Product, Ratio and Single Moments of Lower Record Values of Inverse Weibull Distribution

M. Aleem

Abstract:

In this paper the product, ratio and single moments of the Lower Record values are obtained from Inverse Weibull distribution.

Key Words:

Moments, Recurrence Relations, Lower Record values, Inverse Weibull distribution.

1. Introduction:

A random variable $X$ has an Inverse Weibull distribution with pdf given by

$$f(x) = \frac{m}{\theta x^{m+1}} \exp\left(-\frac{1}{\theta x^m}\right) \quad (1.1)$$

where $x>0$, $(\theta,m)>0$

And the corresponding cdf is given by

$$F(x) = \exp\left(-\frac{1}{\theta x^m}\right) \quad (1.2)$$
where $x > 0$, $(\theta, m) > 0$

If we put $m = 1$, it reduces to the Inverse Exponential distribution. If we put $m = 2$, it reduces to the Inverse Rayleigh distribution. Some work has been done on Inverse Rayleigh distribution by Voda (1972), Ghanaph (1993), Mukargee & Saren (1984) and Mukarjee & Mait (1996). For distributional properties of Record values of inverse weibull distribution see Aleem and Pasha (2003), Aleem (2004). Other references are Ahsanullah (1995), Ahsanullah and Novzorov (2001).

Let $v(x) = \frac{1}{\theta x^m}$ and from (1.1) (1.2), we have

$$F(x) = \frac{1}{v^*(x)} f(x)$$

### 2. Product Moments

The lower record values are respected by $X_{L(1)}$, $X_{L(2)}$, ..., $X_{L(n)}$

The joint pdf of $X_{L(r)}$ and $X_{L(s)}$ $(s > r)$ is

$$f_{(r), (s)}(x, y) = C_{r,s} \left[ H(x) \right]^{r-1} \left[ H(y) - H(x) \right]^{s-r-1} h(x) . f(y)$$

(2.1)

where $C_{r,s} = -\frac{1}{(r-1)!(s-r-1)!}$, $-\infty < y < x < \infty$

and

$$H(x) = -\ln F(x) \quad 0 < F(x) < 1$$

$$h(x) = -\frac{d}{dx} H(x)$$

If $g$ is a Borel measurable function from $\mathbb{R}^2$ to $\mathbb{R}$, then

$$E \{ g(X_{L(r)}, X_{L(s)}) \} = C_{r,s} \int_{0<y<x<\infty} g(x, y) \left[ H(x) \right]^{r-1} \left[ H(y) - H(x) \right]^{s-r-1} h(x) f(y) \, dx \, dy$$

(2.2)

**Theorem 2.1:**

For the distribution function $F(x)$ in (1.2), we have
\[
E \{g(X_{L(r)}, X_{L(s)})\} = E\{u(X_{L(r-1)}, X_{L(s-1)})\} - E\{u(X_{L(r)}, X_{L(s-1)})\}
\]
Where \( u^*(x, y) = \frac{\partial}{\partial x} u(x, y) = g(x, y), v^*(x) \)

And \( v^*(x) = \left| \frac{\partial}{\partial x} v(x) \right| \)

**Proof:** Using (1.3) in (2.2), we have

\[
E \{g(X_{L(r)}, X_{L(s)})\} = C_{r,s} \iint_{0 < y < x < \infty} u^*(x, y)[H(x)]^{r-1}[H(y) - H(x)]^{s-r-1} f(y) \, dx \, dy
\]

Integrating RHS w.r.t “x”, we get as :

\[
= C_{r-1,s-1} \iint_{0 < y < x < \infty} u(x, y)[H(x)]^{r-2}[H(y) - H(x)]^{s-r-2} h(x) f(y) \, dx \, dy
\]

and

\[
E \{g(X_{L(r)}, X_{L(s)})\} = E\{u(X_{L(r-1)}, X_{L(s-1)})\} - E\{u(X_{L(r)}, X_{L(s-1)})\}
\]

Hence the Theorem
Theorem 2.2:

For the distribution function F(x) in (1.2) the recurrence relation for the product Moments of Inverse Weibull distribution is given by:

$$u_{(r),(s)}^{J,K} = \frac{m}{\theta(J-m)} [u_{(r-1),(s-1)}^{(J-m),K} - u_{(r),(s-1)}^{(J-m),K}]$$

**Proof:** Now $v(x) = \frac{1}{\theta x^m}$ and $g(x,y) = x^J y^K$. This gives

$$u(x,y) = \frac{m}{\theta(J-m)} x^{J-m} y^K$$

putting in Theorem (2.1), we get the required recurrence relation.

Note, For the ratio, let $K = -j$, then $u_{(r),(s)}^{J,-j} = E\left(\frac{X_r}{X_s}\right)^j \forall J$

3. Single Moments

The lower record values are represented by $X_{L(1)}$, $X_{L(2)}$, ---, $X_{L(N)}$. The pdf of $X_{L(n)} (n \geq 2)$ is

$$f_{(n)}(x) = \frac{[H(x)]^{n-1}}{(n-1)!} f(x) \quad (3.1)$$

where $H(x) = -\ln F(x)$ \quad $0 < F(x) < 1$

$$h(x) = -\frac{d}{dx} H(x)$$

If $g$ is a Borel measurable function from $\mathbb{R}^2$ to $\mathbb{R}$, then

$$E\{g(X_{L(n)})\} = C_n \int_{0<x<\infty} g(x)[H(x)]^{n-1} f(x) \, dx \quad (3.2)$$

where $C_n = \frac{1}{(n-1)!}$
Theorem 3.1:

For the distribution function $F(x)$ in (1.2), we have

$$E\{g(X_{L(n)})\} = E\{u(x_{L(n-1)})\} - E\{u(x_{L(n)})\}$$

Proof: Using (1.3) in (3.2), we have

$$E\{g(X_{L(n)})\} = C_n \int_{0<x<\infty} u^*(x)[H(x)]^{n-1} F(x) dx$$

Integrating RHS, we get

$$= C_{n-1} \int_{0<x<\infty} u(x)[H(x)]^{n-2} f(x) dx - C_n$$

and

$$E\{g(X_{L(n)})\} = E\{u(x_{L(n-1)})\} - E\{u(x_{L(n)})\}$$

Hence the Theorem

Theorem 3.2:

For the distribution function $F(x)$ in (1.2) the recurrence relation for the single moments of Inverse Weibull Distribution is given by

$$u^J_{(n)} = \frac{m}{\theta(J - m)} \left[ u^{J-m}_{(n-1)} - u^{J-m}_{(n)} \right]$$

Proof: Now $\nu(x) = \frac{1}{\theta x^m}$ and $g(X_{L(n)}) = x^J$ this gives

$$u(X_{L(n)}) = \frac{m}{\theta(J - m)} x^{J-m},$$

putting in (3.1) we get the recurrence relation.

Remark: This recurrence relation in Theorem (2.2) between the moments of ratio of two lower record values, quasi-ranges, joint moment generating function, characteristic functions,
whenever they exist can be obtained by setting respectively \( g(x,y) \) equal to

\[
\left( x^J y^{-K} \right) (y - x) e^{T(x+y)}, e^{T(x+y)}.
\]

**Note:** All the above results goes for single lower record values if one replace the function \( g(.,.) \) by a function of single variable \( g(.) \).