SOME RESULTS ON DERIVATIONS OF BCI-ALGEBRAS

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ABSTRACT: In this note, we investigate some fundamental properties and prove some results on derivations of BCI-algebras.

1. INTRODUCTION

In the theory of rings and near rings, the properties of derivations is an important topic to study [8, 10]. In [7], Jun and Xin applied the notions of rings and near rings theory to BCI-algebras and obtained some properties. In this paper, we prove some results on derivations of BCI-algebras. First, we show that a derivation of a BCK-algebra is regular and prove that if \( d \) is a derivation of a BCI-algebra \( X \) and \( \alpha \in X \) such that \( a \ast d(x) = 0 \) or \( d(x) \ast a = 0 \), for all \( x \in X \), then \( d \) is regular and \( X \) is a BCK-algebra. Also we study a derivation of a \( p \)-semisimple BCI-algebra \( X \), then \( d_1, d_2 \) are derivations of a \( p \)-semisimple BCI-algebra \( X \), then \( d_1 \circ d_2 \) is also a derivation of \( X \) and \( d_1 \circ d_2 = d_2 \circ d_1 \). Finally, we show that \( d_1 \ast d_2 = d_2 \ast d_1 \), where \( d_1, d_2 \) are derivations of a \( p \)-semisimple BCI-algebra \( X \).

2. PRELIMINARIES

Let \( X \) be a set with a binary operation \( \ast \) and a constant 0. Then \( (X, \ast, 0) \) is called a BCI-algebra, if it satisfies the following axioms for all \( x, y, z \in X \):

- **BCI-1:** \( ((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0 \);
- **BCI-2:** \( (x \ast (x \ast y)) \ast y = 0 \);
- **BCI-3:** \( x \ast x = 0 \);
- **BCI-4:** \( x \ast y = 0 \) and \( y \ast x = 0 \) imply \( x = y \) [6].

Define a binary relation \( \leq \) on \( X \) by putting \( x \leq y \) if and only if \( x \ast y = 0 \). Then \( (X, \leq) \) is a partially ordered set. A BCI-algebra \( X \) satisfying \( 0 \leq x \), for all \( x \in X \), is called a BCK-algebra. In any BCI-algebra \( X \), the following hold [6] for all \( x, y, z \in X \):
Let X be a BCI-algebra. We denote \( x \wedge y = y \ast (y \ast x) \), For more details, we refer to [2, 9, 12, 13].

**Definition 2.1** [7] Let X be a BCI-algebra. By a \((\ell, r)\) – deviation of X, we mean a self map \( d \) of X satisfying the identity \( d(x \ast y) = d(x) \ast y \wedge x \ast d(y) \), for all \( x, y \in X \). If X satisfies the identity \( d(x \ast y) = x \ast d(y) \wedge d(x) \ast y \), for all \( x, y \in X \), then we say that \( d \) is a \((r, \ell)\)-derivation of X. Moreover, if \( d \) is both a \((\ell, r)\) and a \((r, \ell)\)-derivation, we say that \( d \) is a derivation of X.

**Definition 2.2** [7]. A self map \( d \) of BCI-algebra X is said to be regular if \( d(0) = 0 \).

**Proposition 2.3** [7] Let \( d \) be a regular derivation of BCI-algebra X. Then the following hold:
3. SOME RESULTS ON DERIVATIONS

First, we study derivations on BCK-algebras:

Proposition 3.1: Every \((r, \ell)\)-derivations (or a \((\ell, r)\)-derivation) of a BCK-algebra is regular.

Proof. Let \(X\) is a BCK-algebra and \(d\) a \((r, \ell)\)-deviation of \(X\). Then for all \(x \in X\), we have:

\[
0 \leq d(0) = d(0^*x) = 0^*d(x) \leq d(0)^*x
\]

and

\[
= 0 \leq d(0)^*x = 0
\]

Now let \(d\) is a \((\ell, r)\)-deviation of \(X\). Then for all \(x \in X\) we have:

\[
0 \leq d(0) = d(0^*x) = d(0)^*x^0d(x)
\]

\[
= d(0)^*x^0d(x) = 0
\]

From Proposition 3.1 we get:

Corollary 3.2: A derivation of a BCK-algebra is regular.

In Proposition 3.3 (resp. Proposition 3.4), we assume that \(d\) is a derivation of a BCI-algebra \(X\) and \(a \in X\) such that \(d(x)^*a = 0\), (resp. \(a^*d(x) = 0\)), for all \(x \in X\), then we show that \(d\) is a regular derivation of \(X\) and \(X\) is a BCK-algebra:

Proposition 3.3. Let \(d\) be a derivation of a BCI-algebra \(X\) and \(a \in X\) such that \(d(x)^*a = 0\), for all \(x \in X\). Then \(d\) is a regular derivation of \(X\). Moreover \(X\) is a BCK-algebra.

Proof. Let \(d\) be a derivation of a BCI-algebra \(X\) and let \(a \in X\) such that \(d(x)^*a = 0\), for all \(x \in X\). Since \(d\) is \((\ell, r)\)-derivation we get:

\[
0 = d(x^*a)^*a = (d(x)^*a^x*d(a))^*a
\]

\[
= 0^*x^*d(a))^*a = 0^*a
\]

thus \(0 \leq a\), and so \(x \in X_+\). This shows that

\[
0 = d(0) = d(0^*a) = d(0)^*a^0d(a)
\]

\[
= 0^*0^*d(a) = 0
\]
Hence \( d \) is a regular derivations of \( X \). So by Proposition 2.3 (i) we have 
\[ d(x) \leq x, \text{ for all } x \in X \] 
and so.

\[ 0 \leq d(x) = (d(x) \ast a) \ast d(x) \]
\[ = 0 \ast a = 0 \]

Thus \( 0 \leq x \) for all \( x \in X \) and so \( 0 = (0 \ast x) \ast 0 = 0 \ast x \). Then we have \( 0 \leq x \), for all \( x \in X \). Which implies that \( X \) is a BCK-algebra.

Similarly we can prove:

Proposition 3.4. Let \( d \) be a derivation of a BCI-algebra \( X \) and \( a \in X \) such that \( a \ast d(x) = 0 \), for all \( x \in X \). Then \( d \) is a regular derivation of \( X \). Moreover, \( X \) is a BCK-algebra.

Finally, we study a derivations of a \( p \)–semisimple BCI-algebra:

Definition 3.5: Let \( X \) be a BCI-algebra and \( d_1,d_2 \) two self maps of \( X \). We define \( d_1 \circ d_2 : X \to X \) as:

\[ d_1 \circ d_2(x) = d_1(d_2(x)), \text{ for all } x \in X. \]

Proposition 3.6 Let \( X \) be a \( p \)–semisimple BCI-algebra and \( d_1 \), \( d_2 \) the \( (\ell,r) \)-derivations of \( X \). Then \( d_1 \circ d_2 \) is also a \( (\ell,r) \)-derivation of \( X \).

Proof. Let \( X \) be a \( p \)–semisimple BCI–algebra and \( d_1,d_2 \) are \( (\ell,r) \)-derivations of \( X \). Then by (14) and proposition 2.3 (ii), we get for all \( x,y \in X \):

\[ (d_1 \circ d_2)(x \ast y) = d_1(d_2(x) \ast y \uparrow x \ast d_2(y)) = d_1(d_2(x) \ast y) \]
\[ = d_1(d_2(x)) \ast y \uparrow d_2(x) \ast d_1(y) = d_1(d_2(x)) \ast y \]
\[ = (x \ast d_1(d_2(y))) \ast ((x \ast d_1(d_2(y))) \ast (d_1(d_2(x)) \ast y)) \]
\[ = (d_1 \circ d_2)(x) \ast y \uparrow (d_1 \circ d_2)(y) \]

Which implies that \( d_1 \circ d_2 \) is a \( (\ell,r) \)-derivation of \( X \).

Similarly, we can prove:

Proposition 3.7. Let \( X \) be a \( p \)–semisimple BCI–algebra and \( d_1,d_2 \) are \( (r,\ell) \)-derivations of \( X \). Then \( d_1 \circ d_2 \) is also a \( (r,\ell) \)-derivation of \( X \).
Combining Propositions 3.6 and 3.7, we get:

**Theorem 3.8.** Let $X$ be a $p$–semisimple BCI–algebra and $d_1, d_2$ derivations of $X$. Then $d_1 \circ d_2$ is also a derivation of $X$.

**Proposition 3.9.** Let $X$ be a $p$–semisimple BCI–algebra and $d_1, d_2$ derivations of $X$. Then $d_1 \circ d_2 = d_2 \circ d_1$.

**Proof.** Let $X$ be a $p$–semisimple BCI–algebra and $d_1, d_2$, the derivations of $X$. Since $d_2$ is a $(l, r)$–derivation of $X$, then for all $x, y \in X$:

$$d_1 \circ d_2 (x \ast y) = d_1 (d_2 (x \ast y)) = d_1 (d_2 (x) \ast y \ast d_2 (y))$$

$$= d_1 (d_2 (x) \ast y)$$

But $d_1$ is a $(r, l)$–derivation of $X$, so

$$(d_1 \circ d_2) (x \ast y) = d_1 (d_2 (x) \ast y)$$

$$= d_2 (x) \ast d_1 (y)$$

thus we have for all $x, y \in X$:

$$(d_1 \circ d_2) (x \ast y) = d_2 (x) \ast d_1 (y) \quad (1)$$

Also, since $d_1$ is a $(r, l)$–derivation of $X$, then for all $x, y \in X$:

$$(d_2 \circ d_1) (x \ast y) = d_2 (x \ast d_1 (y)) \ast d_1 (x) \ast y$$

$$= d_2 (x \ast d_1 (y))$$

But $d_2$ is a $(l, r)$–derivation of $X$, so

$$(d_2 \circ d_1) (x \ast y) = d_2 (x \ast d_1 (y))$$

$$= d_2 (x) \ast d_1 (y) \ast x \ast d_2 (d_1 (y))$$

$$= d_2 (x) \ast d_1 (y)$$

Thus we have for all $x, y \in X$:

$$(d_2 \circ d_1) (x \ast y) = d_2 (x) \ast d_1 (y) \quad (2)$$

From (1) and (2) we get for all $x, y \in X$:

$$d_1 \circ d_2 (x \ast y) = (d_2 \circ d_1) (x \ast y)$$
By putting $y = 0$ we get for all $x \in X$:

$$(d_1 \circ d_2)(x) = (d_2 \circ d_1)(x)$$

Which implies that $d_1 \circ d_2 = d_2 \circ d_1$

**Definition 3.10.** Let $X$ be a BCI-algebra and $d_1, d_2$, two self maps of $X$. We define $d_1 \ast d_2 : X \longrightarrow X$ as:

$$(d_1 \ast d_2)(x) = d_1(x) \ast d_2(x), \text{ for all } x \in X$$

**Proposition 3.11.** Let $X$ be a $p$-semisimple BCI–algebra and $d_1, d_2$ derivations of $X$. Then $d_1 \ast d_2 = d_2 \ast d_1$

**Proof.** Let $X$ is a $p$–semisimple BCI–algebra and $d_1, d_2$ derivations of $X$. Since $d_2$ is a $(l,r)$-derivation of $X$, then for all $x, y \in X$:

$$(d_1 \circ d_2)(x \ast y) = d_1(d_2(x) \ast y \ast d_2(y)) = d_1(d_2(x) \ast y)$$

But $d_1$ is a $(r,l)$-derivation of $X$, so

$$d_1(d_2(x) \ast y) = d_2(x) \ast d_1(y) \ast d_1(d_2(x) \ast y) = d_2(x) \ast d_1(y)$$

hence

$$(d_1 \circ d_2)(x \ast y) = d_2(x) \ast d_1(y) \text{ for all } x, y \in X \quad (3)$$

Also, we have that $d_2$ is an $(r,l)$-derivation of $X$, then for all $x, y \in X$:

$$(d_1 \circ d_2)(x \ast y) = d_1(x \ast d_2(y) \ast d_2(x) \ast y) = d_1(x \ast d_2(y))$$

But $d_1$ is a $(l,r)$-derivation of $X$, so

$$d_1(x \ast d_2(y)) = d_1(x) \ast d_2(y) \ast x \ast d_1(d_2(y)) = d_1(x) \ast d_2(y)$$

Thus

$$(d_1 \circ d_2)(x \ast y) = d_2(x) \ast d_1(y) \text{ for all } x, y \in X \quad (4)$$
From (3) and (4) we get:

\[ d_2(x) * d_1(y) = d_1(x) * d_2(y) \quad \text{for all} \quad x, y \in X \]

By putting \( x = y \) we get for all \( x \in X \):

\[ d_2(x) * d_1(x) = d_1(x) * d_2(x) \]

\[ (d_2 * d_1)(x) = (d_1 * d_2)(x) \]

Which implies that \( d_2 * d_1 = d_1 * d_2 \)

4. REFERENCES

1. B. Ahmad, On Iseki’s Bci-algebras, Jr.of Natural Sciences and Mathematics, 8 (1980), 125-130.