2},1,1-\frac{1}{3},1,...,1,1-\frac{1}{n},1,...\right\}$ **and hence the results in [4, 6-7, 15, 18] are not applicable to this sequence. As $\{\alpha_n\}$ satisfies $(C2)$ so our weak and strong results work for this sequence of parameters.**

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